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APPLICATION OF ADOMIAN DECOMPOSITION METHOD TO A GENERALIZED FRACTIONAL RICCATI DIFFERENTIAL EQUATION (ψ-FRDE)

Asrar Saleh Alsulami, Mariam Al-Mazmumy, Maryam Ahmed Alyami and Mona Alsulami

Department of Mathematics and Statistics
Faculty of Science
University of Jeddah
Jeddah 23218, Saudi Arabia
e-mail: 2200531@uj.edu.sa
mhalmazmumy@uj.edu.sa
maalyami8@uj.edu.sa
mralsolami@uj.edu.sa

Abstract

In this article, we generalize the fractional Riccati differential equations (FRDEs) by using a fractional derivative of a function with respect to another function (ψ-Caputo derivative) and obtain

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ψ-FRDEs. Using the Adomian decomposition method (ADM) with Wazwaz modification, we solve the ψ-FRDEs semi-analytically. Comparing the solutions of the ψ-FRDEs with several functions of $\psi(x)$ and different values of fractional orders, we show that the presented method is efficient.

1. Introduction

Fractional calculus has found extensive applications in the modeling of real-life problems across various fields such as electrical engineering, physics, biology, chemistry, finance, and more [1-4]. The concept of fractional derivatives was first introduced by L'Hospital in 1695. Subsequently, different definitions of fractional derivatives, such as Riemann-Liouville, Hadamard, Atangana-Baleanu, Liouville-Caputo, Riesz, and more, were developed [5-7]. These various formulations provide researchers with a wide range of tools to tackle FDEs and discuss their applications in different fields. Almeida [10] has made considerable contributions by proposing a generalized definition of the Caputo fractional derivative. This new definition involves the concept of the derivative of a function with respect to another function, denoted as ψ. This approach enables more precise in modeling, as the choice of an appropriate function ψ can significantly improve the accuracy of the model [11]. In addition to introducing this new definition, Almeida conducted extensive studies on various properties of fractional calculus. These investigations have proved that fractional calculus has the capacity to create more accurate models, discover hidden aspects in diverse systems [12, 13].

Recently, there has been extensive research on the existence and uniqueness of solutions of differential equations corresponding to the fractional derivatives with respect to other functions. For example, Almeida et al. [12] considered the initial value problem of a fractional differential equation including ψ-Caputo fractional derivative (ψ-FDE). They utilized the fixed point theorem to examine the existence and uniqueness results. Furthermore, Abdo et al. [14] investigated the existence and uniqueness results of the boundary value problem for (ψ-FDE).

The Riccati differential equation (RDE) is an important class of nonlinear differential equations used to describe applications in engineering, physical phenomena, and real-world problems such as the econometric models, flow of rivers, structure formation in dynamic gases, network synthesis, invariant embedding, and financial mathematics [15, 16]. Therefore, the RDE with a fractional derivative is commonly referred to as the fractional Riccati differential equation (FRDE). The FRDE is a generalization of the classical Riccati equation and, due to the additional degree of freedom in the order of the fractional derivative, gives a more flexible description of the experimental data. Hence, an important task is to find a solution to the FRDE. Many studies have been conducted to produce more efficient and approximate results, with many emerging analytical and numerical methods. Öztürk et al. [17] utilized the Taylor collocation method to convert the FDRE into a system of nonlinear algebraic equations and then solved the system. Merdan [18] used the fractional variational iteration method, to obtain an approximate analytical solution for nonlinear FRDE with modified Riemann-Liouville derivative. Liu et al. [19] applied the Laplace transform method on FRDE to obtain its approximate solution with Atangana-Baleanu fractional derivative. Khader et al. introduced a numerical treatment using the generalized Euler method (GEM) for the FRDE and logistic differential equations with Caputo sense in their article [20]. Legendre integral operational matrix, modified homotopy perturbation method, and Bernstein method were utilized for solving FRDE in [21-23].

Considerable effort has been dedicated to developing reliable and consistent numerical and analytical methods for solving FRDE. The Adomian decomposition method (ADM) stands out among the rest. This method distinguishes itself by approximating the issues under consideration without relying on linearization or discretization. The ADM has undergone a number of modifications aimed at increasing its precision, speed, computing efficiency, or even its ability to fit different classes of functional equations [24-26]. It is important to note that several articles have established the convergence of ADM. In [27], Cherruault was the first to prove the convergence of ADM using the fixed point theorem. Furthermore, the convergence of the ADM for solving linear and nonlinear differential equations and integral equations was discussed in [28-30].

In this paper, motivated by the above-cited works, we aim to generalize FRDE in the Caputo sense (ψ-FRDE) and focus on its solution. More specifically, we make consideration of the ψ-FRDE as follows:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\psi}y(x)+a(x)y^2(x)+b(x)y(x)=g(x),\ 0\le x\le 1,\ n-1<\alpha\le n, \\ y_{\psi}^{[k]}(a)=a_k,\ k=0,1,\ldots,n-1, \end{cases} \tag{1.1}$$

where $a(x)$, $b(x)$ and $g(x)$ are continuous functions on $[0, 1]$, a_k, $k = 0, 1, \ldots, n-1$, are given constants, α is a parameter describing the order of the fractional derivative, and ψ is an arbitrary function.

In this study, our focus is on solving the ψ-FRDE (1.1) using the ADM. We have also applied the Wazwaz modification [31], which has proven to converge to exact solutions with only a minor alteration from the standard ADM. This modification can achieve the exact solution using just two iterations, sometimes even without the Adomian polynomials. The main advantage of this method is its ability to obtain the solution in fewer steps compared to ADM. We have compared the solutions of the ψ-FRDE with various functions of $\psi(x)$ and different fractional orders.

The following is an outline of the paper's structure: Section 2 presents some essential definitions and properties of ψ-fractional calculus; Section 3 introduces the proposed method; Section 4 provides test examples for illustrating the method's application steps; and Section 5 summarizes the results.

2. Preliminaries

In this section, we review a few definitions and results related to ψ-fractional operators. We begin by defining ψ-fractional integral (ψ-FI), ψ-fractional derivative (ψ-FD), and ψ-Caputo fractional derivative (ψ-CFD), and list some properties needed in the paper.

