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GENERALIZED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY TWO MUTUALLY INDEPENDENT FRACTIONAL BROWNIAN MOTIONS

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Abstract

This paper deals with a class of generalized backward stochastic differential equations driven by two mutually independent fractional Brownian motions (FGBSDEs in short). The existence and uniqueness

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of solutions for FGBSDE as well as a comparison theorem are obtained.

1. Introduction

Backward stochastic differential equations (BSDEs) were first introduced by Pardoux and Peng in 1990, who established a fundamental existence and uniqueness result under the Lipschitz assumption [14]. This pioneering work laid the groundwork for BSDEs to become a powerful tool in various fields such as financial mathematics, stochastic control, and the stochastic interpretation of solutions to partial differential equations (PDEs). The application of BSDEs in finance, particularly in option pricing and risk management, has been particularly influential, as it allows for the modeling of problems with terminal conditions that naturally arise in these contexts.

Over the years, BSDEs have been extended to accommodate more complex scenarios, including those involving fractional Brownian motion (fBm) $(B_t^H)_{t\geq 0}$, which is characterized by the Hurst parameter *H*. The fBm has garnered significant attention due to its ability to model long-range dependencies and self-similar processes, which are observed in various fields such as finance, telecommunications, and physics [11-13]. However, since fBm is not semimartingale when $H \neq \frac{1}{2}$, the classical stochastic calculus tools, such as the Itô calculus, cannot be directly applied. This presents a significant mathematical challenge when defining the fractional stochastic integral, a key component in BSDEs driven by fBm.

To address these challenges, two primary approaches to defining stochastic integrals with respect to fBm have been developed. The first approach, the pathwise Riemann-Stieltjes integral introduced by Young [16], is based on regularity conditions and exhibits properties similar to the Stratonovich integral. This integral is well-suited for dealing with problems involving pathwise analysis but can be cumbersome in certain stochastic contexts due to the need for regularity conditions.

The second approach, developed by Decreusefond and Ustunel [5], utilizes the divergence operator, also known as the Skorohod integral, which is defined within the framework of Malliavin calculus. This method allows for the definition of an integral with respect to fBm that retains a zero-mean property, making it more analogous to the Itô integral. The Skorohod integral is particularly useful in stochastic control and filtering problems, where anticipative calculus plays a crucial role. These developments have led to a broader understanding and applicability of BSDEs driven by fBm, as seen in the work of Hu and Peng [8], who studied BSDEs driven by fBm and established connections to PDEs.

The mathematical intricacies of fBm-driven BSDEs have further led to the exploration of new classes of equations. Recently, Aidara and Sagna [1] introduced BSDEs driven by two mutually independent fractional Brownian motions with stochastic Lipschitz coefficients. This extension not only broadens the scope of BSDEs in modeling complex systems with multiple sources of uncertainty but also poses new challenges in terms of proving existence and uniqueness results. The interplay between the two fBm processes introduces additional layers of complexity, particularly in the analysis of the dependence structure and the handling of non-Markovian dynamics.

Inspired by these advances, this paper focuses on a more generalized class of BSDEs, namely fractional generalized BSDEs (FGBSDEs). These equations extend the classical BSDE framework by incorporating an integral with respect to a continuous increasing process. This new term adds a dimension of complexity and richness to the study of BSDEs, as it introduces a form of path dependency that is not present in standard BSDEs. The inclusion of this term is motivated by the need to model phenomena where the stochastic process is influenced by cumulative effects over time, which is a common scenario in fields such as finance and physics [9].

The FGBSDEs considered in this paper are represented by the following equations:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds + \int_t^T g(s, \eta_s, Y_s)d\Lambda_s$$

$$- \int_t^T Z_{1,s}dB_{1,s}^{H_1} - \int_t^T Z_{2,s}dB_{2,s}^{H_2}, \quad t \in [0, T], \tag{1.1}$$

where $\{B_{1,t}^{H_1}, t \in [0, T]\}$ and $\{B_{2,t}^{H_2}, t \in [0, T]\}$ are two mutually independent fractional Brownian motions, and $\{\Lambda_t, t \in [0, T]\}$ is a continuous real-valued increasing process. The novelty of this equation lies in the integral with respect to Λ_t, which introduces a new type of memory effect, reflecting cumulative influences that grow over time.

This paper aims to explore the existence and uniqueness of solutions to such FGBSDEs and to establish a comparison theorem. The comparison theorem is particularly important in demonstrating how solutions to FGBSDEs behave under different initial conditions or parameters, which is crucial for applications in finance where such equations are used to model derivative pricing under different market conditions.

Furthermore, we explore the connection between FGBSDEs and associated partial differential equations (PDEs). This connection underscores the deep interplay between stochastic analysis and PDEs, a relationship that has been instrumental in advancing both fields. The study of FGBSDEs also opens up new avenues for research, particularly in areas where the cumulative effects and long-range dependencies modeled by fBm and continuous increasing processes play a crucial role.

In summary, this paper contributes to the growing body of literature on BSDEs by extending the framework to include fractional generalized BSDEs, offering new insights into the behavior of such systems under complex stochastic dynamics. By building on the work of Aidara, Sagna, Borkowska, and others, we provide a rigorous mathematical treatment of these equations, with potential applications in various fields where modeling under uncertainty is paramount.

2. Preliminaries

2.1. Fractional Stochastic calculus

We assume that there are two mutually independent fractional Brownian motions $B^H \in \{B_1^{H_1}, B_2^{H_2}\}$ with Hurst parameter $H \geq \frac{1}{2}$.

