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OPTIMAL HARVESTING STRATEGY FOR PREY-PREDATOR MODEL WITH FISHING EFFORT AS A TIME VARIABLE

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Abstract

We study a prey-predator model with harvesting where the fishing effort is considered as a function of time. The analysis focuses on the equilibrium points and the optimal harvesting strategy.

1. Introduction

For a long time, fishing has been considered as a main activity for large parts of the human population. But the excessive exploitation of fish stocks is a global problem that current management policies are struggling to solve [21]. The implementation of this management requires mostly scientific studies. The authors in [9, 15, 10, 16] conducted studies to better knowledge and understanding the functioning of the fisheries system on the one hand, and on the other hand, to try to improve the fisheries situation. The study of prey-predator models plays an important role in ensuring the survival of species [2, 3, 7, 14, 13]. We are interested in a prey-predator model of aquatic species (fish) with harvesting on prey. We consider that the harvest function per unit effort is proportional to the stock of prey available for harvesting and that the fishing effort is a function of time. The model is presented as follows:

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - \min\left(\dfrac{ax(t)}{y(t) + D}, \gamma\right) y(t) - k(t), \\ \dfrac{dy(t)}{dt} = e \min\left(\dfrac{ax(t)}{y(t) + D}, \gamma\right) y(t) - \mu y(t), \\ \dfrac{dE(t)}{dt} = \beta(pmqx(t) - c)E(t) \end{cases} \tag{1}$$

with $x(0) = x_0 \geq 0$, $y(0) = y_0 \geq 0$ and $E(0) = E_0 \geq 0$.

$x(t)$ and $y(t)$ represent prey and predator densities, respectively, at time t, $E(t)$ is fishing effort at time t, $k(t) = mqE(t)x(t)$ is the harvest function at time t and $r, K, \mu, e, a, D, \gamma, p, m, q, \beta$ are positive constants: r is the growth rate of prey, K is the limiting capacity of the environment, μ

is the mortality rate of predators, e is the conversion rate of prey to predators, a is the predation rate, D measures other causes of prey mortality outside of predation, γ is the maximum amount of food predator needs per unit of time, p is the constant price per unit of biomass, m is the fraction of the prey stock available for harvesting such that $0 < m < 1$, q is the capture coefficient, c is the cost of fishing constant per unit effort and β describes the dynamics of fishing effort. In system (1), $\frac{ax(t)}{y(t) + D}$ is the amount of food a predator has access to and $\min\left(\frac{ax(t)}{y(t) + D}, \gamma\right)$ is the amount of food a predator receives per unit time. In this model, the constant predation rate a is obtained by integration of a continuous and 1-periodic function [12]. Several mathematicians have conducted studies on the model to explain the effects of the water level fluctuations on the life of fish [4-6, 18]. Sarkar et al., in [1], also studied the effects of water level fluctuation taking into account the interactions with invertebrates. The model was recently studied in [17] where harvesting was non-selective.

Remark 1. Without the third equation of system (1) and for $m = 1$, the model (1) is well studied in [12].

The rest of the paper is organized as follows: Section 2 is devoted to the mathematical analysis of the model. We prove the existence and uniqueness of the solution, determine the boundary conditions of the solutions, and study the equilibrium points. In Section 3, we propose an optimal harvesting strategy. Finally, Section 4 provides the conclusion.

2. Mathematical Analysis

We find that if the fishing effort is greater than $\frac{r}{qm}$, then $\frac{dx(t)}{dt} < 0$, and therefore, both the species will disappear. Thus we consider the following hypothesis: (H_1): $E(t) < \frac{r}{qm}, \forall t \geq 0.$

2.1. Existence and uniqueness of the solution

Lemma 2.1. *Let G be a Banach space,* $f : G \to \mathbb{R}$ *be a function and* $\gamma \in \mathbb{R}$. *Set* $h : G \to \mathbb{R}$, $x \mapsto \min(f(x), \gamma)$. *If f is locally Lipschitzian, then the function h is locally Lipschitzian.*

Proof. Since f is locally Lipschitzian, there exists a positive constant k such that

$$|f(x_1) - f(x_2)| \leq k\| x_1 - x_2 \|_G, \quad \forall x_1, x_2 \in G.$$

Since h can be rewritten as

$$h(x) = \frac{1}{2}(f(x) + \gamma - |f(x) - \gamma|),$$

$$\begin{aligned} |h(x_1) - h(x_2)| &= \frac{1}{2}|f(x_1) - f(x_2) - |f(x_1) - \gamma| + |f(x_2) - \gamma|| \\ &\leq \frac{1}{2}[|f(x_1) - f(x_2)| + ||f(x_1) - \gamma| - |f(x_2) - \gamma||] \\ &\leq \frac{1}{2}[|f(x_1) - f(x_2)| + |f(x_1) - f(x_2)|] \\ &\leq |f(x_1) - f(x_2)| \\ &\leq k\| x_1 - x_2 \|_G. \end{aligned}$$