Definition 2.1 [10] (ψ-FI)**.** The *fractional integral of order* $\alpha > 0$ of the function y with respect to another function ψ where $y : I \to \mathbb{R}$ is an integrable function, $I = [a, b]$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $\psi(x) \in C^n(I)$ such that $\psi'(x) \neq 0, \forall x \in I$ is defined as follows:

$$I_a^{\alpha, \psi} y(x) := \{\Gamma(\alpha)\}^{-1} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1+\alpha} y(t) dt, \tag{2.1}$$

where Γ is the gamma function.

Note that equation (2.1) is reduced to the Riemann-Liouville and Hadamard fractional integrals when $\psi(x) = x$ and $\psi(x) = \ln(x)$, respectively.

Definition 2.2 [10] (ψ-FD)**.** The *fractional derivative of order* $\alpha > 0$ of the function y with respect to another function ψ where $y : I \to \mathbb{R}$ is an integrable function, $I = [a, b]$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $\psi(x) \in C^n(I)$ such that $\psi'(x) \neq 0, \forall x \in I$ is defined as follows:

$$D_{a^+}^{\alpha, \psi} y(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n I_{a^+}^{n-\alpha, \psi} y(x) \tag{2.2}$$

$$= \{\Gamma(n - \alpha)\}^{-1} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1-\alpha+n} y(t) dt, \tag{2.3}$$

sequentially, with $n = 1 + [\alpha]$.

Note that equation (2.2) is reduced to the Riemann-Liouville and Hadamard fractional derivative when $\psi(x) = x$ and $\psi(x) = \ln(x)$, respectively.

Definition 2.3 [10] (ψ-FCD)**.** Given the interval $I = [a, b]$ with $\alpha > 0$, $n \in \mathbb{N}$, and let $\psi, y \in C^n(I)$ be two functions such that ψ is increasing and $\psi'(x) \neq 0, \forall x \in I$. Then the ψ-*Caputo fractional derivative of order* α of the function y is defined as follows:

$$ {}^C D_a^{\alpha, \psi} y(x) = I_a^{n-\alpha, \psi}\left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n y(x), \tag{2.4}$$

where $n = 1 + [\alpha]$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify notation, we use the shorthand symbol

$$y_\psi^{[n]} y(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n y(x).$$

From the definition, it is clear that

$$ {}^C D_a^{\alpha, \psi} y(x) = \begin{cases} \{\Gamma(n-\alpha)\}^{-1} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} y_\psi^{[n]}(t)\, dt, & \text{if } \alpha \notin \mathbb{N}, \\ y_\psi^{[n]} y(x), & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{2.5}$$

Note that equation (2.5) is reduced to classical Caputo derivative and Caputo-Hadamard fractional derivative when $\psi(x) = x$ and $\psi(x) = \ln(x)$, respectively.

Proposition 2.4 [10]**.** *For* $\alpha > 0$, *if* $y \in C^{n-1}(I)$, *then*

- $({}^C D_a^{\alpha, \psi} I_a^{\alpha, \psi} y(x)) = y(x)$,
- $I_a^{\alpha, \psi}({}^C D_a^{\alpha, \psi} y(x)) = y(x) - \sum_{k=0}^{n-1} \frac{y_\psi^{[k]}(a)}{k!}(\psi(x) - \psi(a))^k.$

Proposition 2.5 [10]**.** *In addition, considering* $y(x) = (\psi(x) - \psi(a))^\gamma$, *where* $\gamma \in \mathbb{R}$, $\gamma > n$, $\alpha > 0$, *certain features for a* $\psi(x)$*-fractional operator are deduced as follows*:

- ${}^{C}D_{a^+}^{\alpha,\psi} y(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(\psi(x)-\psi(a))^{\gamma-\alpha}$,

- $I_a^{\alpha,\psi} y(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}(\psi(x)-\psi(a))^{\gamma+\alpha}$.

3. Fundamental Description of ADM for Solving ψ-FRDE

To provide a fundamental description of the applied semi-analytic technique, we give a ψ-FRDE as

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\psi} y(x) + a(x)y^2(x) + b(x)y(x) = g(x),\ 0 \le x \le 1,\ n-1 < \alpha \le n, \\ y_\psi^{[k]}(a) = a_k,\ k = 0, 1, \ldots, n-1, \end{cases} \tag{3.1}$$

where $a(x)$, $b(x)$ and $g(x)$ are continuous functions on $[0, 1]$, a_k, $k = 0, 1, \ldots, n-1$, are given constants, α is a parameter describing the order of the fractional derivative, and ψ is an arbitrary function.

To solve the IVP for ψ-FRDE expressed in (3.1) by ADM, we operate by $I_{a^+}^{\alpha,\psi}$ on the governing equation to get

$$I_{a^+}^{\alpha,\psi}[{}^{C}D_{a^+}^{\alpha,\psi} y(x)] + I_{a^+}^{\alpha,\psi}[a(x)y^2(x)] + I_{a^+}^{\alpha,\psi}[b(x)y(x)] = I_{a^+}^{\alpha,\psi}[g(x)]. \tag{3.2}$$

Furthermore, applying the information provided by Proposition 2.4 along with the imposed initial conditions, we get

$$y(x) - \sum_{k=0}^{n-1} \frac{y_\psi^k(a)}{k!}(\psi(x)-\psi(a))^k + I_{a^+}^{\alpha,\psi}[a(x)y^2(x)] + I_{a^+}^{\alpha,\psi}[b(x)y(x)]$$
$$= I_{a^+}^{\alpha,\psi}[g(x)], \tag{3.3}$$

or

$$y(x) = \sum_{k=0}^{n-1} \frac{y_\psi^k(a)}{k!}(\psi(x)-\psi(a))^k - I_{a^+}^{\alpha,\psi}[a(x)y^2(x)]$$

$$- I_{a^+}^{\alpha, \psi}[b(x)y(x)] + I_{a^+}^{\alpha, \psi}[g(x)]. \tag{3.4}$$

Further, the standard ADM defines the solution $y(x)$ by the following infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{3.5}$$

while the nonlinear term $y^2(x)$ is expressed through the decomposed series of Adomian polynomials A_n as follows:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{3.6}$$

which when substituted into (3.4) yields the following:

