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REGULARIZATION OF THE INVERSE PROBLEM WITH THE D'ALEMBERT OPERATOR IN AN UNBOUNDED DOMAIN DEGENERATING INTO A SYSTEM OF INTEGRAL EQUATIONS OF VOLTERRA TYPE

T. D. Omurov and K. R. Dzhumagulov*

Department of Mathematical Analysis
Kyrgyz National University named after J. Balasagyn
Bishkek, Kyrgyzstan
e-mail: omurovtd@mail.ru

School of Data Engineering and Environmental Studies
Technical School of Innovation of the American University of Central Asia
Bishkek, Kyrgyzstan
e-mail: kubatdjumagulov@gmail.com

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*Corresponding author

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Abstract

In this paper, we study the inverse problem for the wave equation with the second-order d'Alembert operator in an unbounded domain in a space with a non-uniform metric. For physical applications, inverse problems for second-order partial differential equations are of particular interest. Such inverse problems are encountered in the study of wave processes, processes of electromagnetic interactions, as well as in various reduction processes. Moreover, if there are external acting forces with respect to the indicated equations that allow additional information about the solution of the original equations, then we obtain classes of inverse problems of a coefficient nature with the d'Alembert operator, which are of particular interest to scientists in this field, in which the results of this article are relevant. Also, the relevance of the problem under study is due to the fact that it is an inverse problem, where the sought quantities are the causes of some known consequences of a particular process. Whereas for direct problems, the methods for solving are well known. Thus, this paper provides a solution to the inverse problem of mathematical physics with a hyperbolic operator and generalizes existing results.

1. Introduction

It is known that for physical applications, direct and inverse problems for second-order partial differential equations, called *equations of mathematical physics* [1, 2], are of particular interest. The classes of these equations are different, and therefore the methods for studying them are different. The noted classes of equations, in particular, are found in the theory of waves, in problems of the hereditary environment, in the theory of interaction [3, 6, 8, 11, 13, 15], for example, as the sin-Gordon equation, nonlinear equations of the Whitham-Schrödinger type, among others. In many applied problems, it is known that there are inverse problems [4, 7-10], where integral equations of Volterra or Fredholm type degenerate. Under certain conditions, these equations are transformed to Volterra-Abel or Fredholm-Abel integral equations of the third kind, for example, refer to [7, 8]. Moreover, based on the theory of these integral equations, it is possible to prove the correctness of [1, 2] or conditional correctness of studied problems in [5, 12]. To this end, this article studies the inverse problem with the second-order d'Alembert operator in an unbounded domain:

$$D^2_{c^2(x)} u = (HZ)(t) + \lambda u u_x, \quad \forall (t, x) \in \overline{Q} \quad (\Omega = (0, T) \times R), \tag{1.1}$$

$$u^{(j)}_{t^j}\Big|_{t=0} = 0 \quad (j = (0, 1)), \tag{1.2}$$

$$(u_t - c(x) u_x)\big|_{x=x_0} = g(t), \quad \forall t \in [0, T] \quad (g(0) = 0), \tag{1.3}$$

where $0 < \lambda = const < 1$, c, H, g are known data satisfying the conditions:

$$\begin{cases} HZ \equiv \int_0^t H(t, s)\gamma(s) Z(s) ds; \ (Z(0) = 0), \\ D^2_{c^2} = \dfrac{\partial^2}{\partial t^2} - c^2(x)\dfrac{\partial^2}{\partial x^2}; \ g(t) \in C^1[0, T]; \ C^{0,1}(Q_0) \ni H(t, s), \\ p(t) = H(t, t); \ p\big|_{t=0, t=T} = 0; \ 0 < p(t), \ \forall t \in (0, T), \\ H_0(t, s) \equiv H_s(t, s); \ Q_0 = [0, T] \times [0, T]; \ p(t) \in C[0, T], \\ \int_0^t |\, H_0(t, s)(p(s))^{-1}\, | ds \le d_0 < 1, \\ 0 \le \gamma(t) \in L^1(0, T); \ \phi(t) = \int_0^t \gamma(s) ds; \ 0 \ne c(x) \in C(R). \end{cases} \tag{1.4}$$

