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EXISTENCE OF MILD SOLUTION FOR (k, Ψ)-HILFER FRACTIONAL CAUCHY VALUE PROBLEM OF SOBOLEV TYPE

Haihua Wang*, Jie Zhao, Junhua Ku and Yanqiong Liu

Department of Mathematics
Qiongtai Normal University
Haikou, Hainan 571100, P. R. China
e-mail: wanghoiwan@163.com

Abstract

In the context of solving (k, Ψ)-Hilfer fractional differential equations with Sobolev type, we initially explore a more generalized version of the (α, β, k)-resolvent family. Subsequently, we present various properties associated with this resolvent family. Specific instances of this resolvent family, such as the C_0 semigroup, sine

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*Corresponding author
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family, cosine family and others, have been previously discussed in other academic papers. By combining this resolvent family with the (k, Ψ)-Hilfer fractional derivative, we examine the existence of mild solutions to (k, Ψ)-Hilfer Sobolev type fractional evolution equations, without requiring the existence of the inverse of E. Ultimately, two existence theorems are derived.

1. Introduction

The theory of fractional order differential equations has garnered increasing interest due to its wide applicability in various fields such as physics, engineering, and medicines, see [2-4, 6, 13, 15, 16]. Hilfer [7] proposed the Hilfer fractional derivative, which exhibits technical properties that make it more general than previous fractional derivatives. This makes the Hilfer fractional derivative a powerful mathematical tool for analyzing real world phenomena and driving technical advancements [5]. Another important contribution is the introduction of the Ψ-Hilfer fractional derivative by Sausa and Oliveira [17], which generalizes key fractional derivatives like the Hilfer, Caputo and Riemann-Liouville. The flexibility of the kernel Ψ is beneficial as it allows for unification and extension of previous studies on fractional differential equations, which are essential in addressing various issues. A more general derivative called the (k, Ψ)-Hilfer fractional derivative was proposed by Kucche and Mali [10], encompassing the Ψ-Hilfer derivative. Special cases like the k-Hilfer-Hadamard fractional derivatives, (k, Ψ)-Caputo, and (k, Ψ)-Riemann-Liouville derivatives can be derived by selecting the appropriate kernel Ψ with $\nu \in [0, 1]$ and $k > 0$.

The existence, uniqueness and continuous dependence of solutions have been extensively studied in fractional abstract evolution equations. Specifically, our focus is on the existence of mild solutions. To establish this, various technical tools such as the fixed point theorem and the iterative technique are employed.

In reference [14], Ponce has made significant contributions by delineating the properties of resolvent families, and, more notably, by applying these properties to explore mild solutions for two types of fractional nonlocal problems. Additionally, in the publication denoted [1], Chang et al. delved into Sobolev type fractional differential equations, achieving the approximate controllability through the employment of resolvent operators. However, the broader studies of more general resolvent families of operators have been relatively neglected. In response to this gap, the present study endeavors to introduce more general resolvent families of operators, aiming to engage in a detailed discourse on the norm continuity and compactness of these operators, thereby extending the scope of existing research. The findings of our research can be seen as an extension of the conclusions presented in the works of various scholars.

The manuscript is organized in a logical and coherent manner. Section 2 presents a compendium of essential definitions and preliminary concepts that form the foundational knowledge for the subsequent sections. Section 3 delves into the properties of continuity and compactness within the context of our proposed generalized resolvent family. Section 4 is dedicated to an in-depth exploration of the existence of mild solution for (k, Ψ)-Hilfer Sobolev type fractional Cauchy problems. Notably, we have formulated two theorems, each derived through distinct methodologies, thereby enriching the discourse with a multifaceted perspective. Section 5 provides an example.

2. Preliminaries

Let J be a finite closed interval of $\mathbb{R}$, and X be a Banach space endowed with the norm $\|\cdot\|$. Let $C(J, X)$ be the Banach space of all continuous functions from J into E with the usual norm $\|x\|_C = \max_{t\in J}\|x(t)\|$, where $x \in C(J, X)$. To avoid confusion, the norm in $C(J, \mathbb{R}^+)$ is defined by $\|x\|_\infty = \max_{t\in J}|x(t)|$, for $x \in C(J, \mathbb{R}^+)$. Moreover, we denote by $\mathfrak{B}(X, Y)$ the space of all bounded linear operators from Banach space X to Banach space Y, and we abbreviate this notation to $\mathfrak{B}(X)$ when $X = Y$.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing odd function with continuous derivative and $\Psi'(t) \neq 0$.

Definition 2.1. The operator family $\{S(t)\}_{t \geq 0} \subset \mathfrak{B}(X)$ is called *general exponentially bounded* (GEB) if there are $M > 0$, $\omega \in \mathbb{R}$ satisfying

$$\| S(t) \| \leq M e^{\omega \Psi(t)}, \quad t \geq 0.$$

We say that a type of $S(t)$ is (M, ω). Set

$$\omega_0(S) = \inf_{\omega \in \mathbb{R}} \{\exists M \geq 0 \text{ such as } \| S(t) \| \leq M e^{\omega \Psi(t)}, t \geq 0\}.$$

Definition 2.2 [8]**.** Let *f* and *h* be two functions. Then the *generalized convolution* of *f* and *h* is defined by

$$(f *_{\Psi} h)(t) = \int_0^t f(r) h(\Psi^{-1}(\Psi(t) - \Psi(r))) \Psi'(r) dr.$$

Definition 2.3. Let *f* and *h* be two functions. Then *another generalized convolution* of *f* and *h* is defined by

$$(f *^{\Psi} h)(t) = \int_{-\infty}^{\infty} f(r) h(\Psi^{-1}(\Psi(t) - \Psi(r))) \Psi'(r) dr.$$

Let $k, \eta > 0$ and $g_{k, \eta}$ denote the function

$$g_{k, \eta}(t) = \begin{cases} \dfrac{\Psi^{\frac{\eta}{k} - 1}(t)}{k \Gamma_k(\eta)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma_k(z)$ denotes the *k*-gamma function

$$\Gamma_k(z) = \int_0^{\infty} s^{z-1} e^{-\frac{s^k}{k}} ds, \quad \mathrm{Re}(z) > 0.$$

Definition 2.4 [11]**.** Let $\eta, k > 0$ and *h* be integrable on $[a, b]$. Then

$$ {}^{k}I_{a+}^{\eta;\Psi}h(t) = \frac{1}{k\Gamma_k(\eta)}\int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{\frac{\eta}{k}-1}h(s)ds $$

is called the (k, Ψ)-*Riemann-Liouville fractional integral* of a function h of order η.

Definition 2.5 [10]**.** Let $k > 0$, $m \in \mathbb{Z}^+$ with $m - 1 < \frac{\eta}{k} \leq m$, and the function h be n times continuously differentiable on $[a, b]$. Then

$$ {}^{k,H}D_{a+}^{\eta,v;\Psi}h(t) = {}^{k}I_{a+}^{v(mk-\eta);\Psi}\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^m {}^{k}I_{a+}^{(1-v)(mk-\eta);\Psi}h(t), \quad v \in [0, 1] $$

is called the (k, Ψ)-*Hilfer fractional derivative* of h with order η and type v.

Definition 2.6 [8]**.** Let $h : [0, \infty) \to \mathbb{R}$. We define the *more general Laplace transform* of h as follows:

$$ \mathcal{L}_{\Psi}(h(t))(\lambda) = \int_0^{\infty} e^{-\lambda\Psi(t)}h(t)\Psi'(t)dt. $$

Definition 2.7. Let $h : \mathbb{R} \to \mathbb{R}$. We define the *more general Fourier transform* of h as follows:

$$ \mathcal{F}_{\Psi}(h(t))(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda\Psi(t)}h(t)\Psi'(t)dt. $$

If h is a function with values in E, then integrals which appear in above definitions mean Bochner integrals.

Theorem 2.1. *Let* $\eta, k > 0$ *with* $m - 1 < \frac{\eta}{k} \leq m$ *and h be a piecewise continuous function. Then*

$$ \mathcal{L}_{\Psi}({}^{k,H}D_{0+}^{\eta,v;\Psi}h(t))(\lambda) $$

$$ = (k\lambda)^{\frac{\eta}{k}}\mathcal{L}_{\Psi}(h(t))(\lambda) $$

$$- (k\lambda)^{-\frac{v(mk-\eta)}{k}} \sum_{i=0}^{m-1} k^{m-i} \lambda^{m-i-1} \left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{i} {}_k I_{0+}^{(1-v)(mk-\eta); \Psi} h(t) \right]_{t=0}.$$

Proof. We first show that

$$\mathcal{L}_{\Psi}\left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{m} h(t) \right](\lambda) = k^m \lambda^m \mathcal{L}_{\Psi}(h(t))(\lambda)$$

$$- \sum_{i=0}^{m-1} k^{m-i} \lambda^{m-i-1} \left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{i} h(t) \right]_{t=0} \tag{2.1}$$

is true, where $m \in \mathbb{Z}^{+}$.