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{k=0}^{n-1}\frac{y_\psi^k(a)}{k!}(\psi(x)-\psi(a))^k + I_{a^+}^{\alpha, \psi}[g(x)]$$

$$- I_{a^+}^{\alpha, \psi}\left[a(x)\sum_{n=0}^{\infty} A_n\right] - I_{a^+}^{\alpha, \psi}\left[b(x)\sum_{n=0}^{\infty} y_n(x)\right], \tag{3.7}$$

upon which the latter admits the following generalized recurrent relations

$$\begin{cases} y_0(x) = \sum_{k=0}^{n-1}\frac{y_\psi^k(a)}{k!}(\psi(x)-\psi(a))^k + I_{a^+}^{\alpha, \psi}[g(x)] = f(x), \\ y_n(x) = -I_{a^+}^{\alpha, \psi}[a(x)A_{n-1}] - I_{a^+}^{\alpha, \psi}[b(x)y_{n-1}(x)], \quad n \geq 1. \end{cases} \tag{3.8}$$

Furthermore, it is worth highlighting here that the ADM defines the zeroth component $y_0(x)$ typically based on the function f derived from the prescribed initial data and the imposed nonhomogeneous term, as detailed earlier. In this context, Wazwaz and El-Sayed [31] presented an effective modification of the standard ADM by decomposing the zeroth component $y_0(x)$ into two components, as shown below

$$y_0 = f_0 + f_1. \tag{3.9}$$

Consequently, using the above assumption, the modified recursive method that was obtained from (3.8) is stated as follows:

$$\begin{cases} y_0(x) = f_0(x), \\ y_n(x) = f_1(x) - I_{a^+}^{\alpha,\psi}[a(x)A_{n-1}] - I_{a^+}^{\alpha,\psi}[b(x)y_{n-1}(x)], \quad n \geq 1. \end{cases} \tag{3.10}$$

Therefore, the closed-form solution found in (3.10) converges rapidly than the standard ADM. The efficiency of Wazwaz's modified method is highly dependent on the selection of the functions $f_0(x)$ and $f_1(x)$, which is discussed in more detail in the next section.

4. Numerical Examples

This section provides a number of numerical examples that illustrate the use of the method and prove its efficiency by providing the accuracy in the results obtained.

Example 4.1 [32]**.** Consider the IVP for ψ-FRDE as follows:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\psi} y(x) - y^2(x) - 1 = 0, \ 0 < \alpha \leq 1, \ x \in [0, 1], \\ y(0) = 0. \end{cases} \tag{4.1}$$

The exact analytical solution of the IVP in (4.1) is obtained as $y(x) = \tan(\psi(x))$.

Consequently, we apply the operator $I_{a^+}^{\alpha,\psi}$ on both sides of the governing equation along with the application of the method in Section 3 to derive the solution to the IVP in (4.1):

$$y(x) = I_{a^+}^{\alpha,\psi}[y^2(x)] + I_{a^+}^{\alpha,\psi}[1], \tag{4.2}$$

such that when the ADM is deployed,

$$\sum_{n=0}^{\infty} y_n(x) = I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} A_n\right] + I_{a^+}^{\alpha,\psi}[1], \tag{4.3}$$

where A_n's are the Adomian polynomials for nonlinear term $y^2(x)$, which when computed iteratively takes the following form:

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0y_1, \\ A_2 = 2y_0y_2 + y_1^2, \\ \quad\vdots \end{cases} \tag{4.4}$$

Moreover, the general recurrent scheme for the governing model is thus given as follows:

$$\begin{cases} y_0(x) = I_{a^+}^{\alpha,\psi}[1], \\ y_n(x) = I_{a^+}^{\alpha,\psi}[A_{n-1}], \ \forall n \geq 1, \end{cases} \tag{4.5}$$

with few components expressed below from the latter recurrent scheme:

$$y_0(x) = \frac{(\psi(x) - \psi(0))^{\alpha}}{\alpha\Gamma(\alpha)},$$

$$y_1(x) = \frac{\alpha\Gamma(\alpha)\Gamma(2\alpha+1)(\psi(x) - \psi(0))^{3\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)},$$

$$y_2(x) = \frac{2\alpha^2\Gamma(\alpha)^2\Gamma(2\alpha+1)\Gamma(4\alpha+1)(\psi(x) - \psi(0))^{5\alpha}}{\Gamma(\alpha+1)^5\Gamma(3\alpha+1)\Gamma(5\alpha+1)},$$

$$\vdots$$

Remarks 4.2.

- Approximate solution of the governing model is obtained upon summing the acquired first six components as follows: $\phi_5 = \sum_{n=0}^{5} y_n(x)$.

• In general, utilizing function ψ is trial; but in this case, use the function that has already been covered in the reference [32].

• We compared our results with the exact solution to the problem by testing various functions ψ in Tables 1-4.

• Tables 1-4 display the function ψ's chosen effect on absolute error. We see that the highest error is in Table 1 (reduction to Caputo derivative), but when we test another function ψ, we see that the absolute error minimizes. Finally, by using $\psi(x) = \sin\left(\frac{x}{4}\right)$ in Table 4, we obtain an analytical solution of the problem.

• Figure 1 shows the behaviour of this solution with respect to various functions ψ and different fractional-orders α.

Table 1. The effect of choosing the function $\psi(x) = x$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.1)

x	$\psi(x) = x$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$2.02710036 \times 10^{-1}$	$2.02710036 \times 10^{-1}$	0
0.4	$4.22793193 \times 10^{-1}$	$4.22793219 \times 10^{-1}$	$2.57000000 \times 10^{-8}$
0.6	$6.84131315 \times 10^{-1}$	$6.84136808 \times 10^{-1}$	$5.49290000e \times 10^{-6}$
0.8	1.02937192	1.02963856	$2.66642000 \times 10^{-4}$
1	1.55136764	1.55740772	$6.04008000 \times 10^{-3}$

Table 2. The effect of choosing the function $\psi(x) = x^3$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.1)