In this case, the unknown is the vector function $U = (u, Z)$ in $W^p_\gamma(Q)$ - space with a weight function with norm:

$$\begin{cases} \| U \|_{W^p_\gamma(Q)} = \| u \|_{C^{2,2}(\overline{Q})} + \| Z \|_{L^p_\gamma(0,T)}, \quad (p > 1), \\ \| Z \|_{L^p_\gamma(0,T)} = \left(\sup\limits_{[0,T]} \int_0^t \gamma(s)(Z(s))^p ds \right)^{\frac{1}{p}}. \end{cases}$$

The study consists of two situations. First, we consider how the original inverse problem is transformed into an integral form. For this, a special version of the auxiliary function method is proposed [8], so that the original problem is transformed to a system of Volterra-Abel integral equations and Volterra integral equations of the first kind, where Volterra integral equations of the first kind are conditionally correct Volterra equations. Next, taking into account the regularization method [5, 8], we study the indicated system. In this case, the regularizability of the original inverse problem is established in $W_\gamma^p(Q)$.

2. Main Results

2.1. Transformation of the inverse problem to integral form

The proposed algorithm modifies the auxiliary function method applied to the study of the direct transfer problem with the d'Alembert operator in an unbounded domain in [9].

Suppose

$$\begin{cases} u_t - c(x)u_x = V_1(t, x), \\ u_t + c(x)u_x = V_2(t, x) \end{cases} \tag{2.1}$$

with conditions

$$\begin{cases} V_1(0, x) = 0, V_2(0, x) = 0, \forall x \in R, \\ V_1(t, x_0) = g(t), \forall t \in [0, T]. \end{cases} \tag{$*$}$$

Then

$$\begin{cases} u = \frac{1}{2}\int_0^t (V_1(s, x) + V_2(s, x))ds \equiv (A_1[V_1, V_2])(t, x), \\ c(x)u_x = \frac{1}{2}(V_2 - V_1) \equiv (A_2[V_1, V_2]), \forall (t, x) \in \overline{\Omega}. \end{cases} \tag{2.2}$$

Therefore, from differential equation (1.1), taking into account (2.1) and (2.2), it follows that

$$\begin{cases} V_{1t} + c(x)V_{1x} = \vartheta(t) + (B_1[V_1, V_2])(t, x), \\ V_{2t} - c(x)V_{2x} = \vartheta(t) + (B_1[V_1, V_2])(t, x), \\ (HZ)(t) \equiv \vartheta(t), \\ (B_1[V_1, V_2])(t, x) \equiv c'(x)A_2[V_1, V_2] - \lambda(c(x))^{-1}A[V_1, V_2], \\ (A[V_1, V_2])(t, x) \equiv (A_1[V_1, V_2])(t, x) \times (A_2[V_1, V_2])(t, x). \end{cases} \tag{2.3}$$

Differential equation (2.3) is a first order equation, and the transformation (2.1) reduces the order of the differential equation. From this equation, it follows that

$$\begin{cases} V_1 = \int_0^t \vartheta(s)ds + \int_0^t (B_1[V_1, V_2])(s, \rho_1(x, t, s))ds \\ \quad \equiv (P_{01}[\vartheta, V_1, V_2])(t, x), \\ V_2 = \int_0^t \vartheta(s)ds + \int_0^t (B_1[V_1, V_2])(s, \rho_2(x, t, s))ds \\ \quad \equiv (P_{02}[\vartheta, V_1, V_2])(t, x), \end{cases} \tag{2.4}$$

where

$$\begin{cases} \rho_i(x, t, t) \equiv x, \ (i = 1, 2), \\ \rho_{1t} + c(x)\rho_{1x} = 0, \\ \rho_{2t} - c(x)\rho_{2x} = 0, \\ \text{or} \\ \rho_1 = x - \int_s^t c(\rho_1(x, t, s'))ds', \\ \rho_2 = x + \int_s^t c(\rho_2(x, t, s'))ds'. \end{cases} \tag{2.5}$$