Let Ω be a non-empty set, $\mathcal{F}$ a σ-algebra of sets Ω, $\mathbb{P}$ a probability measure defined on $\mathscr{F}$ and $\{\mathscr{F}_t, t \in [0, T]\}$ a σ-algebra generated by both fractional Brownian motions.

The triplet $(\Omega, \mathscr{F}, \mathbb{P})$ defines a probability space and $\mathbf{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$.

The fractional Brownian motion B^H is a zero mean Gaussian process with the covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geq 0.$$

Denote

$$\phi(t, s) = H(2H-1)|t-s|^{2H-2}, \quad (t, s) \in \mathbf{R}^2.$$

Let ξ and η be measurable functions on $[0, T]$. It has been defined that

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u, v)\xi(u)\eta(v)dudv \text{ and } \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0, T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$.

Let $\mathscr{P}_T^H$ be the set of all polynomials of fractional Brownian motion $(B_t^H)_{t \geq 0}$. Namely, $\mathscr{P}_T^H$ contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_n(t)dB_t^H\right),$$

where f is a polynomial function of n variables.

The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_n(t)dB_t^H\right)\xi_j(s),$$

$$0 \leq s \leq T.$$

Now, we introduce the Malliavin ϕ-derivative $\mathbb{D}_t^H$ of F by

$$\mathbb{D}_t^H F = \int_0^T \phi(t, s) D_s^H F ds.$$

We have the following theorem (see [[7], Proposition 6.25]):

Theorem 2.1. *Let* $F : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathcal{H}$ *be a stochastic processes such that*

$$\mathbf{E}\left(\| F \|_T^2 + \int_0^T \int_0^T | \mathbb{D}_s^H F_t |^2 dsdt\right) < +\infty.$$

Then, the Itô-Skorohod type stochastic integral denoted by $\int_0^T F_s dB_s^H$ *exists in* $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *and satisfies*

$$\mathbf{E}\left(\int_0^T F_s dB_s^H\right) = 0 \text{ and}$$

$$\mathbf{E}\left(\int_0^T F_s dB_s^H\right)^2 = \mathbf{E}\left(\| F \|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s dsdt\right).$$

Now, we recall the fractional Itô formula (see [[6], Theorem 3.1]).

Theorem 2.2. *Let* $\sigma_1, \sigma_2 \in \mathcal{H}$ *be deterministic continuous functions. Denote*

$$X_t = X_0 + \int_0^t \alpha(s)ds + \int_0^t \sigma_1(s)dB_{1,s}^{H_1} + \int_0^t \sigma_2(s)dB_{2,s}^{H_2},$$

where X_0 *is a constant,* $\alpha(t)$ *is a deterministic function with*

$$\int_0^t |\alpha(s)|ds < +\infty.$$

Let $F(t, x)$ *be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then*

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)dX_s$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)\left[\frac{d}{ds}\|\sigma_1\|_s^2 + \frac{d}{ds}\|\sigma_2\|_s^2\right]ds, \ 0 \le t \le T.$$

Let us finish this section by giving a fractional Itô chain rule (see [[6], Theorem 3.2]).

Theorem 2.3. *Assume that for* $j = 1, 2,$ *the processes* μ_j, α_j *and* ϑ_j, *satisfy*

$$\mathbf{E}\left[\int_0^T \mu_j^2(s)ds + \int_0^T \alpha_j^2(s)ds + \int_0^T \vartheta_j^2(s)ds\right] < +\infty.$$

Suppose that $\mathbb{D}_t^{H_1}\alpha_j(s)$ *and* $\mathbb{D}_t^{H_2}\vartheta_j(s)$ *are continuously differentiable with respect to* $(s, t) \in [0, T]^2$ *for almost all* $\omega \in \Omega$. *Let* X_t *and* Y_t *be two processes satisfying*

$$X_t = X_0 + \int_0^t \mu_1(s)ds + \int_0^t \alpha_1(s)dB_{1,s}^{H_1} + \int_0^t \vartheta_1(s)dB_{2,s}^{H_2}, \ 0 \le t \le T,$$

$$Y_t = Y_0 + \int_0^t \mu_2(s)ds + \int_0^t \alpha_2(s)dB_{1,s}^{H_1} + \int_0^t \vartheta_2(s)dB_{2,s}^{H_2}, \ 0 \le t \le T.$$

If the following conditions hold:

$$\mathbf{E}\left[\int_0^T |\, \mathbb{D}_t^{H_1}\alpha_i(s)\,|^2 dsdt\right] < +\infty \ \textit{and}\ \mathbf{E}\left[\int_0^T |\, \mathbb{D}_t^{H_2}\vartheta_i(s)\,|^2 dsdt\right] < +\infty$$

then

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s$$

$$+ \int_0^t [\alpha_1(s)\mathbb{D}_s^{H_1}Y_s + \alpha_2(s)\mathbb{D}_s^{H_1}X_s + \vartheta_1(s)\mathbb{D}_s^{H_2}Y_s + \vartheta_2(s)\mathbb{D}_s^{H_2}X_s]ds,$$

which may be written formally as

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + [\alpha_1(t)\mathbb{D}_t^{H_1}Y_t + \alpha_2(t)\mathbb{D}_t^{H_1}X_t$$

$$+ \vartheta_1(t)\mathbb{D}_t^{H_2}Y_t + \vartheta_2(t)\mathbb{D}_t^{H_2}X_t]dt.$$

2.2. Definitions and notations

Let G be an open connected subset of $\mathbb{R}^d$ such that for some $\ell \in \mathscr{C}^2(\mathbb{R}^d)$, $G = \{x;\ \ell(x) > 0\}$ and $\partial G = \{x;\ \ell(x) = 0\}$ and $|\nabla\ell(x)| = 1$ for $x \in \partial G$.