In fact, for $m = 1$ using Definition 2.6 and integration by parts, we have

$$\mathcal{L}_{\Psi}\left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right) h(t) \right](\lambda) = \int_0^{\infty} e^{-\lambda \Psi(t)} k \frac{d}{dt} h(t) dt$$

$$= -kh(0) + k\lambda \mathcal{L}_{\Psi}(h(t))(\lambda).$$

Furthermore,

$$\mathcal{L}_{\Psi}\left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{m} h(t) \right](\lambda) = -k \left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{m-1} h(t) \right]_{t=0}$$

$$+ k\lambda \mathcal{L}_{\Psi}\left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right)^{m-1} h(t) \right](\lambda), \tag{2.2}$$

holds by mathematical induction, it is easy to verify that (2.1) is true.

Using Definition 2.5,

$$\mathcal{L}_{\Psi}({}^{k,H} D_{0+}^{\eta, v; \Psi} h(t))(\lambda)$$

$$= \mathcal{L}_{\Psi}({}_k I_{0+}^{v(mk-\eta); \Psi} \, {}^{k,RL} D_{0+}^{\eta+v(mk-\eta); \Psi} h(t))(\lambda)$$

$$= (k\lambda)^{-\frac{v(mk-\eta)}{k}} \mathcal{L}_{\Psi}({}^{k,RL} D_{0+}^{\eta+v(mk-\eta); \Psi} h(t))(\lambda)$$

$$= (k\lambda)^{-\frac{v(mk-\eta)}{k}} \mathcal{L}_{\Psi}\left[\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{m} {}^{k}I_{0+}^{(1-v)(mk-\eta);\Psi} h(t)\right](\lambda).$$

In view of (2.1), we conclude that

$$\mathcal{L}_{\Psi}({}^{k,H}D_{0+}^{\eta,v;\Psi} h(t))(\lambda)$$

$$= (k\lambda)^{-\frac{v(mk-\eta)}{k}} \Bigg\{ (k\lambda)^{m} \mathcal{L}_{\Psi}({}^{k}I_{0+}^{(1-v)(mk-\eta);\Psi} h)(\lambda)$$

$$- \sum_{i=0}^{m-1} k^{m-i}\lambda^{m-i-1}\left[\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{i} {}^{k}I_{0+}^{(1-v)(mk-\eta);\Psi} h(t)\right]_{t=0}\Bigg\}$$

$$= (k\lambda)^{\frac{\eta}{k}} \mathcal{L}_{\Psi}(h(t))(\lambda)$$

$$- (k\lambda)^{-\frac{v(mk-\eta)}{k}} \sum_{i=0}^{m-1} k^{m-i}\lambda^{m-i-1}\left[\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^{i} {}^{k}I_{0+}^{(1-v)(mk-\eta);\Psi} h(t)\right]_{t=0}.$$

This completes the proof. □

For $n > 0$, denoting $D_n(t) = \frac{\sin \Psi(n)\Psi(t)}{\pi\Psi(t)}$, $t \in \mathbb{R}$, we have

$$(D_n *^{\Psi} h)(t) = \int_{-\infty}^{\infty} h(r)\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} e^{is(\Psi(t)-\Psi(r))}\Psi'(r)\,ds\,dr$$

$$= \frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} e^{is\Psi(t)}\left(\int_{-\infty}^{\infty} e^{-is\Psi(r)} h(r)\Psi'(r)\,dr\right)ds$$

$$= \frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} e^{is\Psi(t)}\mathcal{F}_{\Psi}(h)(s)\,ds.$$

For $\omega \in \mathbb{R}$ and a function $S : [0, \infty) \to \mathbb{R}$, we define the shift S_ω by $S_\omega(t) = e^{-\omega\Psi(t)}S(t)$, $t \geq 0$. Let $S : [0, \infty) \to \mathfrak{B}(X, Y)$ be a strongly continuous operator. Then

$$\mathcal{L}_\Psi(S)(\omega + is) = \int_0^\infty e^{-(\omega+is)\Psi(t)}\Psi'(t)S(t)dt$$

$$= \int_0^\infty e^{-is\Psi(t)}\Psi'(t)S_\omega(t)dt$$

$$= \mathcal{F}_\Psi(S_\omega)(s),$$

which leads to

$$K_n(t) = \frac{1}{2\pi i}\int_{\omega-i\Psi(n)}^{\omega+i\Psi(n)} e^{\Psi(t)z}\mathcal{L}_\Psi(S)(z)dz$$

$$= \frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} e^{\Psi(t)\omega}e^{i\Psi(t)s}\mathcal{L}_\Psi(S)(\omega + is)ds$$

$$= e^{\Psi(t)\omega}\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} e^{i\Psi(t)s}\mathcal{L}_\Psi(S)(\omega + is)ds$$

$$= e^{\Psi(t)\omega}\frac{1}{2\pi}\int_{-\Psi(n)}^{\Psi(n)} e^{is\Psi(t)}\mathcal{F}_\Psi(S_\omega)(s)ds$$

$$= e^{\Psi(t)\omega}(D_n *^\Psi S_\omega)(t),$$

namely $(K_n)_\omega = D_n *^\Psi S_\omega$.

Theorem 2.2. *If* $S : [0, \infty) \to \mathfrak{B}(X, Y)$ *is strongly continuous*, $b \in L^1_{loc}([0, \infty), \mathbb{R})$, *functions b and S are finite GBE, then*

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\omega-i\Psi(n)}^{\omega+i\Psi(n)} e^{\Psi(t)z}\mathcal{L}_\Psi\left(\frac{b}{\sqrt{\Psi'}} *^\Psi S\right)(z)dz = b *^\Psi S, \ \omega > \omega_0(S), \omega(b),$$

in $\mathfrak{B}(X, Y)$.

Proof. Replacing the operator S by $b *^\Psi S$, we conclude that $e^{-\omega\Psi(t)}(K_n)(t) = D_n *^\Psi (b *^\Psi S)_\omega$, where

$$(b *^{\Psi} S)_{\omega} = e^{-\omega\Psi(t)} \int_{-\infty}^{\infty} b(r) S(\Psi^{-1}(\Psi(t) - \Psi(r)))\Psi'(r) dr$$

$$= \int_{-\infty}^{\infty} e^{-\omega\Psi(r)} b(r) e^{-\omega(\Psi(t) - \Psi(r))}$$

$$\times S(\Psi^{-1}(\Psi(t) - \Psi(r)))\Psi'(r) dr$$

$$= b_{\omega} *^{\Psi} S_{\omega}.$$

Hence, $e^{-\omega\Psi(t)}(K_n)(t) = D_n *^{\Psi} (b_{\omega} *^{\Psi} S_{\omega}) = D_n *^{\Psi} b_{\omega} *^{\Psi} S_{\omega}$. Using the Plancherel's theorem, $\sqrt{\Psi'}\left(D_n *^{\Psi} \frac{h}{\sqrt{\Psi'}}\right) \to h\sqrt{\Psi'}$ in $L^2(\mathbb{R})$ as $n \to \infty$ for $h\sqrt{\Psi'} \in L^2(\mathbb{R})$, we can get

$$\left(D_n *^{\Psi} \frac{b_{\omega}}{\sqrt{\Psi'}} *^{\Psi} S_{\omega}\right) \to b_{\omega} *^{\Psi} S_{\omega}$$

in $\mathfrak{B}(X, Y)$ uniformly in $t \geq 0$ by using the Young's inequality. The proof is completed. □

Definition 2.8. Let $\alpha, \beta, k > 0$, X be a Banach space, $E : D(E) \subset X \to X$, $A : D(A) \subset X \to X$ be two closed linear operators with $D(E) \cap D(A) \neq \{0\}$, and there be a $\omega \geq 0$ and a strongly continuous GEB operator $S^{E}_{\alpha, \beta, k}(t) : [0, \infty) \to \mathfrak{B}(X)$, and $\{(k\lambda)^{\frac{\alpha}{k}} : ((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1}$ exists in $\mathfrak{B}(X, D(E) \cap D(A))$, $\operatorname{Re} k\lambda > \omega\}$,

$$(k\lambda)^{\frac{\alpha-\beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} x = \int_0^{\infty} e^{-\lambda\Psi(t)} \Psi'(t) S^{E}_{\alpha, \beta, k}(t) x dt, \; x \in X. \quad (2.3)$$

Then we say that (A, E) generates the (α, β, k)-resolvent family $\{S^{E}_{\alpha, \beta, k}(t)\}_{t \geq 0}$.