Hence, h is locally Lipschitzian. □

Theorem 2.1. *Let* $x_0 \geq 0$, $y_0 \geq 0$ *and* $E_0 \geq 0$. *Then the system* (1) *admits a unique positive solution* $(x(t), y(t), E(t))$ *such that* $x(0) = x_0$, $y(0) = y_0$ *and* $E(0) = E_0$.

Proof. Let

$$F(x, y, E) = \begin{pmatrix} f_1(x, y, E) \\ f_2(x, y, E) \\ f_3(x, y, E) \end{pmatrix} = \begin{pmatrix} rx\left(1 - \frac{x}{K}\right) - \min\left(\frac{ax}{y + D}, \gamma\right) y - mqEx \\ e \min\left(\frac{ax}{y + D}, \gamma\right) y - \mu y \\ \beta(pmqx - c)E \end{pmatrix}.$$

From Lemma 2.1 we can say that functions f_1 and f_2 are locally Lipschitzian. The function f_3 is also locally Lipschitzian. According to the Cauchy-Lipschitz's theorem [11], the system (1) admits a unique solution $(x(t), y(t), E(t))$ such that $x(0) = x_0$, $y(0) = y_0$ and $E(0) = E_0$.

For the positivity of the solution, we use the property of isoclines [19].

The axes *x*, *y* and *E* are zero isoclines of the system, so no trajectory can intersect one of these three axes. Thus any trajectory resulting from an initial condition taken in the positive frame remains inside this frame for all $t \geq 0$. So any solution of the system (1) remains in the quadrant $\mathbb{R}_+^3$. This completes the proof. □

Lemma 2.2 [8]**.** *Let a and b be two strictly positive real numbers.*

- *If* $\frac{dx}{dt} \leq x(t)(a - bx(t))$ *with* $x(0) > 0$, *then* $\lim_{t \to +\infty} \sup x(t) \leq \frac{a}{b}$.
- *If* $\frac{dx}{dt} \geq x(t)(a - bx(t))$ *with* $x(0) > 0$, *then* $\lim_{t \to +\infty} \inf x(t) \geq \frac{a}{b}$.

Theorem 2.2. *Assume that the hypotheses* (H_1) *and*

$$(H_2): \ \frac{aeK}{\mu} - D > 0$$

hold. Let the set A be defined by

$$A = \left\{(x, y, E) \in \mathbb{R}_+^3 / 0 \leq x \leq K, 0 \leq y \leq \frac{aeK - \mu D}{\mu}, 0 \leq E < \frac{r}{qm}\right\}.$$

Then the solution of the system (1) *is bounded and belongs to the region A.*

Proof. Since the solution of system (1) is positive, we have $x(t) \geq 0$, $y(t) \geq 0$, and $E(t) \geq 0$ for all $t \geq 0$.

Using the first equation of the system (1), we have

$$\frac{dx(t)}{dt} \leq r\left(1 - \frac{x}{K}\right).$$

According to Lemma 2.2, we obtain

$$\lim_{t \to +\infty} \sup x(t) \leq K.$$

So for an arbitrary $\varepsilon_1 > 0$, there is a $T_1 > 0$ such as

$$x(t) \leq K + \varepsilon_1, \quad \forall t > T_1.$$

Using the second equation of the system (1), we have

$$\frac{dy(t)}{dt} \leq y\left(\frac{ae(K + \varepsilon_1)}{y + D} - \mu\right), \quad \forall t > T_1$$

$$\leq \frac{y}{y + D}(ae(K + \varepsilon_1) - \mu D - \mu y), \quad \forall t > T_1$$

$$\leq \frac{y}{D}(ae(K + \varepsilon_1) - \mu D - \mu y), \quad \forall t > T_1.$$

According to Lemma 2.2, we obtain

$$\lim_{t \to +\infty} \sup y(t) \leq \frac{ae(K + \varepsilon_1) - \mu D}{\mu}, \quad \forall t > T_1.$$

As ε_1 is arbitrary, $\lim_{t \to +\infty} \sup y(t) \leq \frac{aeK - \mu D}{\mu}$, $\forall t > T_1$.