Let $\eta_0 \in G$ and (η_t, Λ_t) be a solution of the following reflected SDE with respect to fractional Brownian motions

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \nabla\ell(s)d\Lambda_s + \int_0^t \sigma_1(s)dB_{1,s}^{H_1}$$

$$+ \int_0^t \sigma_2(s)dB_{2,s}^{H_2}, \quad t \in [0, T], \tag{2.1}$$

where the coefficients η_0, b, σ_1 and σ_2 satisfy:

- η_0 is a given constant and $b : [0, T] \to \mathbb{R}$ is deterministic continuous function;

- $\sigma_1, \sigma_2 : [0, T] \to \mathbb{R}$ are deterministic differentiable continuous functions and $\sigma_1(t) \neq 0$, $\sigma_2(t) \neq 0$ such that

$$|\sigma|_t^2 = \|\sigma_1\|_t^2 + \|\sigma_2\|_t^2, \quad t \in [0, T], \tag{2.2}$$

where

$$\|\sigma_i\|_t^2 = H_i(2H_i - 1)\int_0^t \int_0^t |u - v|^{2H_i - 2} \sigma_i(u)\sigma_i(v)dudv, \; i = 1, 2.$$

- Λ is a nondecreasing process, $\Lambda_0 = 0$, and $\int_0^T (\eta_t - a)d\Lambda_t \leq 0$ for any $a \in G$.

The next Remark will be useful in the sequel.

Remark 2.4. For $i = 1, 2$, the function $|\sigma|_t^2$ defined by equation (2.2) is continuously differentiable with respect to t on $[0, T]$, and for a suitable constant $C_0 > 0$,

$$\inf_{t \in [0, T]} \frac{\hat{\sigma}_i(t)}{\sigma_i(t)} \geq C_0, \text{ where } \hat{\sigma}_i(t) = \int_0^t \phi(t, v)\sigma_i(v)dv.$$

Given ξ a measurable real valued random variable and the functions

$$f : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \; g : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R},$$

we consider the following GFBSDE with parameters (ξ, f, g, Λ):

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds + \int_t^T g(s, \eta_s, Y_s)d\Lambda_s \\ &- \int_t^T Z_{1,s}dB_{1,s}^{H_1} - \int_t^T Z_{2,s}dB_{2,s}^{H_2}, \quad t \in [0, T]. \end{aligned} \tag{2.3}$$

Before giving the definition of the solution for the above equation, we introduce the following (where $\mu, \beta > 0$):

- $\mathscr{C}^{1,2}_{\text{pol}}([0, T] \times \mathbf{R})$ is the space of all $\mathscr{C}^{1,2}$-functions over $[0, T] \times \mathbf{R}$, which together with their derivatives are of polynomial growth,

- $$\mathcal{V}_{[0,T]} = \left\{ Y = \psi(\cdot, \eta); \psi \in \mathscr{C}^{1,2}_{\text{pol}}([0, T] \times \mathbf{R}), \frac{\partial \psi}{\partial t} \text{ is bounded}, t \in [0, T] \right\},$$

- $\widetilde{\mathcal{V}}^2_{[0,T]}$ and $\widetilde{\mathcal{V}}^2_{([0,T],\Lambda)}$ are the completions of $\mathcal{V}_{[0,T]}$ under the following norms, respectively,

$$\| Y \|^2_{\widetilde{\mathcal{V}}^2_{[0,T]}} = \mathbf{E}\int_0^T e^{\mu t + \beta \Lambda_t} | Y_t |^2 dt \text{ and}$$

$$\| Y \|^2_{\widetilde{\mathcal{V}}^2_{([0,T],\Lambda)}} = \mathbf{E}\int_0^T e^{\mu t + \beta \Lambda_t} | Y_t |^2 d\Lambda_t.$$

- $\mathscr{B}^2([0, T], \Lambda) = (\widetilde{\mathcal{V}}^2_{[0,T]} \cap \widetilde{\mathcal{V}}^2_{([0,T],\Lambda)}) \times \widetilde{\mathcal{V}}^2_{[0,T]} \times \widetilde{\mathcal{V}}^2_{[0,T]}$ endowed with the norm

$$\| (Y, Z_1, Z_2) \|^2_{\mathscr{B}^2([0, T], \Lambda)}$$

$$= \mathbf{E}\left[\int_0^T e^{\mu t + \beta \Lambda_t} | Y_t |^2 (dt + d\Lambda_t) + \int_0^T e^{\mu t + \beta \Lambda_t} (| Z_{1,t} |^2 + | Z_{2,t} |^2) dt \right].$$

Note that $\mathscr{B}^2([0, T], \Lambda)$ is a Banach space (see, [2, 3]).

Definition 2.5. A triplet of processes $(Y_t, Z_{1,t}, Z_{2,t})_{0 \le t \le T}$ is called a *solution to fractional GBSDE* (2.3), if $(Y_t, Z_{1,t}, Z_{2,t})_{0 \le t \le T} \in \mathscr{B}^2([0, T], \Lambda)$ and satisfies equation (2.3).

The next proposition will be useful in the sequel.

