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OPTIMIZING ROYALTY PAYMENTS FOR MAXIMUM ECONOMIC BENEFIT: A CASE STUDY UTILIZING MODIFIED SHOOTING AND DISCRETIZATION METHODS

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Abstract

This research delves into the application of the modified shooting method for the numerical resolution of non-standard optimal control (OC) problems. More precisely, it concentrates on scenarios where the final state value component remains unknown and unconstrained, leading to a non-zero final shadow value or costate variable. Moreover, the objective function involved a piecewise royalty function, which poses a challenge due to its lack of differentiability within a specific time interval. Consequently, the novel modified shooting method was employed to ascertain the elusive final state value. The model's differentiability is maintained throughout by adopting a continuous hyperbolic tangent (tanh) approximation. In addition, the construction of the Sufahani-Ahmad-Newton-Golden-Royalty Algorithm (SANGRA) and Sufahani-Ahmad-Powell-Golden-Royalty Algorithm (SAPGRA) was accomplished using the C++ programming language to formulate the problem. The outcomes of these algorithms, satisfying the criteria for optimality, were juxtaposed with non-linear programming (NLP) techniques such as Euler and Runge-Kutta, aside from Trapezoidal and Hermite-Simpson approximations. This groundbreaking discovery carries extensive practical implications as it propels the field forward and ensures the application of contemporary problem-solving methodologies.

Moreover, the study underscores the significance of fundamental theory in effectively tackling current economic challenges.

1. Introduction

In recent times, there has been a growing interest among young researchers in optimal control (OC) problems related to economics, particularly, in the context of royalty payment issues. These problems revolve around finding the OC for a system over a defined period, ensuring that certain optimality criteria are met. Optimization techniques and non-linear programming (NLP) approaches are employed to address these OC challenges.

A royalty payment refers to a monetary compensation provided to the rightful owner of assets, including computer software, copyrights, trademarks, or a patent, in exchange for the utilization of those assets. Users or developers make these payments to the individual or entity that holds legal ownership over the respective item.

In this study, the objective function is distinct from the conventional setup of the OC problem, as it does not depend on the value of the final state, $y(t_f)$, which is typically the default assumption. However, the central emphasis of this study revolves around the non-standard OC problem, where the integral of the objective function depends on the final state variable, $y(t_f)$. As a result, the shadow value at the final time, $p(t_f)$, deviates from zero, which contradicts the standard OC theory that dictates the shadow value, $p(t_f)$, to be zero at the final time.

Within the context of optimization and decision-making, the shadow value denotes the rate at which the objective function's value changes with respect to a small alteration in a constraint or parameter. It can be interpreted as the marginal value of a resource, indicating the extent to which the objective function would improve if the constraint or parameter were relaxed by one unit. The concept of shadow value is crucial in optimization as it aids decision-makers in determining the most efficient allocation of resources to achieve their objectives. In OC theory, the shadow value is also referred to as the costate value.

In this case study, our focus lies on maximizing economic benefit through optimizing royalty payments. To achieve this, we employed modified shooting and discretization methods, which offer innovative approaches to resolving the underlying challenges. By utilizing these techniques, we aim to address non-standard OC problems where the final state value's component is unknown and free. Such scenarios present unique complexities, as the final shadow value or costate variable cannot be assumed to be zero. Additionally, we encountered the intricacy of dealing with a piecewise royalty function that lacks differentiability within specific time frame. A novel modified shooting method was applied to determine the unknown final state value to overcome these obstacles. A continuous hyperbolic tangent (tanh) approximation ensures differentiation of the model at all times.

Furthermore, we compared the results obtained through the utilization of the Sufahani-Ahmad-Newton-Golden-Royalty Algorithm (SANGRA) and Sufahani-Ahmad-Powell-Golden-Royalty Algorithm (SAPGRA) with various discretization methods, including the approximation of Euler and Runge-Kutta aside from the Trapezoidal and Hermite-Simpson method. This study's findings hold significant promise for practical applications and contribute to advancing the academic understanding of problem-solving techniques, thus enriching the field with up-to-date methodologies. Additionally, we explore the value of fundamental theory in effectively addressing real-world economic challenges, showcasing the relevance of this research in bridging theory and practice.