Under the hypothesis (H_2), we have $\frac{\alpha eK - \mu D}{\mu} > 0$.

So for an arbitrary $\varepsilon_2 > 0$, there exists a $T_2 > T_1$ such that

$$y(t) \leq \frac{aeK - \mu D}{\mu} + \varepsilon_2, \quad \forall t > T_2.$$

According to the hypothesis (H_1), we have $E(t) < \frac{r}{qm}, \forall t \geq 0.$

Therefore, E is bounded for all $t \geq 0.$

The proof is complete. □

Proposition 2.1. *Assuming*

$$a < \min\left(\frac{\gamma(y_0 + D)}{x_0}, \frac{4r\gamma\mu D}{K(r+\mu)^2}\right), \qquad (H_3)$$

for all $t \geq 0,$

$$ax(t) < \gamma(y(t) + D).$$

Proof. Let $u(t) = ax(t) - \gamma(y(t) + D).$

Now, we show that if the hypothesis (H_3) is satisfied, then $u(t)$ is strictly negative for all $t \geq 0.$

For $t = 0,$ we have

$$u(0) = ax(0) - \gamma(y(0) + D) = x_0\left(a - \frac{\gamma(y_0 + D)}{x_0}\right),$$

with $x_0 > 0.$ According to the hypothesis (H_3), $a < \frac{\gamma(y_0 + D)}{x_0}.$ Thus $u(0) < 0.$ Suppose there is a positive t_0 such that $u(t_0) = 0$ and $\frac{du(t_0)}{dt} \geq 0.$

Using the system (1), we obtain

$$\frac{du(t_0)}{dt} = a\left[rx(t_0)\left(1 - \frac{x(t_0)}{K}\right) - \frac{ax(t_0)}{y(t_0) + D}\, y(t_0) - mqE(t_0)x(t_0)\right]$$

$$- \gamma\left(e\frac{ax(t_0)}{y(t_0) + D}\, y(t_0) - \mu y(t_0)\right).$$

By using the fact that $y(t_0) = \frac{ax(t_0)}{\gamma} - D,$ we have

$$\frac{du(t_0)}{dt} = -\frac{ar}{K}(x(t_0))^2 + a(r+\mu)x(t_0) - aqmE(t_0)x(t_0)$$

$$- a(a + e\gamma)\frac{y(t_0)}{y(t_0) + D}x(t_0) - \gamma\mu D$$

$$\leq -\frac{ar}{K}(x(t_0))^2 + a(r+\mu)x(t_0) - \gamma\mu D.$$

So under the hypothesis (H_3), $\frac{du(t_0)}{dt} < 0,$ and this constitutes a contradiction.

Thus, for all $t > 0,$ we have $u(t_0) \neq 0$ or $\frac{du(t_0)}{dt} < 0.$

This means that for all $t > 0,$ $u(t) < 0.$ □

Remark 2. Under the hypothesis of the previous proposition, the system (1) can be rewritten in a simplified form as follows:

$$\begin{cases} \frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{axy}{y + D} - mqEx, \\ \frac{dy}{dt} = \frac{eaxy}{y + D} - \mu y, \\ \frac{dE}{dt} = \beta(pmqx - c)E. \end{cases} \tag{2}$$

2.2. Equilibrium points

Proposition 2.2. *The system* (2) *admits five* (05) *points of equilibrium such that*

(i) $P_0(0; 0; 0),$

(ii) $P_1(K; 0; 0),$

(iii) *the equilibrium point* $P_2\left(\frac{c}{pmq}; 0; \frac{r(Kpmq - c)}{Kp(mq)^2}\right)$ *exists under the assumption*

$$(H_4): pmqK > c,$$

(iv) *the equilibrium point* $P_3(\bar{x}; \bar{y}; 0)$ *exists if and only if*

$$r + \sqrt{(r-a)^2 + \frac{4r\mu D}{Ke}} > a + \frac{2r\mu D}{Kae}, \text{ with}$$

$$\bar{x} = \frac{K}{2r}\left(r - a + \sqrt{(r-a)^2 + \frac{4r\mu D}{Ke}}\right) \text{ and}$$

$$\bar{y} = \frac{Kae}{2\mu r}\left(r - a - \frac{2r\mu D}{Kae} + \sqrt{(r-a)^2 + \frac{4r\mu D}{Ke}}\right),$$