Remark 2.1. For $k = 1$, $\alpha = 1$, $\beta = 1$ and $E = I$, $S_{\alpha, \beta, k}(t)$ reduces to C_0-semigroup. For $k = 1$, $\alpha = 2$, $\beta = 2$ and $E = I$, $S_{\alpha, \beta, k}(t)$ reduces to sine family. For $k = 1$, $\alpha = 2$, $\beta = 1$ and $E = I$, $S_{\alpha, \beta, k}(t)$ reduces to cosine family.

Theorem 2.3. *For* $\alpha, \beta, k > 0$, $S^E_{\alpha, \beta, k}(t)$ *satisfies the functional equation*

$$S^E_{\alpha,\beta,k}(s)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t) - (g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s) S^E_{\alpha,\beta,k}(t)$$

$$= g_{k,\beta}(s)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t) - g_{k,\beta}(t)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s),$$

$$t, s \geq 0. \quad (2.4)$$

Proof. For $\lambda, \mu > \omega$ and $\lambda \neq \mu$, taking the Laplace transform on both sides in (2.4), we have

$$\int_0^\infty \int_0^\infty e^{-\lambda\Psi(t) - \mu\Psi(s)} \Psi'(t)\Psi'(s)$$

$$\times [S^E_{\alpha,\beta,k}(s)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t) - (g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s) S^E_{\alpha,\beta,k}(t)] ds dt$$

$$= \int_0^\infty e^{-\lambda\Psi(t)}\Psi'(t)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t) dt \int_0^\infty e^{-\mu\Psi(s)}\Psi'(s) S^E_{\alpha,\beta,k}(s) ds$$

$$- \int_a^\infty e^{-\lambda\Psi(t)}\Psi'(t) S^E_{\alpha,\beta,k}(t) dt \int_0^\infty e^{-\mu\Psi(s)}\Psi'(s)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s) ds$$

$$= k^{-\frac{\alpha}{k}} \lambda^{-\frac{\alpha}{k}} (k\lambda)^{\frac{\alpha-\beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} (k\mu)^{\frac{\alpha-\beta}{k}} E((k\mu)^{\frac{\alpha}{k}} E - A)^{-1}$$

$$- (k\lambda)^{\frac{\alpha-\beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} k^{-\frac{\alpha}{k}} \mu^{-\frac{\alpha}{k}} (k\mu)^{\frac{\alpha-\beta}{k}} E((k\mu)^{\frac{\alpha}{k}} E - A)^{-1}$$

$$= k^{-\frac{2\beta}{k}} \lambda^{-\frac{\beta}{k}} \mu^{-\frac{\beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} E((k\mu)^{\frac{\alpha}{k}} E - A)^{-1} ((k\mu)^{\frac{\alpha}{k}} - (k\lambda)^{\frac{\alpha}{k}})$$

$$= k^{-\frac{2\beta}{k}}\lambda^{-\frac{\beta}{k}}\mu^{-\frac{\beta}{k}}E((k\lambda)^{\frac{\alpha}{k}}E - A)^{-1}((k\mu)^{\frac{\alpha}{k}}E - A)^{-1}((k\mu)^{\frac{\alpha}{k}}E - A)$$

$$- k^{-\frac{2\beta}{k}}\lambda^{-\frac{\beta}{k}}\mu^{-\frac{\beta}{k}}((k\lambda)^{\frac{\alpha}{k}}E - A)^{-1}E((k\mu)^{\frac{\alpha}{k}}E - A)^{-1}((k\lambda)^{\frac{\alpha}{k}}E - A)$$

$$= k^{-\frac{2\beta}{k}}\lambda^{-\frac{\beta}{k}}\mu^{-\frac{\beta}{k}}[E((k\lambda)^{\frac{\alpha}{k}}E - A)^{-1} - E((k\mu)^{\frac{\alpha}{k}}E - A)^{-1}],$$

on the other side,

$$\int_0^\infty\int_0^\infty e^{-\lambda\Psi(t)-\mu\Psi(s)}\Psi'(t)\Psi'(s)$$

$$\times [g_{k,\beta}(s)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t) - g_{k,\beta}(t)(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s)]dtds$$

$$= k^{-\frac{\beta}{k}}\mu^{-\frac{\beta}{k}}k^{-\frac{\alpha}{k}}\lambda^{-\frac{\alpha}{k}}(k\lambda)^{\frac{\alpha-\beta}{k}}E((k\lambda)^{\frac{\alpha}{k}}E - A)^{-1}$$

$$- k^{-\frac{\beta}{k}}\lambda^{-\frac{\beta}{k}}k^{-\frac{\alpha}{k}}\mu^{-\frac{\alpha}{k}}(k\mu)^{\frac{\alpha-\beta}{k}}E((k\mu)^{\frac{\alpha}{k}}E - A)^{-1}.$$

From the uniqueness, we get that (2.4) is true. □

If (A, E) generates the (α, β, k)-resolvent family $S^E_{\alpha,\beta,k}(t)$, then define

$$Ax = \lim_{t\to 0+}\frac{S^E_{\alpha,\beta,k}(t)x - g_{k,\beta}(t)x}{g_{k,\alpha+\beta}(t)}, \quad x \in D(A).$$

Theorem 2.4. *Let* $\alpha, \beta, k > 0$ *and* (A, E) *generate* $\{S^E_{\alpha,\beta,k}(t)\}_{t\geq 0}$. *Then* $S^E_{\alpha,\beta,k}(t)$ *has the following properties*:

(i) $S^E_{\alpha,\beta,k}(t) : D(A) \to D(A)$, *besides*, $S^E_{\alpha,\beta,k}(t)Ax = AS^E_{\alpha,\beta,k}(t)x$;

(ii) *If* $\beta \leq k$, *then* $(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t) : X \to D(A)$, *and for* $x \in X$,

$$S_{\alpha,\beta,k}^{E}(t)x = g_{k,\beta}(t)x + A(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(t)x. \tag{2.5}$$

If $\beta > k$ *and* $x \in D(A)$, *then* (2.5) *holds. Moreover,*

$$S_{\alpha,\beta,k}^{E}(t)x = g_{k,\beta}(t)x + (g_{k,\alpha} *_{\Psi} AS_{\alpha,\beta,k}^{E})(t)x, \quad x \in D(A).$$

Proof. (i) Assume that $x \in D(A)$ and $t \geq 0$. One can see that

$$\lim_{s\to 0+} \frac{S_{\alpha,\beta,k}^{E}(s)S_{\alpha,\beta,k}^{E}(t)x - g_{k,\beta}(s)S_{\alpha,\beta,k}^{E}(t)x}{g_{k,\alpha+\beta}(s)}$$

$$= \lim_{s\to 0+} \frac{S_{\alpha,\beta,k}^{E}(t)[S_{\alpha,\beta,k}^{E}(s)x - g_{k,\beta}(s)x]}{g_{k,\alpha+\beta}(s)}$$

$$= S_{\alpha,\beta,k}^{E}(t)Ax,$$

which implies that $S_{\alpha,\beta,k}^{E}(t) : D(A) \to D(A)$ and

$$S_{\alpha,\beta,k}^{E}(t)Ax = AS_{\alpha,\beta,k}^{E}(t)x.$$

(ii) By Theorem 2.3, we obtain

$$\lim_{s\to 0+} \frac{S_{\alpha,\beta,k}^{E}(s)(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(t)x - g_{k,\beta}(s)(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(t)x}{g_{k,\alpha+\beta}(s)}$$

$$= \lim_{s\to 0+} \frac{(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(s)S_{\alpha,\beta,k}^{E}(t)x - g_{k,\beta}(t)(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(s)x}{g_{k,\alpha+\beta}(s)}$$

$$= \lim_{s\to 0+} \frac{(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(s)[S_{\alpha,\beta,k}^{E}(t)x - g_{k,\beta}(t)x]}{g_{k,\alpha+\beta}(s)}. \tag{2.6}$$

Using that facts that

$$(g_{k,\alpha} *_{\Psi} S_{\alpha,\beta,k}^{E})(s) = \frac{1}{k\Gamma_k(\alpha)}\int_0^s (\Psi(s) - \Psi(r))^{\frac{\alpha}{k}-1}\Psi'(r)S_{\alpha,\beta,k}^{E}(r)dr$$

and

$$g_{k,\alpha+\beta}(s) = \frac{1}{k\Gamma_k(\alpha+\beta)}\Psi^{\frac{\alpha+\beta-k}{k}}(s)$$

$$= {}^kI_{0+}^{\alpha;\Psi}\frac{1}{k\Gamma_k(\beta)}\Psi^{\frac{\beta-k}{k}}(s)$$

$$= \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^s(\Psi(s)-\Psi(r))^{\frac{\alpha}{k}-1}\Psi^{\frac{\beta}{k}-1}(r)\Psi'(r)dr$$

$$= \frac{1}{k\Gamma_k(\alpha)}\int_0^s(\Psi(s)-\Psi(r))^{\frac{\alpha}{k}-1}\Psi'(r)g_{k,\beta}(r)dr,$$

we have

$$\left\|\frac{(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s)u}{g_{k,\alpha+\beta}(s)} - u\right\|$$

$$= \frac{1}{k\Gamma_k(\alpha)g_{k,\alpha+\beta}(s)}\left\|\int_0^s(\Psi(s)-\Psi(r))^{\frac{\alpha}{k}-1}\Psi'(r)[S^E_{\alpha,\beta,k}(r)u - g_{k,\beta}(r)u]dr\right\|.$$