Proposition 2.6. *Let* $(Y_t, Z_{1,t}, Z_{2,t})_{0 \le t \le T} \in \mathscr{B}^2([0, T], \Lambda)$ *be a solution of the fractional GBSDE* (2.3). *Then almost for all* $t \in [0, T]$,

$$\mathbb{D}_t^{H_1} Y_t = \frac{\hat{\sigma}_1(t)}{\sigma_1(t)} Z_{1,t} \text{ and } \mathbb{D}_t^{H_2} Y_t = \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.$$

3. Fractional GBSDE

3.1. Existence and uniqueness of solution

We assume that the coefficients *f* and *g* of the GFBSDE and ξ satisfy the following assumptions:

(H1) $\xi_T = h(\eta_T)$, where $h : \mathbf{R} \to \mathbf{R}$ is a differentiable function and $\mathbf{E}[e^{\mu T + \beta \Lambda_T} | \xi_T |^2] < +\infty.$

(H2) There exists a constant $K > 0$ such that for all $t \in [0, T]$, $x \in \mathbf{R}$, $(y, y') \in \mathbf{R}^2$, $(z_1, z_1') \in \mathbf{R}^2$, $(z_2, z_2') \in \mathbf{R}^2$,

- $$| f(t, x, y, z_1, z_2) - f(t, x, y', z_1', z_2') |^2$$
$$\leq K(| y - y' |^2 + | z_1 - z_1' |^2 + | z_2 - z_2' |^2);$$

- $| g(t, x, y) - g(t, x, y') |^2 \leq K | y - y' |^2.$

We first establish a priori estimate on the solution.

Proposition 3.1. *Under the conditions* **(H1)** *and* **(H2)**, *if* $(Y, Z_1, Z_2) \in \mathscr{B}^2([0, T], \Lambda)$ *is a solution of equation* (2.3), *then there exists a constant C such that*, *for all* $t \in [0, T]$,

$$\mathbf{E}\bigg(e^{\mu t + \beta \Lambda_t} | Y_t |^2 + \int_t^T e^{\mu s + \beta \Lambda_s} | Y_s |^2 (ds + d\Lambda_s)$$

$$+ \int_t^T e^{\mu s + \beta \Lambda_s} | Z_{1,s} |^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} | Z_{2,s} |^2 ds \bigg)$$

$$\leq C\mathbf{E}\bigg(e^{\mu T+\beta\Lambda_T}|\xi_T|^2 + \int_t^T e^{\mu s+\beta\Lambda_s}|f(s, \eta, 0, 0, 0)|^2 ds$$

$$+ \int_t^T e^{\mu s+\beta\Lambda_s}|g(s, \eta, 0)|^2 d\Lambda_s\bigg).$$

Proof. By C we will denote a constant which may vary from line to line. From the Itô formula,

$$e^{\mu t+\beta\Lambda_t}|Y_t|^2$$

$$= e^{\mu T+\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\mu s+\beta\Lambda_s}Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds$$

$$+ 2\int_t^T e^{\mu s+\beta\Lambda_s}Y_s g(s, \eta_s, Y_s)d\Lambda_s - 2\int_t^T e^{\mu s+\beta\Lambda_s}Z_{1,s}\mathbb{D}_s^{H_1}Y_s ds$$

$$- 2\int_t^T e^{\mu s+\beta\Lambda_s}Z_{2,s}\mathbb{D}_s^{H_2}Y_s ds - 2\int_t^T e^{\mu s+\beta\Lambda_s}Y_s Z_{1,s}dB_{1,s}^{H_1}$$

$$- 2\int_t^T e^{\mu s+\beta\Lambda_s}Y_s Z_{2,s}dB_{2,s}^{H_2} - \mu\int_t^T e^{\mu s+\beta\Lambda_s}|Y_s|^2 ds$$

$$- \beta\int_t^T e^{\mu s+\beta\Lambda_s}|Y_s|^2 d\Lambda_s.$$

Taking mathematical expectation on both sides, we have

$$\mathbf{E}\bigg[e^{\mu t+\beta\Lambda_t}|Y_t|^2 + \int_t^T e^{\mu s+\beta\Lambda_s}|Y_s|^2(\mu ds + \beta d\Lambda_s)$$

$$+ 2\int_t^T e^{\mu s+\beta\Lambda_s}(Z_{1,s}\mathbb{D}_s^{H_1}Y_s + Z_{2,s}\mathbb{D}_s^{H_2}Y_s)ds\bigg]$$

$$= \mathbf{E}\bigg[e^{\mu T+\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\mu s+\beta\Lambda_s}Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds$$

$$+ 2\int_t^T e^{\mu s+\beta\Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \Bigg]$$

$$- 2\mathbf{E}\left[\int_t^T e^{\mu s+\beta\Lambda_s} Y_s Z_{1,s} dB_{1,s}^{H_1} + \int_t^T e^{\mu s+\beta\Lambda_s} Y_s Z_{2,s} dB_{2,s}^{H_2}\right].$$

Using the fact that $\mathbf{E}\int_t^T e^{\mu s+\beta\Lambda_s} Y_s Z_{i,s} dB_{i,s}^{H_i} = 0$, we deduce

$$\mathbf{E}\Bigg[e^{\mu t+\beta\Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s+\beta\Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s)$$

$$+ 2\int_t^T e^{\mu s+\beta\Lambda_s} (Z_{1,s}\mathbb{D}_s^{H_1} Y_s + Z_{2,s}\mathbb{D}_s^{H_2} Y_s) ds\Bigg]$$

$$= \mathbf{E}\Bigg[e^{\mu T+\beta\Lambda_T} |\xi|^2 + 2\int_t^T e^{\mu s+\beta\Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds$$

$$+ 2\int_t^T e^{\mu s+\beta\Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s\Bigg].$$