(v) *the positive equilibrium point* $P^*(x^*, y^*; E^*)$ *exists under the assumptions*

$$(H_5): \frac{ace}{\mu pmq} > D \quad \text{and} \quad (H_6): 0 < 1 - \frac{c}{Kpmq} - \frac{\mu}{er} < p,$$

where

$$x^* = \frac{c}{pmq}; \quad y^* = \frac{ace}{\mu pmq} - D \quad \text{and} \quad E^* = \frac{r}{pmq}\left(1 - \frac{c}{Kpmq} - \frac{\mu}{er}\right).$$

Proof. By solving the following system:

$$\begin{cases} rx\left(1 - \frac{x}{K}\right) - \frac{axy}{y + D} - mqEx = 0, \\ \frac{eaxy}{y + D} - \mu y = 0, \\ \beta(pmqx - c)E = 0, \end{cases}$$

we obtain the equilibrium points P_0, P_1, P_2 and P^*. □

Proposition 2.3. (i) *The point* P_0 *is an unstable equilibrium.*

(ii) *The equilibrium points* P_1 *and* P_3 *are also unstable*.

(iii) *The equilibrium point* P_2 *is stable under the assumption* (H_5^c) *and unstable under the assumption* (H_5).

(iv) *The positive equilibrium point* P^* *is locally asymptotically stable under the assumptions of its existence*.

Proof. The Jacobian matrix of the system (2) is

$$J(x, y, E) = \begin{pmatrix} r - \frac{2rx}{K} - \frac{ay}{y+D} - qmE & \frac{-aDx}{(y+D)^2} & -qmx \\ \frac{eay}{y+D} & \frac{eaDx}{(y+D)^2} - \mu & 0 \\ \beta pmqE & 0 & \beta(pmqx - c) \end{pmatrix}.$$

The Jacobian matrix associated with the point P_0 gives:

$$J(P_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\beta c \end{pmatrix}.$$

The matrix $J(P_0)$ has three distinct real eigenvalues $\lambda_1 = r > 0$, $\lambda_2 = -\mu < 0$ and $\lambda_3 = -\beta c < 0$. So the point P_0 is an unstable point.

The Jacobian matrix associated with the point P_1 is

$$J(P_1) = \begin{pmatrix} -r & \frac{-aK}{D} & -qmK \\ 0 & -\mu & 0 \\ 0 & 0 & \beta(pmqK - c) \end{pmatrix}.$$

The eigenvalues of the matrix $J(P_1)$ are: $\lambda_1 = -r > 0$, $\lambda_2 = -\mu < 0$ and $\lambda_3 = \beta(pmqK - c)$. Under the assumption (H_4), $\lambda_3 > 0$, so P_1 is an unstable equilibrium.

Concerning the point P_2, we have

$$J(P_2) = \begin{pmatrix} \frac{-cr}{Kpmq} & \frac{-ac}{pmqD} & \frac{-c}{p} \\ 0 & \frac{eac}{pmqD} - \mu & 0 \\ \frac{\beta r(Kpmq - c)}{Kpmq} & 0 & 0 \end{pmatrix}.$$

The calculation of the characteristic polynomial P gives

$$P(\lambda) = \left(-\mu + \frac{eac}{pmqD} - \lambda\right)\left[\lambda^2 + \frac{cr}{Kpmq}\lambda + \frac{\beta rc}{Kp^2mq}(Kpmq - c)\right].$$

Assuming that $S = \frac{-cr}{Kpmq}$ and $\pi = \frac{\beta rc}{Kp^2mq}(Kpmq - c)$, we have $S < 0$ and $\pi > 0$ under the hypothesis (H_4). Thus under the hypothesis (H_5), that is, $\frac{eac}{\mu pmq} > D$, the point P_2 is unstable and stable if $\frac{eac}{\mu pmq} < D$: (H_5^c).

For $P(\bar{x}, \bar{y}, 0)$, the associated matrix is

$$J(P_3) = \begin{pmatrix} \frac{-r\bar{x}}{K} & \frac{-aD\bar{x}}{(\bar{y} + D)^2} & -qm\bar{x} \\ \frac{ea\bar{y}}{\bar{y} + D} & \frac{ea\bar{x}\bar{y}}{(\bar{y} + D)^2} & 0 \\ 0 & 0 & \beta(pmq\bar{x} - c) \end{pmatrix}.$$