If $\beta \le k$, then

$$\left\|\frac{(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s)u}{g_{k,\alpha+\beta}(s)} - u\right\|$$

$$\le \frac{\sup_{r\in[0,s]}\|S^E_{\alpha,\beta,k}(r)u - g_{k,\beta}(r)u\|}{k\Gamma_k(\alpha)g_{k,\alpha+\beta}(s)}\int_0^s(\Psi(s)-\Psi(r))^{\frac{\alpha}{k}-1}\Psi'(r)dr$$

$$= \frac{k\Gamma_k(\alpha+\beta)}{\alpha\Gamma_k(\alpha)}\Psi^{1-\frac{\beta}{k}}(s)\sup_{r\in[0,s]}\|S^E_{\alpha,\beta,k}(r)u - g_{k,\beta}(r)u\| \to 0, \quad s \to 0+. \tag{2.7}$$

From (2.6) and (2.7), we have

$$(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t)x \in D(A), \quad t \ge 0$$

and

$$S^E_{\alpha,\beta,k}(t)x = g_{k,\beta}(t)x + A(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t)x \text{ for } x \in X.$$

If $\beta > k$, then for $u \in D(A)$,

$$\left\| \frac{(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(s)u}{g_{k,\alpha+\beta}(s)} - u \right\|$$

$$\leq \frac{\sup_{r\in[0,s]} \| S^E_{\alpha,\beta,k}(r)u - g_{k,\beta}(r)u \|}{k\Gamma_k(\alpha) g_{k,\alpha+\beta}(s)} \int_0^s (\Psi(s) - \Psi(r))^{\frac{\alpha}{k}-1} \Psi'(r) dr$$

$$\leq \sup_{r\in[0,s]} \left\| \frac{S^E_{\alpha,\beta,k}(r)u - g_{k,\beta}(r)u}{g_{k,\alpha+\beta}(r)} \right\| \frac{1}{k\Gamma_k(\alpha)}$$

$$\times \int_0^s (\Psi(s) - \Psi(r))^{\frac{\alpha}{k}-1} \Psi'(r) dr \to 0, \ s \to 0+. \tag{2.8}$$

From (i), we know that $(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t)x \in D(A)$, $t \geq 0$,

$$S^E_{\alpha,\beta,k}(t)x = g_{k,\beta}(t)x + A(g_{k,\alpha} *_\Psi S^E_{\alpha,\beta,k})(t)x$$

for $x \in D(A)$ can be given by (2.6) and (2.8). The rest is obvious that follows from (i). □

3. Properties of $S^E_{\alpha,\beta,k}(t)$

Theorem 3.1. *Let* $\gamma > 0$. *If* (A, E) *generates* $\{S^E_{\alpha,\beta,k}(t)\}_{t\geq 0}$, *whose type is* (M, ω). *Then* (A, E) *generates* $\{S^E_{\alpha,\beta+\gamma,k}(t)\}_{t\geq 0}$, *whose type is*

$$(k^{-\frac{\gamma}{k}} \omega^{-\frac{\gamma}{k}} M, \omega).$$

Proof. For $t \geq 0$, using the variable $u = \Psi^{-1}(\Psi(t) - \Psi(r))$, it follows that

$$\|(g_{k,\gamma} *_{\Psi} S^E_{\alpha,\beta,k})(t)\| \le \frac{M}{k\Gamma_k(\gamma)}\int_0^t (\Psi(t)-\Psi(r))^{\frac{\gamma}{k}-1}\Psi'(r)e^{\omega\Psi(r)}dr$$

$$= \frac{Me^{\omega\Psi(t)}}{k\Gamma_k(\gamma)}\int_0^t \Psi^{\frac{\gamma}{k}-1}(u)\Psi'(u)e^{-\omega\Psi(u)}du$$

$$\le Me^{\omega\Psi(t)}\mathcal{L}_{\Psi}(g_{k,\gamma}(t))(\omega) = k^{-\frac{\gamma}{k}}\omega^{-\frac{\gamma}{k}}Me^{\omega\Psi(t)},$$

which implies that the Laplace transform of $(g_{k,\gamma} *_{\Psi} S_{\alpha,\beta,k})(t)$ exists.

For $t \ge 0$ and $\operatorname{Re}\lambda > \omega$, we have

$$\mathcal{L}_{\Psi}(g_{k,\gamma} *_{\Psi} S^E_{\alpha,\beta,k})(\lambda) = k^{-\frac{\gamma}{k}}\lambda^{-\frac{\gamma}{k}}(k\lambda)^{\frac{\alpha-\beta}{k}}E((k\lambda)^{\frac{\alpha}{k}}E - A)^{-1}$$

$$= (k\lambda)^{\frac{\alpha-\beta-\gamma}{k}}E((k\lambda)^{\frac{\alpha}{k}}E - A)^{-1}$$

$$= \mathcal{L}_{\Psi}(S^E_{\alpha,\beta+\gamma,k})(\lambda).$$

Therefore, we see that (A, E) generates the resolvent family $(\alpha, \beta+\gamma, k)$, whose type is $(k^{-\frac{\gamma}{k}}\omega^{-\frac{\gamma}{k}}M, \omega)$. □

Theorem 3.2. *Let* $\alpha > 0$, $\beta > \beta_1 > 0$ *and* $k > 0$. *Assume that* (A, E) *generates* $\{S^E_{\alpha,\beta_1,k}(t)\}_{t\ge 0}$ *whose type is* (M, ω). *If* $t > 0$, *then* $S^E_{\alpha,\beta,k}(t)$ *is norm continuous in* $\mathfrak{B}(X)$.

Proof. From Theorem 3.1, we conclude that (A, E) generates $\{S^E_{\alpha,\beta,k}(t)\}_{t\ge 0}$, whose type is

$$(k^{-\frac{\beta-\beta_1}{k}}\omega^{-\frac{\beta-\beta_1}{k}}M, \omega).$$

There are two cases need to be discussed.

Case 1. $\beta \neq k + \beta_1$

Let $0 \leq t_1 < t_2$, from $S^E_{\alpha,\beta,k}(t) = (g_{k,\beta-\beta_1} *_\Psi S^E_{\alpha,\beta_1,k})(t)$, $t \geq 0$. It follows that

$$\begin{aligned}
& S^E_{\alpha,\beta,k}(t_2) - S^E_{\alpha,\beta,k}(t_1) \\
&= (g_{k,\beta-\beta_1} *_\Psi S^E_{\alpha,\beta_1,k})(t_2) - (g_{k,\beta-\beta_1} *_\Psi S^E_{\alpha,\beta_1,k})(t_1) \\
&= \frac{1}{k\Gamma_k(\beta-\beta_1)} \int_{t_1}^{t_2} (\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} \Psi'(r) S^E_{\alpha,\beta_1,k}(r) dr \\
&\quad + \frac{1}{k\Gamma_k(\beta-\beta_1)} \int_0^{t_1} [(\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} \\
&\qquad\qquad - (\Psi(t_1) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1}] \Psi'(r) S^E_{\alpha,\beta_1,k}(r) dr \\
&= I_1 + I_2. \qquad (3.1)
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
\| I_1 \| &\leq \frac{1}{k\Gamma_k(\beta-\beta_1)} \int_{t_1}^{t_2} (\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} \Psi'(r) \| S^E_{\alpha,\beta_1,k}(r) \| dr \\
&\leq \frac{M}{k\Gamma_k(\beta-\beta_1)} \int_{t_1}^{t_2} (\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} \Psi'(r) e^{\omega\Psi(r)} dr \\
&\leq \frac{Me^{\omega\Psi(t_2)}}{k\Gamma_k(\beta-\beta_1)} \int_{t_1}^{t_2} (\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} \Psi'(r) dr \\
&= \frac{Me^{\omega\Psi(t_2)}}{\Gamma_k(\beta+k-\beta_1)} (\Psi(t_2) - \Psi(t_1))^{\frac{\beta-\beta_1}{k}}. \qquad (3.2)
\end{aligned}$$

From the continuous of $\Psi(t)$, we can see that $\lim_{t_1 \to t_2} \| I_1 \| = 0$.