It is known that, by Proposition 2.6, $\mathbb{D}_s^{H_1} Y_s = \frac{\hat{\sigma}_1(s)}{\sigma_1(s)} Z_{1,s}$ and $\mathbb{D}_s^{H_2} Y_s = \frac{\hat{\sigma}_2(s)}{\sigma_2(s)} Z_{2,s}$. By Remark 2.4, we obtain

$$\mathbf{E}\Bigg[e^{\mu t+\beta\Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s+\beta\Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s)$$

$$+ 2C_0\int_t^T e^{\mu s+\beta\Lambda_s} (|Z_{1,s}|^2 + |Z_{2,s}|^2) ds\Bigg]$$

$$\leq \mathbf{E}\Bigg[e^{\mu T+\beta\Lambda_T} |\xi|^2 + 2\int_t^T e^{\mu s+\beta\Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds$$

$$+ 2\int_t^T e^{\mu s+\beta\Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s\Bigg]. \tag{3.1}$$

Using standard estimates $2ab \leq \lambda a^2 + \frac{1}{\lambda} b^2$ $(\text{where } \lambda > 0)$ and assumption **(H2)**, we obtain

$$\bullet\ 2\mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds$$

$$\leq 2\mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} | Y_s | (| f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})$$

$$- f(s, \eta_s, 0, 0, 0) | + | f(s, \eta_s, 0, 0, 0) |) ds$$

$$\leq \frac{K + 2\lambda^2}{\lambda} \mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} | Y_s |^2 ds$$

$$+ \frac{K}{\lambda} \mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} (| Z_{1,s} |^2 + | Z_{2,s} |^2) ds$$

$$+ \frac{1}{\lambda} \mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} | f(s, \eta_s, 0, 0, 0) |^2 ds$$

$$\bullet\ 2\mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s$$

$$\leq 2\mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} | Y_s | (| g(s, \eta_s, Y_s) - g(s, \eta_s, 0) | + | g(s, \eta_s, 0) |) d\Lambda_s$$

$$\leq \frac{K + 2\lambda^2}{\lambda} \mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} | Y_s |^2 d\Lambda_s$$

$$+ \frac{1}{\lambda} \mathbf{E}\int_t^T e^{\mu s + \beta \Lambda_s} | g(s, 0) |^2 d\Lambda_s.$$

Choosing $\lambda = \lambda_0$ such that $\mu, \beta \geq \frac{K + 2\lambda_0^2 + \lambda_0}{\lambda_0}$ and $C_0 \geq \frac{K + \lambda_0}{2\lambda_0}$, we deduce from (3.1),

$$\mathbf{E}\Big[e^{\mu t+\beta\Lambda_t}|Y_t|^2+\int_t^T e^{\mu s+\beta\Lambda_s}|Y_s|^2(ds+d\Lambda_s)$$

$$+\int_t^T e^{\mu s+\beta\Lambda_s}(|Z_{1,s}|^2+|Z_{2,s}|^2)ds\Big]$$

$$\leq C\mathbf{E}\Big[e^{\mu T+\beta\Lambda_T}|\xi|^2+\int_t^T e^{\mu s+\beta\Lambda_s}|f(s,\eta_s,0,0,0)|^2 ds$$

$$+\int_t^T e^{\mu s+\beta\Lambda_s}|g(s,0)|^2 d\Lambda_s\Big]. \qquad \square$$

The main result of this subsection is the following theorem:

Theorem 3.2. *Assume that the assumptions* **(H1)** *and* **(H2)** *are true. Then the fractional GBSDE* (2.3) *has a unique solution*

$$(Y_t, Z_{1,t}, Z_{2,t})_{0\leq t\leq T}\in\mathscr{B}^2([0,T],\Lambda).$$

Proof. Consider the mapping $\Gamma:\mathscr{B}^2([0,T],\Lambda)\to\mathscr{B}^2([0,T],\Lambda)$ driven by $(U,V_1,V_2)\mapsto\Gamma(U,V_1,V_2)=(Y,Z_1,Z_2)$. We will show that the mapping Γ is a contraction, where (Y,Z_1,Z_2) is a solution of the following fractional GBSDE:

$$Y_t=\xi+\int_t^T f(s,\eta_s,U_s,V_{1,s},V_{2,s})ds+\int_t^T g(s,\eta_s,U_s)d\Lambda_s$$

$$-\int_t^T Z_{1,s}dB_{1,s}^{H_1}-\int_t^T Z_{2,s}dB_{2,s}^{H_2},\quad t\in[0,T]. \tag{3.2}$$

Define for a process $\delta\in\{U,V_1,V_2,Y,Z_1,Z_2\}$, $\overline{\delta}=\delta-\delta'$ and the functions:

$$\Delta f(s)=f(s,\eta_s,Y_s,Z_{1,s},Z_{2,s})-f(s,\eta_s,Y_s',Z_{1,s}',Z_{2,s}'),$$

$$\Delta g(s)=g(s,\eta_s,Y_s)-g(s,\eta_s,Y_s').$$

Then, the triplet $(\overline{Y}, \overline{Z}_1, \overline{Z}_2)$ solves the fractional GBSDE:

$$\overline{Y}_t = \int_t^T \Delta f(s)ds + \int_t^T \Delta g(s)d\Lambda_s - \int_t^T \overline{Z}_{1,s} dB_{1,s}^{H_1}$$

$$- \int_t^T \overline{Z}_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T].$$