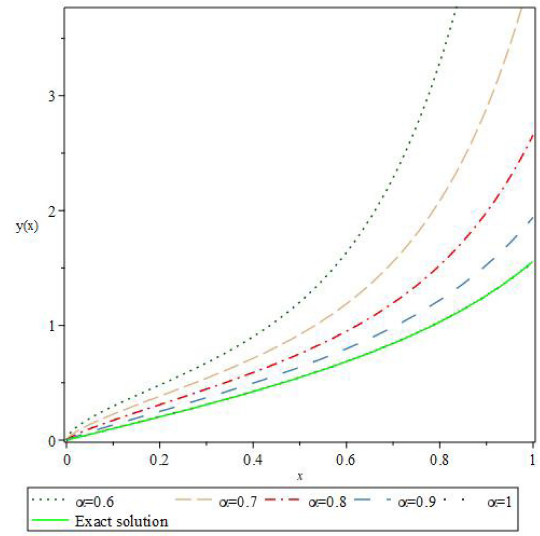
x	$\psi(x) = x^3$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$1.00000033 \times 10^{-3}$	$1.00000033 \times 10^{-3}$	0
0.4	$6.40875247 \times 10^{-2}$	$6.40875247 \times 10^{-2}$	0
0.6	$2.19423130 \times 10^{-1}$	$2.19423130 \times 10^{-1}$	0
0.8	$5.61986756 \times 10^{-1}$	$5.61987424 \times 10^{-1}$	$6.67800000 \times 10^{-7}$
1	1.55136764	1.55740772	$6.04008000 \times 10^{-3}$

Table 3. The effect of choosing the function $\psi(x) = \tan\left(\frac{\pi x}{9}\right)$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.1)

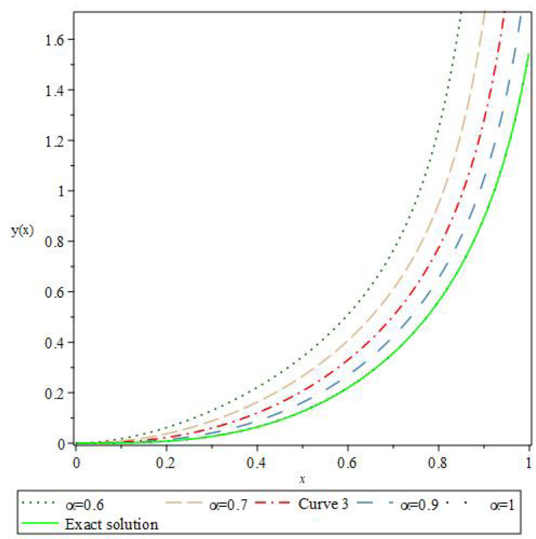
x	$\psi(x) = \tan\left(\frac{\pi x}{9}\right)$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$7.00410104 \times 10^{-2}$	$7.00410104 \times 10^{-2}$	0
0.4	$1.41473512 \times 10^{-1}$	$1.41473512 \times 10^{-1}$	0
0.6	$2.15816613 \times 10^{-1}$	$2.15816613 \times 10^{-1}$	0
0.8	$2.94871774 \times 10^{-1}$	$2.94871775 \times 10^{-1}$	3×10^{-10}
1	$3.80942386 \times 10^{-1}$	$3.80942393 \times 10^{-1}$	7.5×10^{-9}

Table 4. The effect of choosing the function $\psi(x) = \sin\left(\frac{x}{4}\right)$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.1)

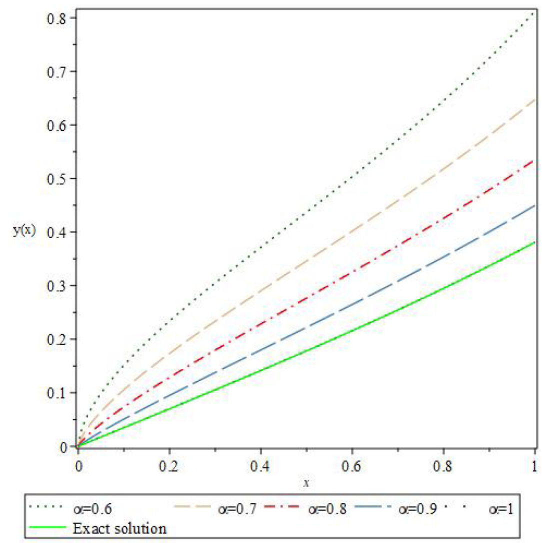
x	$\psi(x) = \sin\left(\frac{x}{4}\right)$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$5.00208255 \times 10^{-2}$	$5.00208255 \times 10^{-2}$	0
0.4	$1.00166414 \times 10^{-1}$	$1.00166414 \times 10^{-1}$	0
0.6	$1.50560565 \times 10^{-1}$	$1.50560565 \times 10^{-1}$	0
0.8	$2.01325060 \times 10^{-1}$	$2.01325060 \times 10^{-1}$	0
1	$2.52578446 \times 10^{-1}$	$2.52578446 \times 10^{-1}$	0



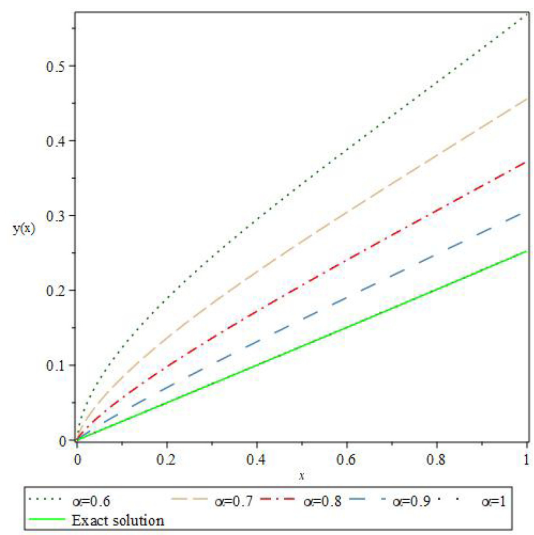
(a) $\psi(x) = x$



(b) $\psi(x) = x^3$



(c) $\psi(x) = \tan\left(\frac{\pi x}{9}\right)$



(d) $\psi(x) = \sin\left(\frac{x}{4}\right)$

Figure 1. Comparison between the ADM and exact solutions for various α's in Example (4.1).

Example 4.3 [33]**.** Consider the IVP for ψ-FRDE as follows:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\psi} y(x) = -y^2(x) + 1,\ 0 < \alpha \le 1,\ x \in [0, 1], \\ y(0) = 0. \end{cases} \tag{4.6}$$

The exact analytical solution of the IVP in (4.6) is obtained as $y(x) = \dfrac{e^{\psi(2x)} - 1}{e^{\psi(2x)} + 1}$.