For I_2, we have

$$\| I_2 \| \leq \frac{Me^{\omega\Psi(t_1)}}{k\Gamma_k(\beta - \beta_1)}$$

$$\times \int_0^{t_1} |(\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} - (\Psi(t_1) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1}| \Psi'(r)dr. \quad (3.3)$$

Subcase 1. $\beta_1 < \beta < k + \beta_1$

We can see that

$$\| I_2 \| \leq \frac{Me^{\omega\Psi(t_1)}}{k\Gamma_k(\beta - \beta_1)} \int_0^{t_1} [(\Psi(t_1) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} - (\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1}]\Psi'(r)dr$$

$$= \frac{Me^{\omega\Psi(t_1)}}{\Gamma_k(\beta - \beta_1 + k)} [\Psi^{\frac{\beta-\beta_1}{k}}(t_1) - \Psi^{\frac{\beta-\beta_1}{k}}(t_2) + (\Psi(t_2) - \Psi(t_1))^{\frac{\beta-\beta_1}{k}}]. \quad (3.4)$$

Subcase 2. $\beta > k + \beta_1$

Similarly, we can get

$$\| I_2 \| \leq \frac{Me^{\omega\Psi(t_1)}}{k\Gamma_k(\beta - \beta_1)} \int_0^{t_1} [(\Psi(t_2) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1} - (\Psi(t_1) - \Psi(r))^{\frac{\beta-\beta_1}{k}-1}]\Psi'(r)dr$$

$$= \frac{Me^{\omega\Psi(t_1)}}{\Gamma_k(\beta - \beta_1 + k)} [\Psi^{\frac{\beta-\beta_1}{k}}(t_2) - (\Psi(t_2) - \Psi(t_1))^{\frac{\beta-\beta_1}{k}} - \Psi^{\frac{\beta-\beta_1}{k}}(t_1)]. \quad (3.5)$$

$\| I_2 \| \to 0$ as $t_2 \to t_1$ holds in all subcases.

Case 2. $\beta = k + \beta_1$

For $0 \leq t_1 < t_2$, from

$$S^E_{\alpha, k+\beta_1, k}(t) = (g_{k,k} *_\Psi S^E_{\alpha, \beta_1, k})(t) = \frac{1}{\Gamma_k(2k)} \int_0^t \Psi'(r) S^E_{\alpha, \beta_1, k}(r)dr,$$

we conclude that

$$\| S^E_{\alpha, k+\beta_1, k}(t_2) - S^E_{\alpha, k+\beta_1, k}(t_1) \|$$

$$\leq \frac{1}{\Gamma_k(2k)} \int_{t_1}^{t_2} \Psi'(r) \| S^E_{\alpha, \beta_1, k}(r) \| dr$$

$$\leq \frac{M e^{\omega \Psi(t_2)}}{\Gamma_k(2k)} (\Psi(t_2) - \Psi(t_1)) \to 0, \text{ as } t_2 \to t_1.$$

This completes the proof. □

Definition 3.1. $\{S^E_{\alpha, \beta, k}(t)\}_{t \geq 0}$ is said to be *compact* if $S^E_{\alpha, \beta, k}(t)$ is a compact for $t > 0$.

Theorem 3.3. *Let* $\alpha > 0$, $\beta > \beta_1 > 0$ *and* $k > 0$. *Assume that* (A, E) *generates* $\{S^E_{\alpha, \beta_1, k}(t)\}_{t \geq 0}$, *whose type is* (M, ω). *Then the following are equivalent*:

(i) $S^E_{\alpha, \beta, k}(t)$ *is compact for* $t > 0$.

(ii) $E(k\mu E - A)^{-1}$ *is compact for* $k\mu > \omega^{k/\alpha}$.

Proof. Assume that $S^E_{\alpha, \beta, k}(t)$ is compact. Let $k\lambda > \omega$. Then

$$(k\lambda)^{\frac{\alpha - \beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} = \int_0^\infty e^{-\lambda \Psi(t)} \Psi'(t) S^E_{\alpha, \beta, k}(t) dt \tag{3.6}$$

from Definition 2.8. Noting that the uniform continuity of $\{S^E_{\alpha, \beta, k}(t)\}_{t \geq 0}$ by Theorem 3.2, implies that $E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1}$ is compact from Corollary 2.3 in [18].

Conversely, for $g_{k, \beta - \beta_1} \in L^1_{loc}[0, \infty)$, using Theorem 2.2,

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\omega - i\Psi(n)}^{\omega + i\Psi(n)} e^{\Psi(t) z} \mathcal{L}_\Psi \left(\frac{g_{k, \beta - \beta_1}}{\sqrt{\Psi'}} *^\Psi S^E_{\alpha, \beta_1, k} \right)(z) dz = g_{k, \beta - \beta_1} *^\Psi S^E_{\alpha, \beta_1, k}.$$

However, for $g_{k,\beta-\beta_1} *^{\Psi} S^E_{\alpha,\beta_1,k} = g_{k,\beta-\beta_1} *_\Psi S^E_{\alpha,\beta_1,k}$, we obtain that

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\omega-i\Psi(n)}^{\omega+i\Psi(n)} e^{\Psi(t)z}\mathcal{L}_\Psi\left(\frac{g_{k,\beta-\beta_1}}{\sqrt{\Psi'}} *_\Psi S^E_{\alpha,\beta_1,k}\right)(z)dz = g_{k,\beta-\beta_1} *_\Psi S^E_{\alpha,\beta_1,k},$$

namely

$$\frac{1}{2\pi i}\int_L e^{\Psi(t)z}\mathcal{L}_\Psi\left(\frac{g_{k,\beta-\beta_1}}{\sqrt{\Psi'}}\right)(\lambda)(k\lambda)^{\frac{\alpha-\beta_1}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} dz$$

$$= S^E_{\alpha,\beta,k},\ t > 0, \tag{3.7}$$

where L is the line $\operatorname{Re}(z) = \omega$. $S^E_{\alpha,\beta,k}(t)$ is compact for $\alpha > 0$, $\beta > \beta_1 > 0$, $k > 0$ can be found by using Corollary 2.3 in [18] again. □

4. Mild Solution for (k, Ψ)-Hilfer Fractional Initial Value Problem

We study the following (k, Ψ)-Hilfer fractional derivative:

$$\begin{cases} {}^{k,H}D^{\alpha,v;\Psi}_{0+}(Ex)(t) = Ax(t) + f(t, x(t)), \quad t \in [0, T], \\ ({}^kI^{2k-\alpha_k;\Psi}_{0+}Ex(t))_{t=0} = Ex_0, \\ \left[\left(\frac{k}{\Psi'(t)}\frac{d}{dt}\right)^k I^{2k-\alpha_k;\Psi}_{0+}Ex(t)\right]_{t=0} = Ex_1 \end{cases} \tag{4.1}$$

with initial value problem, where $x_0, x_1 \in X$, $1 < \frac{\alpha}{k} < 2$, $0 \le v \le 1$, $\alpha_k = \alpha + v(2k - \alpha)$, A and E are closed linear operators on X generating the resolvent family $(\alpha, \alpha_k - k, k)$.

Taking the generalized Laplace transform on both sides of the equation, and in view of Theorem 2.1, we can get

$$x(t) = \int_0^t S^E_{\alpha,\alpha,k}(\Psi^{-1}(\Psi(t) - \Psi(s)))\Psi'(s)f(s, x(s))ds$$

$$+ kS^E_{\alpha, \alpha_k - k, k}(t)x_0 + kS^E_{\alpha, \alpha_k, k}(t)x_1.$$

Definition 4.1. Let $1 < \frac{\alpha}{k} < 2,\ 0 \le \nu \le 1,$ and (A, E) generate

$$\{S^E_{\alpha, \alpha_k - k, k}(t)\}_{t \ge 0}.$$

Then $x(t)$ is called a *mild solution* of (4.1) if

$$x(t) = \int_0^t S^E_{\alpha, \alpha, k}(\Psi^{-1}(\Psi(t) - \Psi(s)))\Psi'(s) f(s, x(s)) ds$$

$$+ kS^E_{\alpha, \alpha_k - k, k}(t)x_0 + kS^E_{\alpha, \alpha_k, k}(t)x_1.$$

Lemma 4.1. *Assume that* $a, y : [0, T] \to \mathbb{R}^+$ *are locally integrable functions*, *b is a constant satisfying*

$$y(t) \le a(t) + be^{\omega\Psi(t)}\int_0^t e^{-\omega\Psi(s)} y(s)\Psi'(s) ds. \tag{4.2}$$

Then

$$y(t) \le a(t) + e^{\omega\Psi(t)}\int_0^t e^{-\omega\Psi(s)} a(s)\Psi'(s)\sum_{i=1}^{\infty}\frac{b^i}{(i-1)!}(\Psi(t) - \Psi(s))^{i-1} ds.$$

Proof. Let

$$By(t) = be^{\omega\Psi(t)}\int_0^t e^{-\omega\Psi(s)} y(s)\Psi'(s) ds, \quad t \in [0, T].$$

Then

$$\begin{aligned} y(t) &\le a(t) + By(t) \\ &= a(t) + Ba(t) + B^2 y(t) \\ &= \cdots = \sum_{i=0}^{n-1} B^i a(t) + B^n y(t). \end{aligned}$$

Now, we show that

$$B^n y(t) = \frac{b^n e^{\omega\Psi(t)}}{(n-1)!}\int_0^t e^{-\omega\Psi(s)} y(s)\Psi'(s)(\Psi(t)-\Psi(s))^{n-1} ds \tag{4.3}$$

and $B^n y(t) \to 0$ as $n \to \infty$.