From this, it is clear that (2.4) contains three unknown functions: ϑ, V_1, V_2. Therefore, taking into account $(*)$ from (2.4), we define the function:

$$\begin{cases} x = x_0: \\ \int_0^t \vartheta(s)ds = g(t) - \int_0^t (B_1[V_1, V_2])(s, \rho_1(x_0, t, s))ds \equiv (P_{03}[V_1, V_2])(t). \end{cases} \tag{2.6}$$

Then from the system (2.4), we obtain

$$\begin{cases} V_1 = (P_{03}[V_1, V_2])(t) \\ \qquad + \int_0^t (B_1[V_1, V_2])(s, \rho_1(x, t, s))ds \equiv (P_1[V_1, V_2])(t, x), \\ V_2 = (P_{03}[V_1, V_2])(t) \\ \qquad + \int_0^t (B_1[V_1, V_2])(s, \rho_2(x, t, s))ds \equiv (P_2[V_1, V_2])(t, x), \end{cases} \tag{2.7}$$

where (2.7) is a system of Volterra integral equations of the second kind with respect to the variable. Therefore, if executed

$$\begin{cases} \sum_{i=1}^{2} L_{P_i} = q_0 < 1, \\ V_{i,n+1} = (P_i[V_{1,n}, V_{2,n}])(t, x), \ (i = 1, 2; \ n = 0, 1, \ldots), \end{cases} \tag{2.8}$$

where L_{P_i}'s are the coefficients of the initial approximation operators.

Based on the Picard method, we have

Lemma 1. *Under conditions* $(*)$, (2.8), *system* (2.7) *is uniquely solvable in* $C^{1,1}(\overline{\Omega})$, *and*, *based on* (2.1), (2.2), *the solution is a unique function* $u \in C^{2,2}(\overline{\Omega})$.

2.2. Regularizability of the Volterra integral equation

If the conditions of Lemma 1 are met, taking into account (1.4) from (2.6), we have

$$\begin{cases} (HZ)(z) \equiv \int_0^t H(t, s)h(s)Z(s)ds = F(t), \\ \frac{d}{dt}[(P_{03}[V_1, V_2])(t)] \equiv F(t) \in C[0, T] : F(0) = 0. \end{cases} \tag{2.9}$$

Next, carrying out integration by parts from (2.9), we obtain

$$\begin{cases} p(t)\theta(t) = F(t) + \int_0^t H_0(t, s)\theta(s)ds, \\ \int_0^t \gamma(s)Z(s)ds = \theta(t), \ (\theta(0) = 0, \ Z(0) = 0), \\ H_s(t, s) \equiv H_0(t, s); \ H(t, t) \equiv p(t); \ p\,|_{t=0, t=T} = 0; \ p > 0, \ \forall t \in (0, T) \end{cases}$$

or

$$\begin{cases} \psi(t) = F(t) + \int_0^t H_0(t, s)(p(s))^{-1}\psi(s)ds \equiv (G\psi)(t), \\ p(t)\theta(t) = \psi(t), \ (\psi(0) = 0, \ \psi(T) = 0), \\ \int_0^t \gamma(s)Z(s)ds = \theta(t). \end{cases} \tag{2.10}$$

The first integral equation of the system (2.10) is a Volterra-Abel integral equation of the second kind and is subject to the conditions:

$$\begin{cases} G: \\ d_0 = L_G < 1, \\ \psi_{n+1} = (G\psi_n)(t), \ (\psi_0 = 0; \ n = 0, 1, ...). \end{cases} \tag{2.11}$$

The indicated equation is uniquely solvable in $C[0, T]$.