Consequently, we apply the operator $I_{a^+}^{\alpha,\psi}$ on both sides of the governing equation along with the application of the method in Section 3 to derive the solution to the IVP in (4.6):

$$y(x) = -I_{a^+}^{\alpha,\psi}[y^2(x)] + I_{a^+}^{\alpha,\psi}[1], \tag{4.7}$$

such that when the ADM is deployed,

$$\sum_{n=0}^{\infty} y_n(x) = -I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} A_n\right] + I_{a^+}^{\alpha,\psi}[1], \tag{4.8}$$

where A_n's are the resulting polynomials by Adomian for the nonlinear term $y^2(x)$ as earlier portrayed in (4.4). Moreover, the general recurrent scheme for the governing model is given by:

$$\begin{cases} y_0(x) = I_{a^+}^{\alpha,\psi}[1], \\ y_n(x) = -I_{a^+}^{\alpha,\psi}[A_{n-1}], \quad \forall n \ge 1, \end{cases} \tag{4.9}$$

with few components expressed from the latter recurrent scheme as follows:

$$y_0(x) = \frac{(\psi(x) - \psi(0))^{\alpha}}{\alpha\Gamma(\alpha)},$$

$$y_1(x) = -\frac{\Gamma(2\alpha + 1) \cdot (\psi(t) - \psi(0))^{3\alpha}}{\Gamma(\alpha)^2 \alpha^2 \cdot \Gamma(3\alpha + 1)},$$

$$y_2(x) = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)(\psi(t)-\psi(0))^{5\alpha}}{\Gamma(\alpha)^3\alpha^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)},$$

$$\vdots$$

Remarks 4.4.

• Approximate solution of the governing model is obtained upon summing the acquired first six components as follows: $\phi_5 = \sum_{n=0}^{5} y_n(x)$.

• In general, utilizing function ψ is trial; but in this case, use of the function that has already been covered in the reference [33].

• We compared our results with the exact solution to the problem by testing various functions ψ in Tables 5-8.

• Tables 5-8 display the function ψ's chosen effect on absolute error. We see that the highest error is in Table 5 (reduction to Caputo derivative), but when we test another function ψ, we see that the absolute error minimizes. Finally, by using $\psi(x) = \sin\left(\frac{x}{4}\right)$ in Table 8, we obtain most accurate solution of the problem.

• Figure 2 shows the behaviour of this solution with respect to various functions ψ and different fractional-orders α.

Table 5. The effect of choosing the function $\psi(x) = x$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.3)

x	$\psi(x) = x$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$1.97375320 \times 10^{-1}$	$1.97375320 \times 10^{-1}$	0
0.4	$3.79949311 \times 10^{-1}$	$3.79948962 \times 10^{-1}$	$3.49100000 \times 10^{-7}$
0.6	$5.37077628 \times 10^{-1}$	$5.37049567 \times 10^{-1}$	$2.80614000 \times 10^{-5}$
0.8	$6.64641310 \times 10^{-1}$	$6.64036770 \times 10^{-1}$	$6.04539700 \times 10^{-4}$
1	$7.67901234 \times 10^{-1}$	$7.61594156 \times 10^{-1}$	$6.30707840 \times 10^{-3}$

Table 6. The effect of choosing the function $\psi(x) = x^3$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.3)

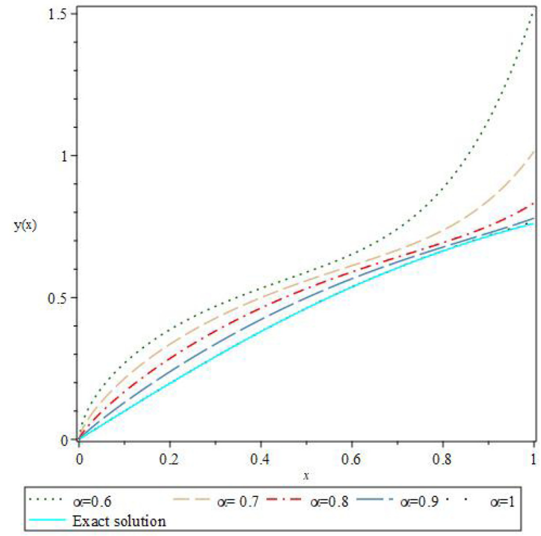
x	$\psi(x) = x^3$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$7.99982934 \times 10^{-3}$	$7.99982914 \times 10^{-3}$	2×10^{-10}
0.4	$6.39127616 \times 10^{-2}$	$6.39127617 \times 10^{-2}$	1×10^{-10}
0.6	$2.12702298 \times 10^{-1}$	$2.12702297 \times 10^{-1}$	4×10^{-10}
0.8	$4.71507115 \times 10^{-1}$	$4.71502037 \times 10^{-1}$	$5.07810000 \times 10^{-5}$
1	$7.67901234 \times 10^{-1}$	$7.61594156 \times 10^{-1}$	$6.30707840 \times 10^{-3}$

Table 7. The effect of choosing the function $\psi(x) = \tan\left(\frac{\pi x}{9}\right)$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.3)

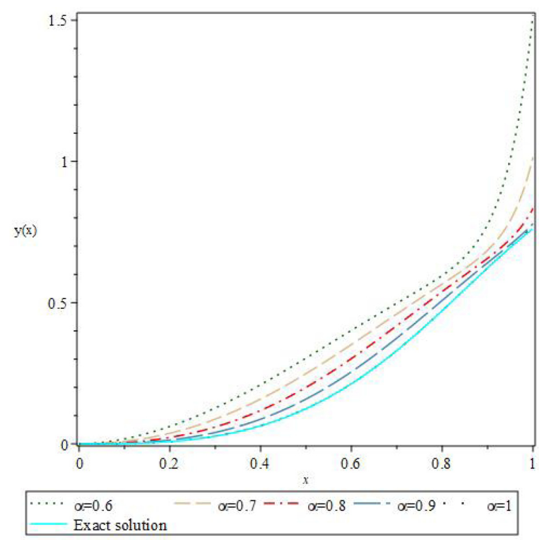
x	$\psi(x) = \tan\left(\frac{\pi x}{9}\right)$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$6.98130593 \times 10^{-2}$	$6.98130595 \times 10^{-2}$	2×10^{-10}
0.4	$1.39622779 \times 10^{-1}$	$1.39622779 \times 10^{-1}$	0
0.6	$2.09412252 \times 10^{-1}$	$2.09412251 \times 10^{-1}$	2×10^{-10}
0.8	$2.79136532 \times 10^{-1}$	$2.79136523 \times 10^{-1}$	9×10^{-9}
1	$3.48706445 \times 10^{-1}$	$3.48706320 \times 10^{-1}$	$1.24800000 \times 10^{-7}$