(4.3) is true in the case when $n = 1$. Suppose that it holds for $n = l$. Then for $n = l + 1$,

$$B^{l+1} y(t)$$

$$= B(B^l y(t))$$

$$= be^{\omega\Psi(t)}\int_0^t e^{-\omega\Psi(s)}\Psi'(s)\left[\frac{b^l e^{\omega\Psi(s)}}{(l-1)!}\int_0^s e^{-\omega\Psi(r)} y(r)\Psi'(r)(\Psi(s)-\Psi(r))^{l-1} dr\right] ds$$

$$= \frac{b^{l+1} e^{\omega\Psi(t)}}{(l-1)!}\int_0^t \left[\int_0^s e^{-\omega\Psi(r)} y(r)\Psi'(r)(\Psi(s)-\Psi(r))^{l-1} dr\right] d\Psi(s)$$

$$= \frac{b^{l+1} e^{\omega\Psi(t)}}{(l-1)!}\Psi(t)\int_0^t e^{-\omega\Psi(r)} y(r)\Psi'(r)(\Psi(t)-\Psi(r))^{l-1} dr$$

$$- \frac{b^{l+1} e^{\omega\Psi(t)}}{(l-2)!}\int_0^t \Psi(s)\left[\int_0^s e^{-\omega\Psi(r)} y(r)\Psi'(r)(\Psi(s)-\Psi(r))^{l-2} dr\right]\Psi'(s) ds.$$

By mathematical induction, we have

$$B^{l+1} y(t) = \frac{b^{l+1} e^{\omega\Psi(t)}}{(l-1)!}\int_0^t e^{-\omega\Psi(r)} y(r)\Psi'(r)$$

$$\times\left[\sum_{i=1}^{l}(-1)^{i-1}\frac{C_l^i}{l}\Psi^i(t)(\Psi(t)-\Psi(r))^{l-i} + (-1)^l \frac{1}{l}\Psi^l(r)\right] dr$$

$$= \frac{b^{l+1} e^{\omega\Psi(t)}}{(l-1)!}\int_0^t e^{-\omega\Psi(r)} y(r)\Psi'(r)$$

$$\times\left[\frac{(\Psi(t)-\Psi(r)-\Psi(t))^{l}-(\Psi(t)-\Psi(r))^{l}}{-l}+(-1)^{l}\frac{1}{l}\Psi^{l}(r)\right]dr$$

$$=\frac{b^{l+1}e^{\omega\Psi(t)}}{l!}\int_0^t e^{-\omega\Psi(r)}y(r)\Psi'(r)(\Psi(t)-\Psi(r))^{l}dr.$$

The relation (4.3) is proved. Since

$$B^n y(t)\le\frac{b^n e^{\omega\Psi(T)}}{(n-1)!}\int_0^t y(s)\Psi'(s)(\Psi(t)-\Psi(s))^{n-1}ds\to 0 \text{ as } n\to\infty$$

for $t\in[0, T]$ we have

$$y(t)\le\sum_{i=0}^{n-1}B^i a(t)+B^n y(t)$$

$$\le\sum_{i=0}^{\infty}B^i a(t)$$

$$=a(t)+e^{\omega\Psi(t)}\int_0^t e^{-\omega\Psi(s)}a(s)\Psi'(s)\sum_{i=1}^{\infty}\frac{b_i}{(i-1)!}(\Psi(t)-\Psi(s))^{i-1}ds. \quad\square$$

Corollary 4.1. *Suppose* $a:[0, T]\to\mathbb{R}^+$ *is a nondecreasing and locally integrable*, *b is a constant*, $y:[0, T]\to\mathbb{R}^+$ *is locally integrable with* (4.1) *hold*, *then*

$$y(t)\le a(t)\left(1+e^{\omega\Psi(t)}\sum_{i=1}^{\infty}\frac{b^i\Psi^i(t)}{i!}\right).$$

Theorem 4.1. *Assume that* (A, E) *generates* $\{S^E_{\alpha,\alpha_k-k,k}(t)\}_{t\ge 0}$, *whose type is* (M, ω). *Further*, $S^E_{\alpha,\alpha_k-k,k}(t)$ *is norm continuous for* $t>0$ *and* $E((k\lambda)^{\frac{\alpha}{k}}E-A)^{-1}$ *is compact in the case* $k\lambda>\omega^{k/\alpha}$. *Suppose that*

(H1) $f : [0, T] \times X \to X$ *and it satisfies the Carathéodory conditions*;

(H2) *There exist two scalar functions* $a \in C([0, T], \mathbb{R}^+)$, $b \in C([0, T], (0, \infty))$ *and a scalar nondecreasing function* $\Phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ *such that for* $x \in X$, $t \in [0, T]$,

$$\| f(t, x) \| \leq a(t) + b(t)\Phi(\| x \|),$$

with $\Phi(t) \leq \frac{1}{\| b \|_\infty} t$, *then problem* (4.1) *possesses one mild solution on* $[0, T]$.

Proof. Set $B_r = \{x \in C([0, T], X) : \| x \| \leq r\}$, where

$$r = \left[(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \frac{e^{\omega\Psi(T)} - 1}{\omega} \| a \|_\infty + kMe^{\omega\Psi(T)} \| x_0 \| + \frac{1}{\omega} Me^{\omega\Psi(T)} \| x_1 \| \right]$$

$$\times [1 + e^{\omega\Psi(T)}(e^{(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M\Psi(T)} - 1)] + 1.$$

The operator F defined on B_r is given by

$$Fx(t) = \int_0^t (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k-k, k})(\Psi^{-1}(\Psi(t) - \Psi(s)))\Psi'(s) f(s, x(s)) ds$$

$$+ kS^E_{\alpha, \alpha_k-k, k}(t)x_0 + kg_{k, k} *_\Psi S^E_{\alpha, \alpha_k-k, k}(t)x_1, \ t \in [0, T].$$

Step 1. If x satisfies $x = \mu Fx$ for $x \in B_r$ and $\mu \in (0, 1)$, then $\| x \| \neq r$.

From $x = \mu Tx$, we have

$$\| x(t) \| \leq \int_0^t \| (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k-k, k})(\Psi^{-1}(\Psi(t) - \Psi(s))) \| \Psi'(s)$$

$$\times \| f(s, x(s)) \| ds + k\| S^E_{\alpha, \alpha_k-k, k}(t) \| \| x_0 \|$$

$$+ k\| g_{k, k} *_\Psi S^E_{\alpha, \alpha_k-k, k}(t) \| \| x_1 \|$$

$$\leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} [a(s) + b(s)\Phi(\| x(t) \|)] \Psi'(s) ds$$

$$+ kMe^{\omega\Psi(t)} \| x_0 \| + \frac{1}{\omega} Me^{\omega\Psi(t)} \| x_1 \|$$

$$\leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} a(s) \Psi'(s) ds$$

$$+ (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t)} \frac{1}{\| b \|_\infty} \int_0^t e^{-\omega\Psi(s)} b(s) \| x(s) \| \Psi'(s) ds$$

$$+ kMe^{\omega\Psi(t)} \| x_0 \| + \frac{1}{\omega} Me^{\omega\Psi(t)} \| x_1 \|$$

$$\leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \frac{e^{\omega\Psi(T)} - 1}{\omega} \| a \|_\infty$$

$$+ kMe^{\omega\Psi(T)} \| x_0 \| + \frac{1}{\omega} Me^{\omega\Psi(T)} \| x_1 \|$$

$$+ (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} \| x(s) \| \Psi'(s) ds.$$

Applying Corollary 4.1, it holds

$$\| x(t) \|$$

$$\leq \left[(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \frac{e^{\omega\Psi(T)} - 1}{\omega} \| a \|_\infty + kMe^{\omega\Psi(T)} \| x_0 \| + \frac{1}{\omega} Me^{\omega\Psi(T)} \| x_1 \| \right]$$

$$\times \left(1 + e^{\omega\Psi(t)} \sum_{i=1}^{\infty} \frac{(k\omega)^{-i\frac{k-(\alpha_k-\alpha)}{k}} M^i \Psi^i(t)}{i!} \right)$$

$$\leq \left[(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \frac{e^{\omega\Psi(T)} - 1}{\omega} \| a \|_\infty + kMe^{\omega\Psi(T)} \| x_0 \| + \frac{1}{\omega} Me^{\omega\Psi(T)} \| x_1 \| \right]$$

$$\times [1 + e^{\omega\Psi(T)}(e^{(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M\Psi(T)} - 1)] < r,$$

which implies that $\| x \| \neq r.$

Step 2. F is a continuous operator on B_r.