Applying Itô formula to $e^{\mu t + \beta\Lambda_t} |\overline{Y}_t|^2$, we obtain that

$$e^{\mu t + \beta\Lambda_t} |\overline{Y}_t|^2 = 2\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Y}_s \Delta f(s)ds + 2\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Y}_s \Delta g(s)d\Lambda_s$$

$$- 2\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Z}_{1,s} \mathbb{D}_s^{H_1} \overline{Y}_s ds - 2\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Z}_{2,s} \mathbb{D}_s^{H_2} \overline{Y}_s ds$$

$$- 2\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Y}_s \overline{Z}_{1,s} dB_{1,s}^{H_1} - 2\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Y}_s \overline{Z}_{2,s} dB_{2,s}^{H_2}$$

$$- \mu\int_t^T e^{\mu s + \beta\Lambda_s} |\overline{Y}_s|^2 ds - \beta\int_t^T e^{\mu s + \beta\Lambda_s} |\overline{Y}_s|^2 d\Lambda_s.$$

Taking mathematical expectation on both sides, we have

$$\mathbf{E}\Bigg[e^{\mu t + \beta\Lambda_t} |\overline{Y}_t|^2 + \int_t^T e^{\mu s + \beta\Lambda_s} |\overline{Y}_s|^2 (\mu ds + \beta d\Lambda_s)$$

$$+ 2\int_t^T e^{\mu s + \beta\Lambda_s} (\overline{Z}_{1,s} \mathbb{D}_s^{H_1} \overline{Y}_s + \overline{Z}_{2,s} \mathbb{D}_s^{H_2} \overline{Y}_s) ds\Bigg]$$

$$= 2\mathbf{E}\left[\int_t^T e^{\mu s + \beta\Lambda_s} \overline{Y}_s \Delta f(s)ds + \int_t^T e^{\mu s + \beta\Lambda_s} \overline{Y}_s \Delta g(s)d\Lambda_s\right].$$

By the same computations as in Proposition 3.1, we deduce that

$$\mathbf{E}\Bigg[e^{\mu t + \beta\Lambda_t} |\overline{Y}_t|^2 + \int_t^T e^{\mu s + \beta\Lambda_s} |\overline{Y}_s|^2 ((\mu - \lambda)ds + (\beta - \lambda)d\Lambda_s)$$

$$+ 2C_0\int_t^T e^{\mu s + \beta\Lambda_s} (|\overline{Z}_{1,s}|^2 + |\overline{Z}_{2,s}|^2) ds\Bigg]$$

$$\leq \frac{K}{\lambda}\mathbf{E}\left[\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{U}_s\,|^2(ds+d\Lambda_s)+\int_t^T e^{\mu s+\beta\Lambda_s}(|\,\overline{V}_{1,s}\,|^2+|\,\overline{V}_{2,s}\,|^2)ds\right],$$

$$t\in[0,T]. \quad (3.3)$$

Taking $\lambda = \lambda_0$ such that $\frac{1}{2}\min\{\mu-\lambda_0, \beta-\lambda_0\} \geq 2C_0 = \frac{4K}{3\lambda_0}$, we get

$$\mathbf{E}\left[\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{Y}_s\,|^2(ds+d\Lambda_s)+\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{Z}_{1,s}\,|^2 ds\right.$$

$$\left.+\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{Z}_{2,s}\,|^2 ds\right]$$

$$\leq \frac{3}{4}\mathbf{E}\left[\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{U}_s\,|^2(ds+d\Lambda_s)+\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{V}_{1,s}\,|^2 ds\right.$$

$$\left.+\int_t^T e^{\mu s+\beta\Lambda_s}|\,\overline{V}_{2,s}\,|^2 ds\right], \quad t\in[0,T].$$

Thus, the mapping $(U, V_1, V_2)\mapsto\Gamma(U, V_1, V_2) = (Y, Z_1, Z_2)$ determined by the fractional GBSDE (2.3) is a strict contraction on $\mathscr{B}^2([0, T], \Lambda)$. Using the fixed point principle, we deduce that the solution to the fractional GBSDE (2.3) exists and is unique. This completes the proof. □

3.2. Comparison theorem

In this subsection, our objective is to establish the comparison theorem for the following fractional GBSDEs, for all $t\in[0, T]$,

$$Y_t^{(i)} = h^i(\eta_T)+\int_t^T f^i(s, \eta_s, Y_s^{(i)}, Z_{1,s}^{(i)}, Z_{2,s}^{(i)})ds+\int_t^T g^i(s, \eta_s, Y_s^{(i)})d\Lambda_s$$

$$-\int_t^T Z_{1,s}^{(i)}dB_s^{H_1}-\int_t^T Z_{2,s}^{(i)}dB_s^{H_2}, \quad (3.4)$$

where $i = 1, 2$. By Theorem 3.2, equation (3.4) has a unique solution $(Y^{(i)}, Z_1^{(i)}, Z_2^{(i)}) \in \mathscr{B}^2([0, T], \Lambda)$.

Now, we give a comparison theorem of GBSDEs under Lipschitz assumptions.