This means, based on the last conclusion regarding the first integral equation of the system (2.10), while assuming the conditions:

$$\tilde{\theta}(t) = \begin{cases} \dfrac{\psi(t)}{p(t)}, & t \neq 0, \ (t \neq T), \\ 0, & t = 0, \\ k_0, & t = T \end{cases} \tag{2.12}$$

and

$$\lim_{t\to 0} \frac{\psi(t)}{p(t)} = 0, \quad \left(\lim_{t\to T} \frac{\psi}{p} = k_0\right), \tag{2.13}$$

we can conclude that the function $\tilde{\theta}(t)$ is continuous. Thus the function $\theta(t)$ is continuous, and

$$\theta(t) \equiv \tilde{\theta}(t), \quad t \in [0, T]. \tag{2.14}$$

Taking into account the results obtained regarding the first and second relations of the system (2.10), we study the Volterra integral equation of the first kind:

$$\int_0^t \gamma(s) Z(s) ds = \theta(t) \tag{2.15}$$

with the condition:

$$\begin{cases} Z(0) = 0; \ \theta(0) = 0; \ C_\phi^1[0, T] \ni \theta(t) : |\theta(t) - \theta(s)| \leq C_0 |\phi(t) - \phi(s)|, \\ \phi(t) = \int_0^t \gamma(s) ds; \ 0 < C_0 = \text{const}, \\ \theta_\varepsilon(t) : \|\theta_\varepsilon - \theta(t)\|_{L_\gamma^p(0,T)} \leq \Delta(\varepsilon), \left(\dfrac{\Delta(\varepsilon)}{\varepsilon^{2-\frac{1}{q}}} \xrightarrow[\varepsilon\to 0]{} 0, \ \dfrac{1}{p} + \dfrac{1}{q} = 1 \right), \end{cases} \tag{2.16}$$

in particular, for example:

$$\Delta(\varepsilon) = \varepsilon^{\tilde{\beta}}, \ \left(2 - \frac{1}{q} < \tilde{\beta} \le 2 - \frac{1}{q} + \mu, \ (0 < \mu < 1)\right), \tag{2.17}$$

and since it is required that the solution (2.15) exists in a weighted space $L_\gamma^p(0, T)$, regularizability is proved in this space.

In order to prove regularizability, we introduce an equation with a small parameter:

$$\varepsilon Z_\varepsilon(t) + \int_0^t \gamma(s) Z_\varepsilon(s) ds = \theta_\varepsilon(t). \tag{2.18}$$

Since for the kernel $\left(-\frac{1}{\varepsilon}\gamma(s)\right)$, it has a resolvent of the form:

$$R_0(t, s, \varepsilon) \equiv -\frac{1}{\varepsilon}\gamma(s)\exp\left(-\frac{1}{\varepsilon}(\phi(t) - \phi(s))\right), \ \ (s \le t),$$

(2.18) is transformed to the form

$$Z_\varepsilon = -\frac{1}{\varepsilon^2}\int_0^t \gamma(s)\exp\left(-\frac{1}{\varepsilon}(\phi(t) - \phi(s))\right)[\theta_\varepsilon(s) - \theta(s) + \theta(s) - \theta(t)]ds$$

$$+ \frac{1}{\varepsilon}(\theta_\varepsilon(t) - \theta(t)) + \frac{1}{\varepsilon}\theta(t)\exp\left(-\frac{1}{\varepsilon}\phi(t)\right). \tag{2.19}$$

Evaluating (2.19), we have

$$\begin{cases} |Z_\varepsilon| \le \dfrac{1}{\varepsilon^2}\left(\displaystyle\int_0^t \gamma(s)\exp\left(-q\frac{1}{\varepsilon}(\phi(t)-\phi(s))\right)ds\right)^{\frac{1}{q}} \|\theta_\varepsilon(t)-\theta(t)\|_{L_\gamma^p(0,T)} \\ \quad + \dfrac{1}{\varepsilon^2}\displaystyle\int_0^t \gamma(s)\exp\left(-\frac{1}{\varepsilon}(\phi(t)-\phi(s))\right)C_0|\phi(t)-\phi(s)|ds \\ \quad + \dfrac{1}{\varepsilon}|\theta_\varepsilon(t)-\theta(t)| + \dfrac{1}{\varepsilon}C_0\phi(t)\exp\left(-\dfrac{1}{\varepsilon}\phi(t)\right) \\ \quad \le \Delta(\varepsilon)\dfrac{1}{\varepsilon^{2-\frac{1}{q}}(q)^{\frac{1}{q}}} + C_0(1+e^{-1}) + \dfrac{1}{\varepsilon}|\theta_\varepsilon(t)-\theta(t)|, \\ 1+e^{-1} < 2;\ \sup\limits_{\theta\ge 0}\chi\exp(-\chi) = e^{-1},\ \left(\chi = \dfrac{1}{\varepsilon}\phi(t)\right). \end{cases} \tag{2.20}$$