Table 8. The effect of choosing the function $\psi(x) = \sin\left(\frac{x}{4}\right)$ on the numerical and analytical solutions with $\alpha = 1$ in Example (4.3)

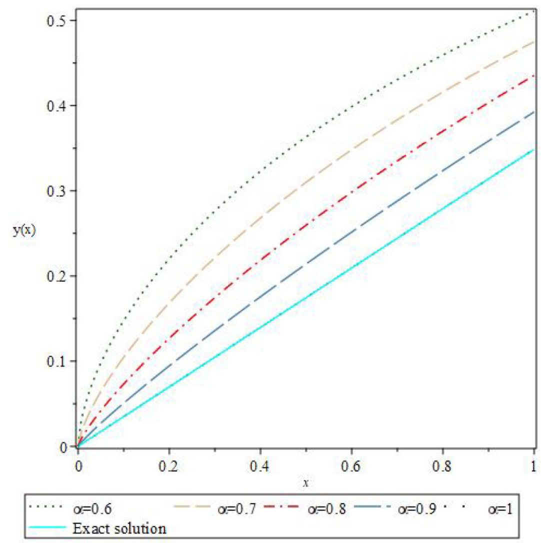
x	$\psi(x) = \sin\left(\frac{x}{4}\right)$		Absolute Error
	Numerical results	Exact results	
0	0	0	0
0.2	$4.99375962 \times 10^{-2}$	$4.99375962 \times 10^{-2}$	0
0.4	$9.95030633 \times 10^{-2}$	$9.95030632 \times 10^{-2}$	7×10^{-11}
0.6	$1.48335575 \times 10^{-1}$	$1.48335575 \times 10^{-1}$	0
0.8	$1.96096155 \times 10^{-1}$	$1.96096155 \times 10^{-1}$	0
1	$2.42476800 \times 10^{-1}$	$2.42476798 \times 10^{-1}$	$1.70800000 \times 10^{-9}$



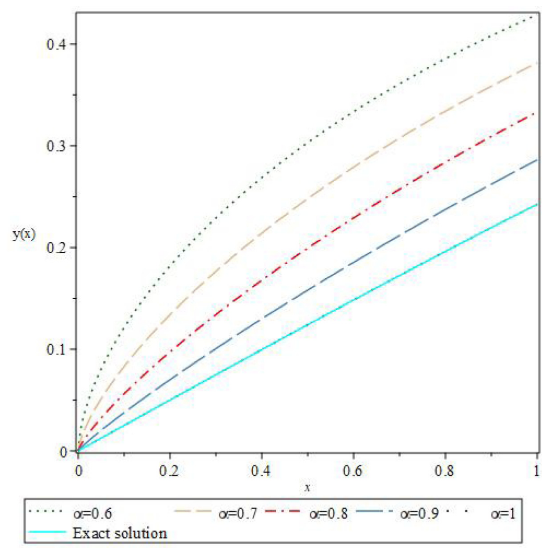
(a) $\psi(x) = x$



(b) $\psi(x) = x^3$



(c) $\psi(x) = \tan\left(\frac{\pi x}{9}\right)$



(d) $\psi(x) = \sin\left(\frac{x}{4}\right)$

Figure 2. Comparison between the ADM and exact solutions for various α's in Example (4.3).

Example 4.5 [32]**.** Consider the IVP for ψ-FRDE as follows:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\psi} y(x) + y(x) + y^2(x) = g(x),\ 0 < \alpha \le 1,\ 0 \le x \le 1, \\ y(0) = 0, \end{cases} \tag{4.10}$$

where the nonhomogeneous term $g(x)$ is given by

$$g(x) = (\psi(x))^2 + (\psi(x))^4 + \frac{2}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha}.$$

Besides, the exact solution for the governing ψ-fractional model takes the form (4.10) is $y(x) = (\psi(x))^2$. Further, to solve the problem in (4.10) using the adopted ADM, we apply the operator $I_{a^+}^{\alpha,\psi}$ on both sides of the equation to obtain

$$\sum_{n=0}^{\infty} y_n(x) = I_{a^+}^{\alpha,\psi}\left[\frac{2}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha} + (\psi(x))^2 + (\psi(x))^4\right] - I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} y_n(x)\right] - I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} A_n\right], \tag{4.11}$$

or, equivalently

$$\sum_{n=0}^{\infty} y_n(x) = (\psi(x))^2 + \frac{\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} + \frac{\Gamma(5)}{\Gamma(5+\alpha)}(\psi(x))^{4+\alpha} - I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} y_n(x)\right] - I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} A_n\right], \tag{4.12}$$

where A_n's are the resulting polynomials by Adomian for the nonlinear term $y^2(x)$ as earlier portrayed in (4.4).

Further, the resultant recursive formula for the governing ψ-fractional IVP is obtained using the Wazwaz's modification algorithm from (3.10) as follows:

$$\begin{cases} y_0(x) = (\psi(x))^2, \\ y_n(x) = -I_{a^+}^{\alpha,\psi}[y_{n-1}(x)] - I_{a^+}^{\alpha,\psi}[A_{n-1}] + \dfrac{\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} \\ \qquad + \dfrac{\Gamma(5)}{\Gamma(5+\alpha)}(\psi(x))^{4+\alpha}, \quad \forall n \geq 1 \end{cases} \tag{4.13}$$

or more unequivocally as follows:

$$\begin{cases} y_0(x) = (\psi(x))^2, \\ y_1(x) = -I_{a^+}^{\alpha,\psi}[y_0(x)] - I_{a^+}^{\alpha,\psi}[A_0] + \dfrac{\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} \\ \qquad + \dfrac{\Gamma(5)}{\Gamma(5+\alpha)}(\psi(x))^{4+\alpha} \\ \quad = -I_{a^+}^{\alpha,\psi}[(\psi(x))^2] - I_{a^+}^{\alpha,\psi}[(\psi(x))^4] + \dfrac{\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} \\ \qquad + \dfrac{\Gamma(5)}{\Gamma(5+\alpha)}(\psi(x))^{4+\alpha} \\ \quad = -\dfrac{\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} - \dfrac{\Gamma(5)}{\Gamma(5+\alpha)}(\psi(x))^{4+\alpha} \\ \qquad + \dfrac{\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} + \dfrac{\Gamma(5)}{\Gamma(5+\alpha)}(\psi(x))^{4+\alpha} \\ \quad = 0. \end{cases}$$

Significantly, the iterative solution above sums up to yield the following closed-form expression

$$y_n(x) = 0, \quad \forall n \geq 1, \tag{4.14}$$

upon which the exact solution for the model is obtained as follows:

$$y(x) = (\psi(x))^2. \tag{4.15}$$

Remarks 4.6.