Theorem 3.3. *For* $i = 1, 2$, *assume that* f^i *and* g^i *satisfy* **(H1)**. *Let* h^i *be continuously differentiable. Let* $(Y^{(i)}, Z_1^{(i)}, Z_2^{(i)})$ *be solutions of fractional GBSDEs* (3.4) *respectively. Suppose that*:

(i) $h^1(\eta_T) \geq h^2(\eta_T)$, *a.s.*;

(ii) *For all* $t \in [0, T]$,

$$f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \geq f^2(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \text{ or}$$

$$f^1(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) \geq f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) \text{ a.s.}$$

(iii) *For all* $t \in [0, T]$, $g^1(t, \eta_t, Y_t^{(1)}) \geq g^2(t, \eta_t, Y_t^{(1)})$, *or*

$$g^1(t, \eta_t, Y_t^{(2)}) \geq g^2(t, \eta_t, Y_t^{(2)}) \text{ a.s.}$$

Then

$$Y_t^{(1)} \geq Y_t^{(2)}, \text{ a.s.}, \; t \in [0, T].$$

Proof. For all $t \in [0, T]$, we denote

$$\hat{Y}_t = Y_t^{(2)} - Y_t^{(1)}, \; \hat{Z}_{1,t} = Z_{1,t}^{(2)} - Z_{1,t}^{(1)}, \; \hat{Z}_{2,t} = Z_{2,t}^{(2)} - Z_{2,t}^{(1)},$$

$$\hat{h}(\eta_T) = h^2(\eta_T) - h^1(\eta_T);$$

$$\Delta f(t) = f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) - f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}),$$

$$\Delta g(t) = g^2(t, \eta_t, Y_t^{(2)}) - g^1(t, \eta_t, Y_t^{(1)}).$$

Then $(\hat{Y}, \hat{Z}_1, \hat{Z}_2)$ satisfies

$$\hat{Y}_t = \hat{h}(\eta_T) + \int_t^T \Delta f(s)ds + \int_t^T \Delta g(s)d\Lambda_s \int_t^T \hat{Z}_{1,s}dB_s^{H_1}$$

$$- \int_t^T \hat{Z}_{2,s}dB_s^{H_2}, \quad t \in [0, T]. \tag{3.5}$$

Applying the extension of the Itô formula to $e^{\mu t + \beta\Lambda_t} | \hat{Y}_t^+ |^2$, we have

$$\mathbf{E}\Bigg[e^{\mu t + \beta\Lambda_t} | \hat{Y}_t^+ |^2 + \int_t^T e^{\mu s + \beta\Lambda_s} | \hat{Y}_s^+ |^2 (\mu ds + \beta d\Lambda_s)$$

$$+ 2C_0 \int_t^T e^{\mu s + \beta\Lambda_s} \mathbf{1}_{\{\hat{Y}_s > 0\}} (| \hat{Z}_{1,s} |^2 + | \hat{Z}_{2,s} |^2) ds \Bigg]$$

$$\leq 2\mathbf{E}\int_t^T e^{\mu s + \beta\Lambda_s} \hat{Y}_s^+ \Delta f(s)ds + 2\mathbf{E}\int_t^T e^{\mu s + \beta\Lambda_s} \hat{Y}_s^+ \Delta g(s)d\Lambda_s. \tag{3.6}$$

In view of **(H1)** and (ii), Young's inequality and Jensen's inequality, for any $\lambda > 0$, we have:

•

$$2\mathbf{E}\int_t^T e^{\mu s + \beta\Lambda_s} \hat{Y}_s^+ \Delta f(s)ds$$

$$= 2\mathbf{E}\Bigg(\int_t^T e^{\mu s + \beta\Lambda_s} \hat{Y}_s^+ (f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)})$$

$$- f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}))ds \Bigg)$$

$$\leq 2\mathbf{E}\int_t^T e^{\mu s + \beta\Lambda_s} \hat{Y}_s^+ (f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)})$$

$$- f^2(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}))ds$$

$$\leq \frac{\lambda^2 + K}{\lambda} \mathbf{E} \int_t^T e^{\mu s + \beta \Lambda_s} | \hat{Y}_s^+ |^2 ds$$

$$+ \frac{K}{\lambda} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\hat{Y}_s > 0\}} (| \hat{Z}_{1,s} |^2 + | \hat{Z}_{2,s} |^2) ds$$

•

$$2\mathbf{E} \int_t^T e^{\mu s + \beta \Lambda_s} \hat{Y}_s^+ \Delta s(s) d\Lambda_s$$

$$= 2\mathbf{E} \int_t^T e^{\mu s + \beta \Lambda_s} \hat{Y}_s^+ (g^2(t, \eta_t, Y_t^{(2)}) - g^1(t, \eta_t, Y_t^{(1)})) d\Lambda_s$$

$$\leq 2\mathbf{E} \int_t^T e^{\mu s + \beta \Lambda_s} \hat{Y}_s^+ (g^2(t, \eta_t, Y_t^{(2)}) - g^2(t, \eta_t, Y_t^{(1)})) d\Lambda_s$$

$$\leq \frac{\lambda^2 + K}{\lambda} \mathbf{E} \int_t^T e^{\mu s + \beta \Lambda_s} | \hat{Y}_s^+ |^2 d\Lambda_s.$$

Then, due to the above inequalities, we obtain

$$\mathbf{E}\left[e^{\mu t + \beta \Lambda_t} | \hat{Y}_t^+ |^2 + \int_t^T e^{\mu s + \beta \Lambda_s} | \hat{Y}_s^+ |^2 (\mu ds + \beta d\Lambda_s) \right.$$

$$\left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\hat{Y}_s > 0\}} (| \hat{Z}_{1,s} |^2 + | \hat{Z}_{2,s} |^2) ds \right]$$

$$\leq \frac{\lambda^2 + K}{\lambda} \mathbf{E} \int_t^T e^{\mu s + \beta \Lambda_s} | \hat{Y}_s^+ |^2 (ds + d\Lambda_s)$$

$$+ \frac{K}{\lambda} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\hat{Y}_s > 0\}} (| \hat{Z}_{1,s} |^2 + | \hat{Z}_{2,s} |^2) ds. \tag{3.7}$$

Choosing $\lambda = \lambda_0$ such that $\min\{\mu, \beta\} \geq \frac{\lambda_0^2 + K}{\lambda_0}$ and $\lambda_0 \geq \frac{K}{2C_0}$, we have

$$\mathbf{E}[e^{\mu t+\beta\Lambda_t} | \hat{Y}_t^+ |^2] \le 0, \quad t \in [0, T].$$

This implies

$$Y_t^{(1)} \ge Y_t^{(2)}, \; a.s., \; t \in [0, T].$$

This completes the proof of the theorem.