2. The Problem of Royalty Payments and its Practical Application

The subject of royalty payment is of great significance in the domain of intellectual property, especially concerning individuals such as inventors, musicians, authors, artists and musicians who have rightful claims to receive a portion of the earnings generated from their creative works. It is crucial to prioritize equitable royalty distribution as it plays a vital role in fairly compensating the creators for their valuable contributions. This practice, in turn, fosters a conducive environment for innovation and creativity to thrive.

The issue of royalty payment emerges when there is a requirement to allocate the revenue generated from the use of copyrighted works or sale for instant films, books and music among the various stakeholders engaged in their distribution and creation. This encompasses individuals such as artists, composers, producers, and music labels within the music industry context. The primary objective is to ensure a fair and accurate allocation of these royalties to all the relevant parties involved.

The problem arises when numerous parties are involved in the production and success of work and may have various contractual arrangements for income sharing. Furthermore, the dynamic shifts in the evolving business model and technology within the music industry, for example, the emergence of digital downloads and streaming services, have introduced complexities in accurately tracking and distributing revenue.

Numerous mathematical models and techniques, such as game theory, multi-criteria decision-making models and linear programming (LP), have been proposed to handle the royalty payment problem. By considering several parameters, including sales, demand and discount rate, this study seeks to design an open and equitable method of income sharing. The aim is to develop a method that ensures fairness and transparency in distributing royalties among the relevant stakeholders.

Cai et al. [1] conducted a study on the impact of a fixed royalty payment in the context of sustainable fashion brand franchising. The model that captures the interactions between franchisors and franchisees was developed using game theory. The franchisor is faced with the decision of when to invest in sustainability, while the franchisee chooses whether to make sustainability investments or not. For both parties, the set royalty payment can result in a win-win situation that motivates them to make investments in sustainability. The importance of choosing a franchisee is emphasized in the study since a good franchisee can increase the efficiency of the fixed royalty payment method. In the context of Ukraine's decentralization process, Horal et al. [5] explored the issue of defining and justifying the distribution ratio of oil and gas royalties. The nation's current method for distributing royalties needs to be updated. Yahya and Habbal [11] highlight the advantages of blockchain's decentralized nature, emphasizing its potential to offer a transparent, including a secure platform for the music industry's royalty distribution.

The existing body of literature on royalty payment problems offers diverse approaches to tackle the challenge of distributing royalties in a fair and efficient manner. These proposed models take into account multiple factors, including sales, airplay, digital downloads, and patent licensing, to determine the appropriate allocation for each party within the royalty pool.

However, scholars have not paid much attention to the use of mathematical modelling techniques in problems involving royalty payments. Additionally, the incorporation of fundamental theory to compute a fair royalty payment has been overlooked. These shortcomings can have an impact on the received royalty payment, potentially hindering the achievement of a mutually beneficial "win-win" circumstance for both the developer and the landowner.

3. Royalty Payment Problem in the Literature

Spence [9] examined the learning curve theory, which contends that as businesses produce more frequently and gain expertise, they can increase productivity and cut costs. Additionally, he also investigated how the learning curve affected competition in various areas. The author posits that companies with a steeper learning curve, experiencing more significant productivity improvements with experience, possess a competitive edge over those with a shallower learning curve. Moreover, Spence [9] puts forward the notion that firms may have the motivation to disclose information regarding their production processes to competitors as a means to decrease their own learning costs. This information sharing can result in a more effective distribution of resources within the industry and potentially yield advantages for consumers. In a previous study, Spence [9] introduced an economic model to explore this concept further:

$$J[u(t)] = \int_{t_0}^{t_f} F(t,\ y(t),\ u(t),\ y(t_f))\,dt \tag{1}$$

subject to

$$F(t,\ y(t),\ u(t),\ y(t_f)) = (au^{1-\alpha} - (\rho + m_0 + c_0 e^{-wy})u)e^{-rt}. \tag{2}$$

Later, Zinober and Kaivanto [12] made an endeavour to address the economic model by employing a matrix formulation. Their approach takes into account pertinent parameters, such as the present demand value and discount factor, which play a role in determining the optimal royalty payment for maximizing the objective function. However, they encountered challenges in calculating the optimal objective function when the differentiation process became infeasible due to variations in the royalty payment level. Cruz et al. [2] proposed a collocation approach to solve a variational non-standard OC problem.