Let $\{x_n\},\ x \in B_r$ with $\lim_{n\to\infty} x_n = x.$ Then

$$\| (Fx_n)(t) - (Fx)(t) \|$$

$$\leq \int_0^t \| (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t) - \Psi(s))) \|$$

$$\times \Psi'(s) \| f(s, x_n(s)) - f(s, x(s)) \| ds.$$

Noting that $f(s, x_n(s)) \to f(s, x(s)),$ a.e. $t \in [0, T]$ as $n \to \infty.$ By the Lebesgue dominated convergence theorem, we have

$$\sup_{t\in[0, T]} \| (Fx_n)(t) - (Fx)(t) \| \to 0$$

when $n \to \infty,$ namely F is a continuous operator.

Step 3. $\{Fx : x \in B_r\}$ is equicontinuous on $[0, T].$

Let $0 \leq t_1 < t_2 \leq T$ and $x \in B_r.$ Then

$$\| Fx(t_2) - Fx(t_1) \|$$

$$\leq \left\| \int_{t_1}^{t_2} (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t_2) - \Psi(s)))\Psi'(s) f(s, x(s)) ds \right\|$$

$$+ \left\| \int_0^{t_1} [(g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t_2) - \Psi(s))) \right.$$

$$\left. - (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t_1) - \Psi(s)))]\Psi'(s) f(s, x(s)) ds \right\|$$

$$+ k \| S^E_{\alpha, \alpha_k - k, k}(t_2) - S^E_{\alpha, \alpha_k - k, k}(t_1) \| \| x_0 \|$$

$$+ k\| S^E_{\alpha, \alpha_k, k}(t_2) - S^E_{\alpha, \alpha_k, k}(t_1) \| \| x_1 \|$$

$$\leq I_1 + I_2 + I_3 + I_4.$$

Observe that

$$I_1 \leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t_2)} \frac{e^{-\omega\Psi(t_1)} - e^{-\omega\Psi(t_2)}}{\omega} (\| a \|_\infty + r),$$

hence $\lim_{t_1 \to t_2} I_1 = 0$ independently of $x \in B_r$,

$$I_2 \leq (\| a \|_\infty + r) \int_0^{t_1} \| (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t_2) - \Psi(s)))$$

$$- (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t_1) - \Psi(s))) \| \Psi'(s) ds,$$

observe that

$$I_2 \leq 2(\| a \|_\infty + r)(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(T)} \frac{1}{\omega}$$

and $g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k} = S^E_{\alpha, \alpha, k}$ is norm continuous by Theorem 3.2, we have $\lim_{t_1 \to t_2} I_2 = 0$ independently of $x \in B_r$. Moreover, we see that $\lim_{t_1 \to t_2} I_3 = 0$ and $\lim_{t_1 \to t_2} I_4 = 0$ independently of $x \in B_r$ by using the hypothesis and Theorem 3.2, respectively.

Step 4. $\{Fx(t) : x \in B_r\}$ is precompact.

Firstly, $\{(Fx)(0) : x \in B_r\}$ is precompact. Secondly, for $0 < t \leq T$, we define F^ε on B_r by

$$(F^\varepsilon x)(t) = \int_0^{t-\varepsilon} (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k - k, k})$$

$$\times (\Psi^{-1}(\Psi(t) - \Psi(s)))\Psi'(s) f(s, x(s)) ds, \quad \forall \varepsilon \in (0, t).$$

From the hypothesis, $S_{\alpha, \alpha, k}(t)$ is compact for $t > 0$, and hence $\{(F^{\varepsilon}x)(t) : x \in B_r\}$ is precompact in B_r. Furthermore, for every $x \in B_r$,

$$\| Fx(t) - F^{\varepsilon}x(t) \|$$

$$\leq \int_{t-\varepsilon}^{t} \| (g_{k, k-(\alpha_k - \alpha)} *_{\Psi} S^{E}_{\alpha, \alpha_k - k, k})(\Psi^{-1}(\Psi(t) - \Psi(s)))\Psi'(s) f(s, x(s)) \| ds$$

$$\leq (k\omega)^{-\frac{k-(\alpha_k - \alpha)}{k}} M e^{\omega\Psi(t)} \frac{e^{-\omega\Psi(t-\varepsilon)} - e^{-\omega\Psi(t)}}{\omega} (\| a \|_{\infty} + r) \to 0, \ \ \varepsilon \to 0,$$

which implies that $\{Fx(t) : x \in B_r, t \in (0, T]\}$ is precompact in B_r.

Hence, $\{Fx : x \in B_r\}$ is precompact. Furthermore, F is compact on B_r. By using the Leray-Schauder fixed point theorem, then F has a fixed point on B_r, namely (4.1) has a mild solution. □

Theorem 4.2. *Assume that* (A, E) *generates* $\{S^{E}_{\alpha, \alpha_k - k, k}(t)\}_{t \geq 0}$, *whose type is* (M, ω). *Further,* $S^{E}_{\alpha, \alpha_k - k, k}(t)$ *is norm continuous for* $t > 0$ *and* (H1) *holds. Suppose that*

(H2)' *There exist a scalar nondecreasing function* $\Phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and two scalar functions* $a, b \in C([0, T], \mathbb{R}^+)$ *such that*

$$\| f(t, x) \| \leq a(t) + b(t)\Phi(\| x \|)$$

for $x \in X$, $t \in [0, T]$;

(H3) *There exists a scalar function* $c \in C([0, T], \mathbb{R}^+)$ *such that*

$$\alpha(f(t, D)) \leq L\alpha(D)$$

for every bounded set D in X and a.e. $t \in [0, T]$.

If

$$(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \max\left\{\frac{e^{\omega\Psi(T)}-1}{\omega}\| b \|_\infty \liminf_{r\to+\infty}\frac{\Phi(r)}{r},\ 4L\frac{1}{\omega}\right\} < 1, \quad (4.4)$$

then (4.1) *possesses one mild solution on* $[0, T]$.

Proof. We define an operator F by $Fx(t) = F_1x(t) + F_2x(t)$ on X, where

$$F_1x(t) = \int_0^t (g_{k,k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha,\alpha_k-k,k})(\Psi^{-1}(\Psi(t)-\Psi(s)))\Psi'(s) f(s, x(s))ds,$$

$$F_2x(t) = kS^E_{\alpha,\alpha_k-k,k}(t)x_0 + kg_{k,k} *_\Psi S^E_{\alpha,\alpha_k-k,k}(t)x_1,\ t \in [0, T].$$

Assume that $FB_{r_0} \subseteq B_{r_0}$ for some $r_0 > 0$. Otherwise, for every $r > 0$, $\exists x_r \in B_r$, $t_r \in [0, T]$ such that $\|(Fx_r)(t_r)\| > r$. Firstly, we observe that for $x \in X$,

$$\begin{aligned}\| Fx(t) \| &\leq \int_0^t \| (g_{k,k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha,\alpha_k-k,k})(\Psi^{-1}(\Psi(t)-\Psi(s)))\|\Psi'(s)\\
&\quad \times \| f(s, x(s))\| ds + k\| S^E_{\alpha,\alpha_k-k,k}(t)\|\| x_0 \|\\
&\quad + k\| g_{k,k} *_\Psi S^E_{\alpha,\alpha_k-k,k}(t)\|\| x_1 \|\\
&\leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t)}\int_0^t e^{-\omega\Psi(s)}[a(s)+b(s)\Phi(\| x \|)]\Psi'(s)ds\\
&\quad + kMe^{\omega\Psi(t)}\| x_0 \| + \frac{1}{\omega}Me^{\omega\Psi(t)}\| x_1 \|\\
&\leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M\frac{e^{\omega\Psi(T)}-1}{\omega}[\| a \|_\infty + \| b \|_\infty \Phi(\| x \|)]\\
&\quad + kMe^{\omega\Psi(T)}\| x_0 \| + \frac{1}{\omega}Me^{\omega\Psi(T)}\| x_1 \|, \quad t \in [0, T].\end{aligned}$$

Consequently,

$$r < \| Tx_r(t_r) \|$$
$$\leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \frac{e^{\omega\Psi(T)}-1}{\omega} [\| a \|_\infty + \| b \|_\infty \Phi(r)] + kMe^{\omega\Psi(T)} \| x_0 \|$$
$$+ \frac{1}{\omega} Me^{\omega\Psi(T)} \| x_1 \|.$$

Multiplying above inequality by $\frac{1}{r}$, then taking the lower limit as $r \to \infty$, we have

$$1 \leq (k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M \frac{e^{\omega\Psi(T)}-1}{\omega} \| b \|_\infty \liminf_{r\to+\infty} \frac{\Phi(r)}{r},$$

a contradiction.