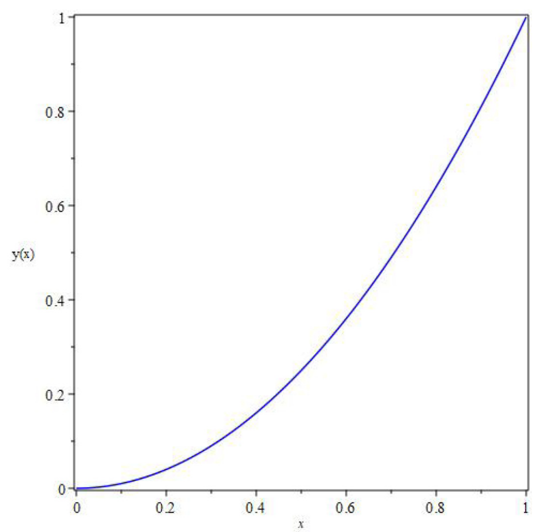
- The Wazwaz modification appears to slightly accelerate the convergence of the series solution when compared to the standard ADM.

• It is important to recall that the Wazwaz method's success depends mainly on selection of the correct parts, f_0 and f_1. We have not been able to establish any rule to determine which types of f_0 and f_1 will probably result in needed acceleration.

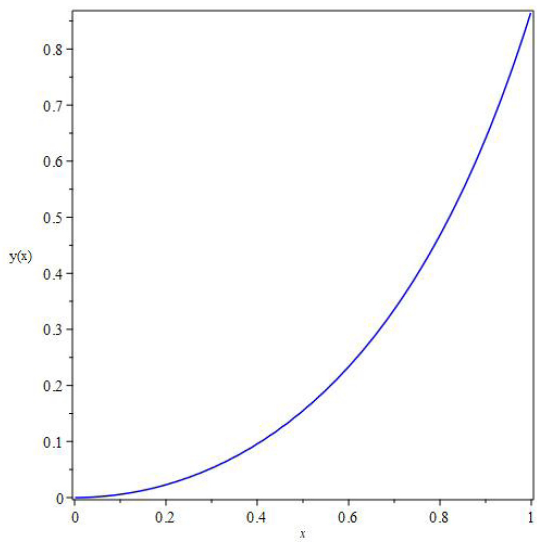
• Note that modified Wazwaz avoids the calculation of Adomian polynomials $(A_1, A_2, \ldots)$, which reduces calculation complexity.

• In the same vein, Ali and Minamoto [32] equally solved the governing ψ-FRDE (4.10) through the application of the ψ-Haar wavelet operational matrix, and obtained an approximate solution by calculating 6 terms. However, we obtain an exact solution using only 2 terms.

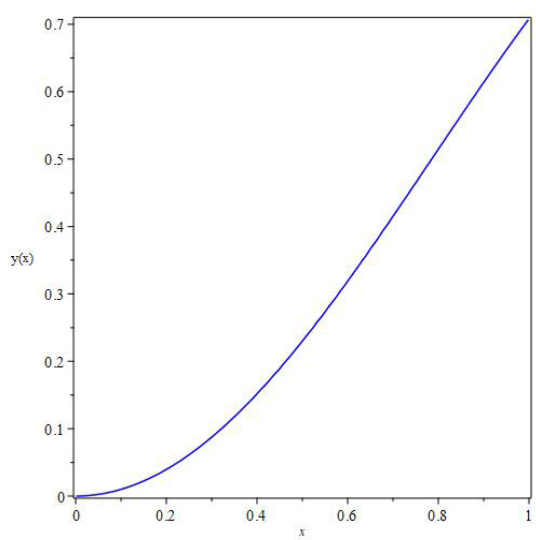
• Figure 3 displays the acquired ADM solution for different choices of ψ for the ψ-fractional IVP (4.10).



(a) $\psi(x) = x$



(b) $\psi(x) = \tan\left(\frac{3x}{4}\right)$



(c) $\psi(x) = \sin(x)$

Figure 3. ADM solution for Example (4.10) with respect to different kernels ψ.

Example 4.7 [32]**.** Consider the ψ-RFDE as follows:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\psi} y(x) + y^2(x) = g(x),\ 0 < \alpha \leq 1,\ 0 \leq x \leq 1, \\ y(0) = 0, \end{cases} \tag{4.16}$$

with the nonhomogeneous term $g(x)$ defined by

$$g(x) = \frac{2}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha} + \frac{\alpha}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha}$$

$$+ (\psi(x))^4 + (\alpha)^2(\psi(x))^2 + 2\alpha(\psi(x))^3.$$

The exact analytical solution expressed for the fractional IVP (4.16) is reported to be $y(x) = \alpha(\psi(x)) + (\psi(x))^2$. As we proceed to solve (4.16) using ADM, we apply the operator $I_{a^+}^{\alpha,\psi}$ on both sides of the equation to obtain the following:

$$\sum_{n=0}^{\infty} y_n(x) = I_{a^+}^{\alpha,\psi}\left[\frac{2}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha} + \frac{\alpha}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha}\right.$$

$$\left. + (\psi(x))^4 + (\alpha)^2(\psi(x))^2 + 2\alpha(\psi(x))^3\right] - I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} A_n\right], \tag{4.17}$$

or, equivalently

$$\sum_{n=0}^{\infty} y_n(x) = (\psi(x))^2 + \alpha(\psi(x)) + \frac{\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha}$$

$$+ \frac{\alpha^2\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} + \frac{2\alpha\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha}$$