A recent study was conducted by Zinober and Sufahani [13] to address a particular OC problem where a certain function involved the two-stage percentage function. The issue at hand involved an economics application and required the creation of a strategy for solving it using Pontryagin's Maximum Principle, a proven method in the field of OC. The researchers concentrated on a case where a company engages in market competition with other companies. The goal was to maximize company profit within a limited time frame while taking into account production and investment choice limits. They made use of Pontryagin's maximum principle to derive the essential condition prerequisites. Subsequently, a coupled system of partial differential equations (PDEs) was formulated, which could be solved quantitatively through the application of finite difference techniques.

A noteworthy observation made by the researchers is the need to represent the piecewise function in a continuous form to enable differentiation at each stage. To address this challenge, they propose the utilization of a continuous hyperbolic tangent (tanh) approach, which presents an innovative solution. Our research into tackling the economic model first put forth by Spence [9] has been motivated by this brilliant idea.

4. Non-standard Optimal Control through Modified Shooting Method

The modified shooting method finds widespread application across various scientific and engineering disciplines. It is invaluable in solving boundary value problems that cannot be resolved analytically, including areas such as fluid dynamics, heat transfer, and mechanics. The shooting method in this study, known as SANGRA, combined the Newton method and the Golden Section Search approach. The flowchart illustrating these combined methods is presented in Figure 1.

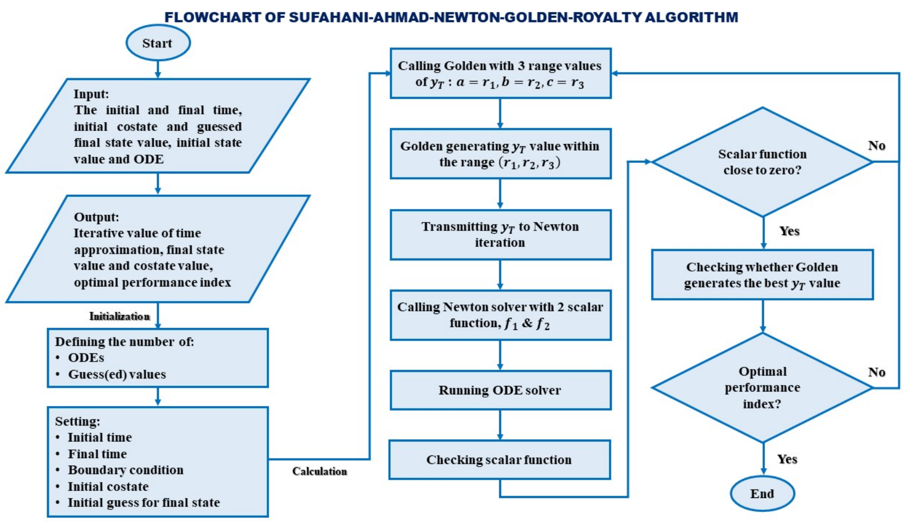


Figure 1. Flowchart for modified shooting method (SANGRA).

The initialization procedure of the programme establishes the number of ordinary differential equations (ODEs) and the estimated value(s). The initial time (which is normally set to zero), the final time, and the boundary condition are among the specific parameters that are established during this stage. The golden section search method is used in the computation to generate the ideal value of the final state. The Newton method is then applied to this value, and the program uses the estimated value to run the ODE solver. At the final step, the software checks if the optimality requirements are satisfied. Otherwise, the program iterates and reruns the process until an optimal solution is obtained, satisfying the optimality condition.

Later, to determine the best option, we used a modified shooting technique that combined the Powell and Golden Section Search approaches. Figure 2, referred to as the SAPGRA, illustrates this combined approach's functioning.