Similar to steps in Theorem 4.1, we have F is continuous but also $\overline{co}FB_{r_0}$ is equicontinuous.

If $B \subset B_{r_0}$, then using the properties with measure of noncompactness, for $\varepsilon > 0$, there is $\{x_n\} \subset B$ such that

$$\alpha(F_1 B(t)) \leq 2\alpha\Bigg(\Bigg\{\int_0^t (g_{k, k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha, \alpha_k-k, k})$$
$$\times (\Psi^{-1}(\Psi(t) - \Psi(s)))\Psi'(s) f(s, \{x_n(s)\}) ds\Bigg\}\Bigg) + \varepsilon$$
$$\leq 4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} Me^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} \Psi'(s) \alpha(f(s, \{x_n(s)\})) ds + \varepsilon$$
$$\leq 4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML\alpha(\{x_n\}) e^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} \Psi'(s) ds + \varepsilon$$
$$\leq 4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML \frac{e^{\omega\Psi(t)}-1}{\omega} \alpha(B) + \varepsilon,$$

and thus

$$\alpha(F_1 B(t)) \leq 4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML \frac{e^{\omega\Psi(t)}-1}{\omega}\alpha(B),$$

according to the arbitrariness of $\varepsilon > 0$.

Similarly, there is $\{y_n\} \subset \overline{co}F_1 B$ such that

$$\begin{aligned}
\alpha(F_1^2 B(t)) &= \alpha(F_1(\overline{co}F_1 B(t))) \\
&\leq 2\alpha\left(\left\{\int_0^t (g_{k,k-(\alpha_k-\alpha)} *_\Psi S^E_{\alpha,\alpha_k-k,k}) \right.\right. \\
&\quad \left.\left. \times (\Psi^{-1}(\Psi(t)-\Psi(s)))\Psi'(s) f(s, \{y_n(s)\}) ds\right\}\right) + \varepsilon \\
&\leq 4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} M e^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} \Psi'(s) \alpha(f(s, \{y_n(s)\})) ds + \varepsilon \\
&\leq 4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML e^{\omega\Psi(t)} \int_0^t e^{-\omega\Psi(s)} \Psi'(s) \alpha(F_1 B(s)) ds + \varepsilon \\
&\leq [4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML]^2 \alpha(B) e^{\omega\Psi(t)} \int_0^t \Psi'(s) \frac{1-e^{-\omega\Psi(s)}}{\omega} ds + \varepsilon \\
&= [4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML]^2 \left(\frac{\Psi(t)e^{\omega\Psi(t)}}{\omega} + \frac{e^{\omega\Psi(t)}}{\omega^2} - \frac{1}{\omega^2}\right)\alpha(B).
\end{aligned}$$

Using the mathematical induction, we obtain

$$\begin{aligned}
\alpha(F_1^n B(t)) &\leq [4(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML]^n \left(\sum_{k=0}^{n-1} \frac{\Psi^k(t)e^{\omega\Psi(t)}}{k!\,\omega^{n-k}} - \frac{1}{\omega^n}\right)\alpha(B) \\
&= [4\omega^{-1}(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML]^n \left(\sum_{k=0}^{n-1} \frac{\omega^k \Psi^k(t) e^{\omega\Psi(t)}}{k!} - 1\right)\alpha(B),
\end{aligned}$$

from

$$4\omega^{-1}(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML < 1$$

and

$$\sum_{k=0}^{n-1} \frac{\omega^k \Psi^k(t) e^{\omega\Psi(t)}}{k!} \leq e^{2\omega\Psi(T)},$$

we get

$$\lim_{n\to\infty} [4\omega^{-1}(k\omega)^{-\frac{k-(\alpha_k-\alpha)}{k}} ML]^n \left(\sum_{k=0}^{n-1} \frac{\omega^k \Psi^k(t) e^{\omega\Psi(t)}}{k!} - 1 \right) = 0.$$

Moreover, $\alpha(F_2^n B(t)) = 0$, and consequently,

$$\alpha(F^n B) \leq \alpha(F_1^n B) + \alpha(F_2^n B),$$

by Lemma 2.4 in [12]. Thus F has a fixed point, namely (4.1) has a mild solution.

5. An Example

Example 5.1. Set $X = L^2([0, \pi], \mathbb{R})$, $k < \alpha < 2k$, $0 \leq v \leq 1$. We study the following (k, Ψ)-Hilfer fractional initial value problem:

$$\begin{cases} {}^{k,H}D_{0+}^{\alpha, v; \Psi}[x(t, \xi) - x_{\xi\xi}(t, \xi)] = Ax(t, \xi) + f(t, x(t, \xi)), & t \in [0, 1], \\ ({}^{k}I_{0+}^{2k-\alpha_k; \Psi} x(t, \xi))_{t=0} = x_0(\xi), & \xi \in [0, \pi], \\ \left[\left(\frac{k}{\Psi'(t)} \frac{d}{dt} \right) {}^{k}I_{0+}^{2k-\alpha_k; \Psi} x(t, \xi) \right]_{t=0} = x_1(\xi), & \xi \in [0, \pi], \\ x(t, 0) = x(t, \pi) = 0, \end{cases} \tag{5.1}$$

where ${}^{k,H}D_{0+}^{\alpha, v; \Psi}$ is the (k, Ψ)-Hilfer fractional derivative. Operators A and E are given by $Ax = x_{\xi\xi\xi\xi}$, $Ex = x - x_{\xi\xi}$, respectively, let

$$\{x \in E : x \in H^4([0, \pi]),\ x(t, 0) = x(t, \pi) = 0\}$$

be the domain of A and E. Obviously, the eigenvalues and eigenvectors of A are $-n^4$ and $x_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$, the eigenvalues of E are $1 + n^2$.

For $x \in X$ and $k \le \alpha \le 2k$, we have

$$(k\lambda)^{\frac{\alpha-\beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1} x$$

$$= \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \frac{(k\lambda)^{\frac{\alpha-\beta}{k}}(1 + n^2)}{(k\lambda)^{\frac{\alpha}{k}}(1 + n^2) + n^4}$$

$$= \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \int_0^{\infty} e^{-\lambda\Psi(t)} \Psi^{\frac{\beta}{k}-1}(t) E_{\frac{\alpha}{k}, \frac{\beta}{k}}\left(-\frac{n^4}{1 + n^2} \Psi^{\frac{\alpha}{k}}(t)\right) dt$$

$$= \int_0^{\infty} e^{-\lambda\Psi(t)} \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \Psi^{\frac{\beta}{k}-1}(t) E_{\frac{\alpha}{k}, \frac{\beta}{k}}\left(-\frac{n^4}{1 + n^2} \Psi^{\frac{\alpha}{k}}(t)\right) dt.$$

Hence, (A, E) generates the family $\{S^E_{\alpha, \beta, k}(t)\}_{t \ge 0}$,

$$S^E_{\alpha, \beta, k}(t) x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \Psi^{\frac{\beta}{k}-1}(t) E_{\frac{\alpha}{k}, \frac{\beta}{k}}\left(-\frac{n^4}{1 + n^2} \Psi^{\frac{\alpha}{k}}(t)\right), \quad x \in X,$$

$S^E_{\alpha, \beta, k}(t)$ is norm continuous from the continuity of $E_{\frac{\alpha}{k}, \frac{\beta}{k}}(\cdot)$. Also, we have $(k\lambda)^{\frac{\alpha-\beta}{k}} E((k\lambda)^{\frac{\alpha}{k}} E - A)^{-1}$ is compact on X by

$$\lim_{n \to \infty} \frac{(k\lambda)^{\frac{\alpha-\beta}{k}}}{(k\lambda)^{\frac{\alpha}{k}} + \frac{n^4}{1 + n^2}} = 0.$$

Assume that for $x \in X$, we have $\| S^E_{\alpha, \beta, k}(t)x \| \leq M\| x \|$. Therefore, $S^E_{\alpha, \beta, k}(t)$ is type $(M, 1)$.

If $f(t, x) = \frac{1}{1+t}x$ and $a(t) = 0$, $b(t) = 1$ and $\Phi(r) = r$, then there exists one mild solution on $[0, 1]$ for (5.1) by Theorem 4.1.

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