Moreover, based on the norm $L_\gamma^p(0, T)$, from (2.16), it follows that

$$\begin{cases} \|Z_\varepsilon\|_{L_\gamma^p} \le 2\left[\Delta(\varepsilon)\dfrac{1}{\varepsilon^{2-\frac{1}{q}}(q)^{\frac{1}{q}}} + 2C_0\right]\left(\sup\limits_{[0,T]}\displaystyle\int_0^t\gamma(s)ds\right)^{\frac{1}{p}} \\ \quad + \dfrac{1}{\varepsilon}2\|\theta_\varepsilon(t)-\theta(t)\|_{L_\gamma^p(0,T)} \\ \quad \le \left[2\left(\Delta(\varepsilon)\dfrac{1}{\varepsilon^{2-\frac{1}{q}}(q)^{\frac{1}{q}}} + 2C_0\right)N_1 + 2\dfrac{1}{\varepsilon}\Delta(\varepsilon)\right] \le N_2, \\ \left(\sup\limits_{[0,T]}\displaystyle\int_0^t\gamma(s)ds\right)^{\frac{1}{p}} = N_1;\ \dfrac{\Delta(\varepsilon)}{\varepsilon^{2-\frac{1}{q}}}\xrightarrow[\varepsilon\to 0]{} 0,\ (0 < N_i = const,\ i = 1,\ 2). \end{cases} \tag{2.21}$$

Next, based on

$$Z_\varepsilon(t) = Z(t) + \xi_\varepsilon(t), \quad \forall t \in C[0, T], \tag{2.22}$$

from (2.19), we get

$$\begin{cases} \xi_\varepsilon = \dfrac{1}{\varepsilon^2}\displaystyle\int_0^t \gamma(s)\exp\left(-\dfrac{1}{\varepsilon}(\phi(t)-\phi(s))\right)[\theta_\varepsilon(s)-\theta(s)]ds \\ \qquad + \dfrac{1}{\varepsilon}(\theta_\varepsilon(t)-\theta(t)) - \Delta(\varepsilon, Z), \\ \Delta(\varepsilon, Z) \equiv Z(t)\exp\left(-\dfrac{1}{\varepsilon}\phi(t)\right) \\ \qquad + \dfrac{1}{\varepsilon}\displaystyle\int_0^t \gamma(s)\exp\left(-\dfrac{1}{\varepsilon}(\phi(t)-\phi(s))\right)[Z(t)-Z(s)]ds, \end{cases} \tag{2.23}$$

where $\xi_\varepsilon(t)$ is the residual function, and the following is assumed:

$$\begin{cases} \|\Delta(\varepsilon, Z)\|_{L_\gamma^p} \\ \leq \left[\displaystyle\int_0^{\varepsilon^\beta} |Z(\phi^{-1}(\nu))|^p d\nu + \|Z(t)\|_{L_\gamma^p}^p e^{-\frac{p}{\varepsilon^{1-\beta}}}\right]^{\frac{1}{p}} \\ \quad + (2q^{-1})^{\frac{1}{q}} k_0 \left[2p^{-1}\omega_Z^p(\varepsilon^\beta) + 2^{p+1}p^{-1}\|Z(t)\|_{L_\gamma^p}^p e^{-\frac{p}{2\varepsilon^{1-\beta}}}\right]^{\frac{1}{p}} \\ = N_0(\varepsilon), \ (1 < k_0 = const, \ 0 < \beta < 1), \\ \omega_Z^p(\varepsilon^\beta) = \sup\limits_{0 \leq h \leq \varepsilon^\beta} \|Z(\phi^{-1}(u+h)) - Z(\phi^{-1}(u))\|_{L^p}, \\ Z(\phi^{-1}(u)) = 0, \text{ at } u \in [0, \phi(T)], \ \varepsilon^\beta < \phi(T). \end{cases} \tag{2.24}$$