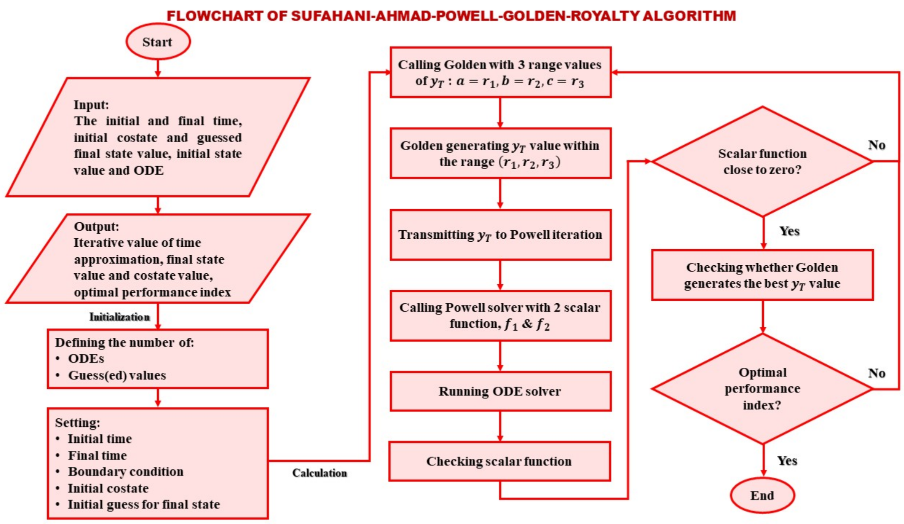


Figure 2. Flowchart for modified shooting method (SAPGRA).

5. Discretization Method

Through the division of continuous functions, equations, or systems into a finite number of discrete elements, the discretization approach was used to approximate them. Numerous fields, such as engineering, computational physics, and numerical analysis, use this technique. A numerical solution to the issue at hand is made possible through discretization, which converts a continuous function or equation into a finite set of equations or variables. The continuous domain is divided into a finite collection of points or intervals in this process, and the values of the function or equation are calculated at each individual point or interval.

For validation purposes, this research employed multiple discretization methods, including Euler and Runge-Kutta methods, aside from the Trapezoidal and Hermitte-Simpson approximations. The program was constructed in the AMPL programming language by implementing the MINOS solver [8, 9]. It was decided to use the direct technique, which is recognized for its adaptability, simplicity and capacity to handle a broad variety of issue types, including non-linear and non-convex situations. Incorporating path limitations and endpoint constraints, which are frequently seen in real-world applications, is also possible using the direct technique. However, it is crucial to be aware that the direct technique could run into difficulties, including convergence problems and a lack of global optimality guarantees, particularly, when working with extremely non-linear situations.

6. The Illustrative Example

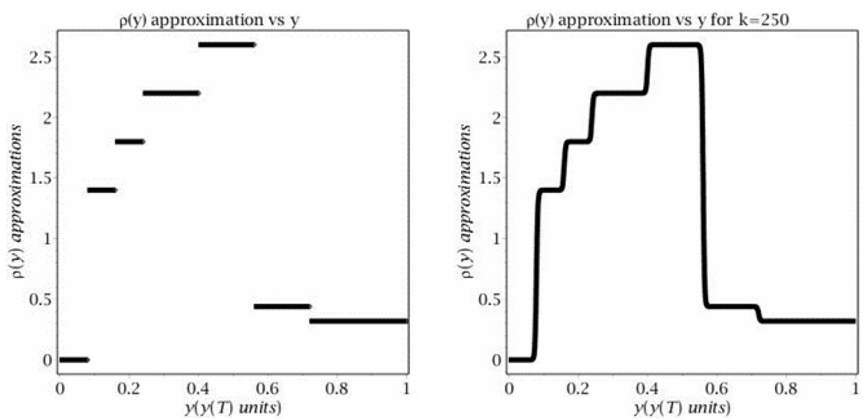
The payment can be in any number of stages. In our case, a seven-stage piecewise function was implemented. Let us examine the following piecewise function consisting of seven constant steps:

$$\rho(y) = \begin{cases} 0 & \text{for } 0 \le y < .08z \\ 1.4 & \text{for } .08z \le y < .16z \\ 1.8 & \text{for } .16z \le y < .24z \\ 2.2 & \text{for } .24z \le y < .4z \\ 2.6 & \text{for } .4z \le y < .56z \\ .44 & \text{for } .56z \le y < .72z \\ .32 & \text{for } .72z \le y < z. \end{cases} \tag{3}$$

The resultant function is shown below after transforming the system into a hyperbolic tangent (tanh) representation. The function is now in continuous form as follows:

$$\rho = .16 + .7\tanh(k(y - .08z)) + .2\tanh(k(y - .16z))$$
$$+ .2\tanh(k(y - .24z)) + .2\tanh(k(y - .4z))$$
$$- 1.08\tanh(k(y - .56z)) - .06\tanh(k(y - .72z)). \quad (4)$$

Equations (3) and (4) will be visually represented in the form of a graph, as depicted in the following Figure 3.