3.3. Connection with partial differential equations

Consider the following fractional GBSDEs:

$$\begin{cases} dY_t = -f(t, \eta_t, Y_t, Z_{1,t}, Z_{2,t})dt - g(t, \eta_t, Y_t)d\Lambda_t \\ \qquad + Z_{1,t}dB_t^{H_1} + Z_{2,t}dB_t^{H_2} \\ Y_T = \xi_T. \end{cases} \tag{3.8}$$

Denote

$$\widetilde{\sigma}(t) = \left[\frac{d}{ds}\| \sigma_1 \|_s^2 + \frac{d}{ds}\| \sigma_2 \|_s^2\right].$$

Then the function $u(t, x)$ satisfies the following partial differential equation (PDE in short):

$$\begin{cases} u_t'(t, x) + \frac{1}{2}\widetilde{\sigma}(t)u_{xx}''(t, x) + b(t)u_x'(t, x) \\ \qquad + f(t, x, u(t, x), \sigma_1(t)u_x'(t, x), \sigma_2(t)u_x'(t, x)) = 0; \\ \nabla\ell(t)u_x'(t, x) + g(t, x, u(t, x)) = 0; \\ u(T, x) = h(x). \end{cases} \tag{3.9}$$

Now, we have the following theorem:

Theorem 3.4. *If the PDE* (3.9) *has a solution* $u(t, x)$ *which is continuously differentiable in t and twice is continuously differentiable in x, then*

$$(Y_t, Z_{1,t}, Z_{2,t}) = (u(t, \eta_t), \sigma_1(t)u_x'(t, \eta_t), \sigma_2(t)u_x'(t, \eta_t))$$

satisfies the fractional GBSDE (3.8).

Proof. From the fractional Itô formula (Theorem 2.2), we get

$$\begin{aligned}
du(t, \eta_t) &= u'_t(t, \eta_t)dt + u'_x(t, \eta_t)[b(t)dt + \nabla\ell(t)d\Lambda_t + \sigma_1(t)dB_{1,t}^{H_1} \\
&\quad + \sigma_2(t)dB_{2,t}^{H_2}] + \frac{1}{2}\tilde{\sigma}(t)u''_{xx}(t, \eta_t)dt \\
&= \left[u'_t(t, \eta_t) + b(t)u'_x(t, \eta_t) + \frac{1}{2}\tilde{\sigma}(t)u''_{xx}(t, \eta_t)\right]dt \\
&\quad + u'_x(t, \eta_t)\nabla\ell(t)d\Lambda_t + \sigma_1(t)u'_x(t, \eta_t)dB_{1,t}^{H_1} \\
&\quad + \sigma_2(t)u'_x(t, \eta_t)dB_{2,t}^{H_2}.
\end{aligned}$$

Since $u(t, x)$ satisfies the PDE (3.9), we have

$$\begin{aligned}
du(t, \eta_t) &= -f(t, \eta_t, u(t, \eta_t), \sigma_1(t)u'_x(t, \eta_t), \sigma_2(t)u'_x(t, \eta_t))dt \\
&\quad - g(t, \eta_t, u(t, \eta_t))d\Lambda_t + \sigma_1(t)u'_x(t, \eta_t)dB_{1,t}^{H_1} \\
&\quad + \sigma_2(t)u'_x(t, \eta_t)dB_{2,t}^{H_2}.
\end{aligned}$$

Thus, the proof is complete. □

Remark 3.5. From the above proof, we also see that if the nonlinear partial differential equation (3.9) has a unique solution, then the fractional generalized backward stochastic differential equation (3.8) also has a unique solution.

3.4. Application in physics: modeling heat transfer in composite materials

Context: Heat transfer in composite materials, where different layers of materials are combined to create a system with unique thermal properties, is a key area of applied physics. These materials can exhibit long-term dependency behaviors and memory effects, particularly in systems where thermal conductivity varies complexly with time and space, such as in materials exposed to fluctuating thermal conditions or extreme environments.

Modeling with FGBSDEs: Fractional generalized BSDEs (FGBSDEs) can be used to model heat diffusion in these composite systems, accounting for both non-Markovian stochastic processes (modeled by fractional Brownian motions) and cumulative effects (modeled by a continuous increasing process). In this application, the two fractional Brownian motions $B_{1,t}^{H_1}$ and $B_{2,t}^{H_2}$ represent temperature fluctuations within different layers of the material, each characterized by a Hurst parameter H_1 or H_2, reflecting the roughness and long-term correlations of the thermal diffusion processes.

The FGBSDE equation takes the following form to model the temperature Y_t at a given time *t*:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds + \int_t^T g(s, \eta_s, Y_s)d\Lambda_s$$

$$- \int_t^T Z_{1,s}dB_{1,s}^{H_1} - \int_t^T Z_{2,s}dB_{2,s}^{H_2}. \tag{3.10}$$

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