In this case, it is taken into account that if $Z \in L_\gamma^p(0, T)$, then $Z(\phi^{-1}(u)) \in L^p(0, \phi(T))$, and

$$\|Z(t)\|_{L_\gamma^p} = \|Z(\phi^{-1}(u))\|_{L^p},$$

where $\|\cdot\|_{L^p}$ is the norm in $L^p(0, \phi(T))$. Then, estimating (2.23) in the sense of $L^p_\gamma(0, T)$, taking into account (2.24), we have

$$\begin{cases} \|\xi_\varepsilon\|_{L^p_\gamma} \le 2^{p+1}\left\{\Delta(\varepsilon)\left[\dfrac{1}{\varepsilon^{2-\frac{1}{q}}(q)^{\frac{1}{q}}}N_1 + \dfrac{1}{\varepsilon}\right] + N_0(\varepsilon)\right\} = Q_*(\varepsilon), \\ \Delta(\varepsilon)\dfrac{1}{\varepsilon^{2-\frac{1}{q}}} \xrightarrow[\varepsilon\to 0]{} 0, \left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right). \end{cases} \tag{2.25}$$

Therefore, based on the results obtained, it follows that equation (2.15) is regularizable in $L^p_\gamma(0, T)$, and

$$\begin{cases} \|Z_\varepsilon - Z\|_{L^p_\gamma} = \|\xi_\varepsilon\|_{L^p_\gamma} \le O(Q_*(\varepsilon)) \xrightarrow[\varepsilon\to 0]{} 0, \\ \|Z\|_{L^p_\gamma} \le 2[\|Z - Z_\varepsilon\|_{L^p_\gamma} + \|Z_\varepsilon\|_{L^p_\gamma}] \le 2(O(Q_*(\varepsilon)) + N_2) \le N_3. \end{cases} \tag{2.26}$$

Lemma 2. *Under conditions* (2.16), (2.24) *and* (2.26), *the permissible estimation error between the functions* Z_ε, Z *is of the order of* $O(Q_*(\varepsilon))$ *in* $L^p_\gamma(0, T)$.

Therefore, taking into account the results of Lemmas 1, 2 relative to the vector function $U = (u, Z) \in W^p_\gamma(\Omega)$ holds, and

$$\|U\|_{W^p_\gamma(\Omega)} = \|u\|_{C^{2,2}(\overline{\Omega})} + \|Z\|_{L^p_\gamma(0,T)} \le M_0. \tag{2.27}$$

In addition, we note that based on the theorem of Friedrichs [14], the function $u \in L^2(\Omega)$ (no going back), the function $U = (u, Z)$ is evaluated in $\widetilde{W}^p_\gamma(\Omega)$, and

$$\|U\|_{W^p_\gamma(\Omega)} = \|u\|_{L^p(\Omega)} + \|Z\|_{L^p_\gamma(0,T)} \le M_*. \tag{2.28}$$

Theorem 1. *Under the conditions of Lemmas* 1, 2 *and* (2.27) (*or* (2.28)), *the inverse problem* (1.1)-(1.3) *is regularizable in* $W_{\gamma}^{p}(\Omega)$ (*or* $\tilde{W}_{\gamma}^{p}(\Omega)$).

3. Conclusion

In this paper, we considered the inverse problem with the second-order d'Alembert operator in an unbounded domain, which, using the modified auxiliary function method, is transformed into a system of integral equations of Volterra type. Using the regularization method, the regularizability of the original inverse problem in a space with a non-uniform metric is proved.

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