(a) (b)

Figure 3. Comparison of the piecewise function in discrete and continuous forms: (a) Discrete piecewise function, (b) Continuous piecewise function.

Equation (3) was initially expressed as a discrete piecewise function, which presented challenges in differentiability when the function level increased at specific time frame. To solve this problem, the hyperbolic tangent (tanh) function, as shown in equation (4), was used as a continuous approximation. Figure 3(b) illustrates the resulting continuous piecewise function. Equation (4) is used here to solve Equations (1) and (2) in order to solve for the integrand F.

Malinowska and Torres [7] provided a thorough framework that demonstrated the use of natural boundary conditions to produce more precise solutions for CoV problems. The relationship between OC issues and natural boundary conditions was also investigated. Their work underlined the value of taking into account natural boundary conditions when researching variational problems. Building upon this significant contribution, we applied their insightful findings to address the issue of a non-zero final costate value, $p(t_f)$, and established a novel necessary condition for our non-standard OC problem.

Given that $p(t_f) \neq 0$ mathematically, it necessitates the introduction of a new equation to tackle the non-standard problem. By taking into account Equation (1) along with the constraint on the unknown $y(t_f)$ as discussed by Malinowska and Torres [7], we can proceed to address the problem:

$$\frac{d}{dy} F_{\dot{y}}(t, y(t), \dot{y}(t), y(t_f)) = F_y(t, y(t), \dot{y}(t), y(t_f)). \tag{5}$$

Moreover, as discussed by Malinowska and Torres [7],

$$\begin{aligned} & F_{\dot{y}}(\rho(t_f), y(\rho(t_f)), \dot{y}(\rho(t_f)), y(t_f)) \\ & + \mu(\rho(t_f) F_y(\rho(t_f)), y(\rho(t_f)), \dot{y}(\rho(t_f)), y(t_f)) \\ & + \int_a^{t_f} F_z(t, y(t), \dot{y}(t), y(t_f))\, dt = 0. \end{aligned} \tag{6}$$

By considering the properties of the delta integral, we can conclude that

$$\int_{\rho(t_f)}^{t_f} F_y(\cdots)\, dt = \mu(\cdots). \tag{7}$$

Since the function $F_y(\cdots)$ is continuous, it can be inferred that

$$\int_{\rho(t_f)}^{t_f} F_y(\cdots)\, dt = 0. \tag{8}$$

When the second term of Equation (6) becomes zero, as indicated in Equation (8), it can be deduced that

$$F_{\dot{y}}(t_f, y(t_f), \dot{y}(t_f), y(t_f)) = -\int_a^{t_f} F_z(t, y(t), \dot{y}(t), y(t_f))\, dt. \tag{9}$$

Equation (9) simplifies to $F_z(t, y(t), \dot{y}(t), y(t_f)) = 0$ since in the standard OC problem, the function is independent of the $y(t_f)$. Considering that the standard setting has a final costate value of $p(t_f) = 0$, the additional natural boundary condition can be formulated as follows:

$$p(t_f) = -\int_{t_i}^{t_f} F_z\, dt. \tag{10}$$

By taking into account the ODE system:

$$\dot{y}(t) = u(t),\ y(0) = 0, \tag{11}$$

the integrand component in Equation (2) is governed by the following condition:

$$a(t) = e^{.025t},\ \alpha = .5,\ m_0 = 1,\ c_0 = 1,\ w = .12,\ r = .1. \tag{12}$$

Now, the integrand of the objective function can be formulated as shown below:

$$F = \left(\begin{array}{l} e^{.025t} u^{.5} - \\ \left(\begin{array}{l} .7 \tanh(k(y - .08z)) + .2 \tanh(k(y - .16z)) \\ + .2 \tanh(k(y - .24z)) + .2 \tanh(k(y - .4z)) \\ - 1.08 \tanh(k(y - .56z)) - .06 \tanh(k(y - .72z)) \\ + e^{-.12y} + 1.16 \end{array} \right) u \end{array} \right) e^{-.1t}. \tag{13}$$