$$- I_{a^+}^{\alpha,\psi}\left[\sum_{n=0}^{\infty} A_n\right], \tag{4.18}$$

with A_n's equally denoting the polynomials by Adomian earlier outlined for nonlinear term $y^2(x)$. Further, with the help of the modified recursive algorithm by Wazwaz, we obtain from (3.10), the following recurrent scheme for the model:

$$\begin{cases} y_0(x) = (\psi(x))^2 + \alpha\psi(x), \\ y_1(x) = \dfrac{\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} + \dfrac{\alpha^2\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} \\ \qquad + \dfrac{2\alpha\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} - I_{a^+}^{\alpha,\psi}[A_0] \\ \quad = \dfrac{\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} + \dfrac{\alpha^2\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} \\ \qquad + \dfrac{2\alpha\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} - I_{a^+}^{\alpha,\psi}[((\psi(x))^2 + \alpha\psi(x))^2] \\ \quad = \dfrac{\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} + \dfrac{\alpha^2\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} \\ \qquad + \dfrac{2\alpha\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} - \dfrac{\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} \\ \qquad - \dfrac{\alpha^2\Gamma(3)}{\Gamma(3+\alpha)}(\psi(x))^{2+\alpha} - \dfrac{2\alpha\Gamma(4)}{\Gamma(4+\alpha)}(\psi(x))^{3+\alpha} \\ \quad = 0. \end{cases}$$

Moreover, as in the previous model, the iterative components above sum to the following concise expression

$$y_n(x) = 0, \quad \forall n \geq 1, \tag{4.19}$$

upon which the exact solution is obtained as follows:

$$y(x) = \alpha\psi(x) + (\psi(x))^2. \tag{4.20}$$

Remarks 4.8.

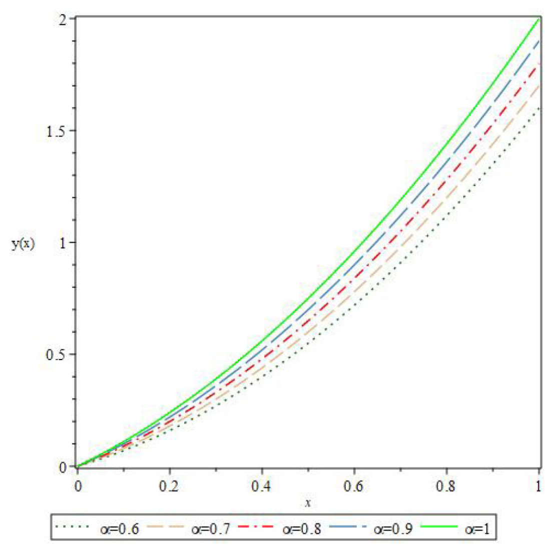
- The Wazwaz modification appears to slightly accelerate the convergence of the series solution when compared to the standard ADM.

• It is important to recall that the Wazwaz method's success depends mainly in selection of the correct parts, f_0 and f_1. We have not been able to establish any rule to determine which types of f_0 and f_1 will probably result in needed acceleration.

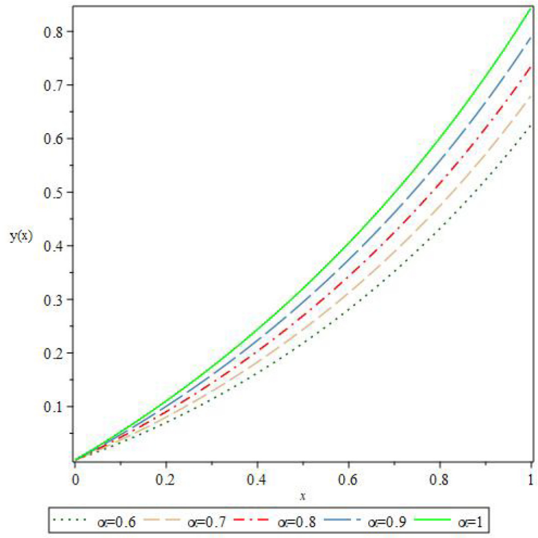
• Note that modified Wazwaz avoids the calculation of Adomian polynomials $(A_1, A_2, \ldots)$, which reduces calculation complexity.

• Additionally, Ali and Minamoto [32] equally solved the present ψ-fractional model (4.16) with the aid of the ψ-Haar wavelet operational matrix, and obtained an approximate solution by calculating 6 terms. However, we obtain an exact solution using only 2 terms.

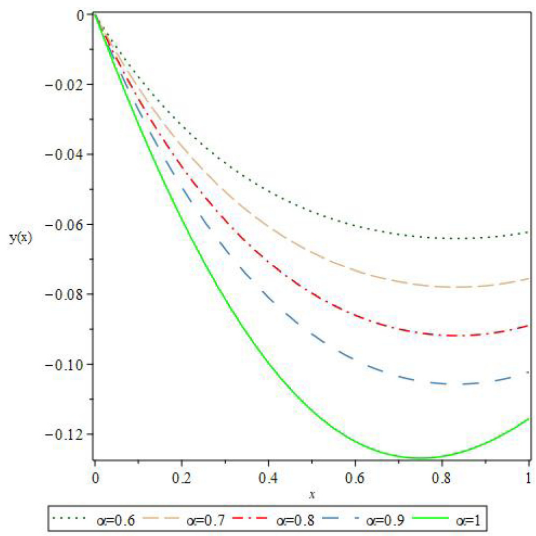
• Figure 4 represents the acquired ADM solution for different choices of ψ and different fractional-orders α in case $\psi(x) = x$ featuring the state of the Caputo fractional derivative for (4.16).



(a) $\psi(x) = x$



(b) $\psi(x) = \tan\left(\frac{x}{2}\right)$



(c) $\psi(x) = \frac{x^2}{5} - \frac{x}{3}$

Figure 4. ADM solution for Example (4.10) with respect to different kernels.

5. Conclusion

In this study, we have presented a new generalization of fractional Riccati equations by using a fractional derivative of a function with respect to a function $\psi(x)$. The main goal of this article is to obtain the semi-analytical solutions to the ψ-Caputo fractional Riccati equations using Adomian decomposition method with Wazwaz modification. Our solutions are successfully applied to find the solutions of a class of ψ-FRDE. We have investigated these semi-analytical solutions with different functions of $\psi(x)$ for various fractional orders. The study concludes by recommending the extension of the employed method to address more intricate physical models with real-life applications.

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