The final costate value can, therefore, be explained as follows:

$$p(t_f) = \int_{t_0}^{T} \begin{bmatrix} (-.056k(1-\tanh(k(y-.08z))^2) \\ -.032k(1-\tanh(k(y-.16z))^2) \\ -.048k(1-\tanh(k(y-.24z))^2) \\ -.08k(1-\tanh(k(y-.4z))^2) \\ +.6048k(1-\tanh(k(y-.56z))^2) \\ +.0432k(1-\tanh(k(y-.72z))^2))ue^{-.1t} \end{bmatrix} dt. \tag{14}$$

In line with Kirk [6], the usefulness of a Hamiltonian function, denoted as $H = F(t, y(t), \dot{y}(t), y(t_f)) + p(t)u(t)$, in solving the non-standard OC problem was emphasized. Consequently,

$$H = \begin{pmatrix} e^{.025t}u^{.5} - \\ \begin{pmatrix} .7\tanh(k(y-.08z)) + .2\tanh(k(y-.16z)) \\ + .2\tanh(k(y-.24z)) + .2\tanh(k(y-.4z)) \\ - 1.08\tanh(k(y-.56z)) - .06\tanh(k(y-.72z)) \\ + e^{-.12y} + 1.16 \end{pmatrix} u \end{pmatrix} e^{-.1t}$$
$$+ p(t)u(t), \tag{15}$$

where the control variable $u(t)$ and the costate variable $p(t)$ are both present. Kirk [6] elaborates on the behaviour of a Hamiltonian function, which is a mathematical function utilized in mathematics and physics to capture the dynamics of a physical system using generalized coordinates and their conjugate momenta. This function holds significant importance in classical mechanics, serving as a cornerstone in its formulation. Denoted by the symbol H, the Hamiltonian function provides a robust mathematical framework for describing and analyzing the dynamics of diverse physical systems. By employing the Hamiltonian function, researchers gain valuable insights into the fundamental principles underlying mechanics and other interconnected fields:

$$\dot{y}(t) = H_p(t, y(t), u(t), p(t)),$$
$$\dot{p}(t) = -H_y(t, y(t), u(t), p(t)),$$

$$H_u(t,\ y(t),\ u(t),\ p(t)) = 0. \tag{16}$$

With the initial time set to zero and the final time set to ten, our objective is to maximize Equation (1). The function that we seek to optimize can be represented as follows:

$$\text{Maximize } J[u(t)] = \int_0^{10} \left[\left(\begin{array}{l} e^{.025t} u^{.5} - \\ \left(\begin{array}{l} .7 \tanh(k(y - .08z)) \\ + .2 \tanh(k(y - .16z)) \\ + .2 \tanh(k(y - .24z)) \\ + .2 \tanh(k(y - .4z)) \\ - 1.08 \tanh(k(y - .56z)) \\ - .06 \tanh(k(y - .72z)) \\ + e^{-.12y} + 1.16 \end{array} \right) u \end{array} \right) e^{-.1t} \right] dt. \tag{17}$$

Table 1. Optimal results

Methods	Final state value	Objective function	Initial shadow value	Final shadow value
Modified shooting method				
SANGRA	0.317978	0.606969	-1.254410	-0.356795
SAPGRA	0.317988	0.606969	-1.254380	-0.356766
Discretization method				
Euler	0.319074	0.611833	-1.26011	-
Runge-Kutta	0.323087	0.612860	-1.28002	-
Trapezoidal	0.322194	0.614032	-1.28354	-
Hermite-Simpson	0.328815	0.612950	-1.25833	-

Table 1 shows the desired outcomes for the shooting, including the discretization method.

When comparing the modified shooting method with the discretization method, Table 1 reveals that the ideal state value at the final time is similar up to one decimal place. At the same time, the computation of the optimal outcome for the objective function using the shooting method matches the discretization results up to one decimal place. Similarly, the initial costate values obtained from the Hermite-Simpson approximation and the modified shooting method are comparable up to two decimal places. In conclusion, the objective function can be maximized by up to 61% to satisfy the optimality condition.

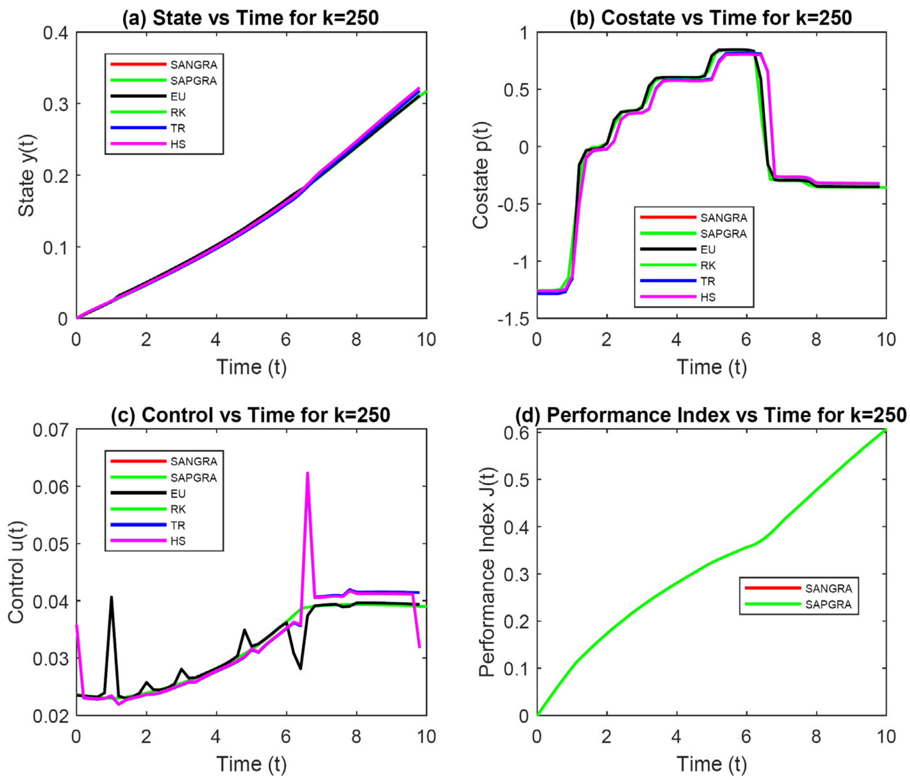


Figure 4. Graphical representation of the optimal state, costate, control, and objective function.

In contrast to the modified shooting approach, the discretization method exhibited less accuracy in the computational process, as depicted in Figure 4. The observed discrepancy is most likely attributable to the discretization error inherent in the process [8, 10]. Nevertheless, the method remains applicable for solving the OC problem, as it yielded optimal results that were comparable to those obtained through the modified shooting method.

7. Concluding Remark

In conclusion, our case study focused on optimizing royalty payments to maximize economic benefit. Through the utilization of modified shooting and discretization methods, we tackled the complexities associated with non-standard OC problems in this context. By addressing scenarios where the final state value's component is unknown and free, we successfully determined the unknown final state value using the novel modified shooting method. This method, accompanied by the continuous hyperbolic tangent (tanh) approximation, enabled differentiation of the model at all times. In addition, we compared the results obtained from the SANGRA and SAPGRA with various discretization methods, such as Euler and Runge-Kutta methods. At the same time, Trapezoidal and Hermite-Simpson approximations were also implemented for the discretization methods. This comparative analysis provided valuable insights into the effectiveness of different techniques in optimizing royalty payments and maximizing economic benefit. The findings of this study have significant practical implications, offering innovative solutions to real-world challenges in the economic domain. Moreover, by advancing problem-solving techniques and incorporating fundamental theory, this research contributes to the academic discipline and ensures the continuous development of practical methodologies for resolving economic problems. Generally, this case study highlights the importance of optimizing royalty payments to achieve maximum economic benefit and emphasizes the value of modified shooting and discretization methods in addressing complex OC problems. Furthermore, this case study's contributions lie in the development of innovative methodologies, comparative analysis of different techniques, practical implications for economic benefit maximization, and academic advancement in the field of OC. These contributions enhance both theoretical understanding and practical decision-making in industries where royalty payments play a crucial role.

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