The characteristic polynomial of $J(P_3)$ is

$$P(\lambda) = [\beta(pmq\bar{x} - c) - \lambda]$$
$$\times\left[\lambda^2 + \left(\frac{r\bar{x}}{K} + \frac{ea\bar{x}\bar{y}}{(\bar{y} + D)^2}\right)\lambda + \frac{rea\bar{x}\bar{y}}{K(\bar{y} + D)^2} + \frac{ea^2D\bar{x}\bar{y}}{K(\bar{y} + D)^3}\right].$$

Assuming that $\pi = \frac{rea\bar{x}\bar{y}}{K(\bar{y}+D)^2} + \frac{ea^2D\bar{x}\bar{y}}{K(\bar{y}+D)^3}$ and $S = -\left(\frac{r\bar{x}}{K} + \frac{ea\bar{x}\bar{y}}{(\bar{y}+D)^2}\right);$ we have: $\pi > 0$ and $S < 0$.

Moreover, under the hypothesis (H_4), $\beta(pmqK - c) > 0$. Therefore, P_3 is an unstable equilibrium.

For P^*, the associated matrix is

$$J(P^*) = \begin{pmatrix} \frac{-rx^*}{K} & \frac{-aDx^*}{(y^*+D)^2} & -qmx^* \\ \frac{eay^*}{y^*+D} & \frac{-eax^*y^*}{(y^*+D)^2} & 0 \\ \beta pmqE^* & 0 & 0 \end{pmatrix}.$$

Consider $J(P^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ with

$$\begin{cases} a_{11} = \frac{-rx^*}{K}; & a_{12} = \frac{-aDx^*}{(y^*+D)^2}; & a_{13} = -qmx^*, \\ a_{21} = \frac{eay^*}{y^*+D}; & a_{22} = \frac{-eax^*y^*}{(y^*+D)^2}; & a_{23} = 0, \\ a_{31} = \beta pmqE^*; & a_{32} = 0; & a_{33} = 0. \end{cases}$$

By calculating the determinant of the matrix $J(P^*)$, we have

$$\begin{aligned} \det(J(P^*)) &= a_{31}(a_{11}a_{22} - a_{12}a_{21}) \\ &= \beta pmqE^*\left(\frac{rx^*}{K} \times \frac{eax^*y^*}{(y^*+D)^2} + \frac{aDx^*}{(y^*+D)^2} \times \frac{eay^*}{y^*+D}\right) \\ &\neq 0. \end{aligned}$$

The characteristic polynomial of $J(P^*)$ is of the form:

$$P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,$$

where

$$\begin{cases} A_1 = -(a_{11} + a_{22}) = \dfrac{rx^*}{K} + \dfrac{eax^*y^*}{(y^* + D)^2}, \\ A_2 = -(a_{12}a_{21} + a_{13}a_{31} - a_{11}a_{22}) \\ \quad = \dfrac{eDa^2x^*y^*}{(y^* + D)^2} + \beta p(mq)^2 x^* E^* + \dfrac{arex^*y^*}{K(y^* + D)^2}, \\ A_3 = a_{13}a_{22}a_{31} = \beta eap(mq)^2 \dfrac{(x^*)^2 y^* E^*}{(y^* + D)^2}. \end{cases}$$

We have $A_1 > 0$ and $A_3 > 0$.

Now, we determine the sign of $A_1A_2 - A_3$:

$$A_1A_2 - A_3 = \frac{rx^*}{K}\left(\frac{eDa^2x^*y^*}{(y^* + D)^2} + \beta p(mq)^2 x^*E^* + \frac{arex^*y^*}{K(y^* + D)^2}\right)$$

$$+ \frac{eax^*y^*}{(y^* + D)^2}\left(\frac{eDa^2x^*y^*}{(y^* + D)^2} + \frac{arex^*y^*}{K(y^* + D)^2}\right).$$

Thus, $A_1A_2 - A_3 > 0$.

According to the Routh-Hurwitz criterion [19], the eigenvalues of the matrix $J(P^*)$ have negative real parts. Therefore, the point P^* is locally asymptotically stable. □

Remark 3. According to the hypothesis (H_5), if $m < \dfrac{ace}{\mu pqD}$, then the trajectories converge towards equilibrium P^* (Figures 1 and 2). However, if $m > \frac{ace}{\mu pqD}$, then the trajectories converge towards the equilibrium P_2 (Figure 3). In this case, the predator disappears.

For figures below, the constants are as: $r = 12$, $K = 40$, $a = 11$, $D = 5$, $q = 6$, $e = 1.20$, $\mu = 0.45$, $\beta = 1.25$, $p = 1.3$, $c = 1.15$, $(x(0),\ y(0),\ E(0)) = (7, 15, 5)$.

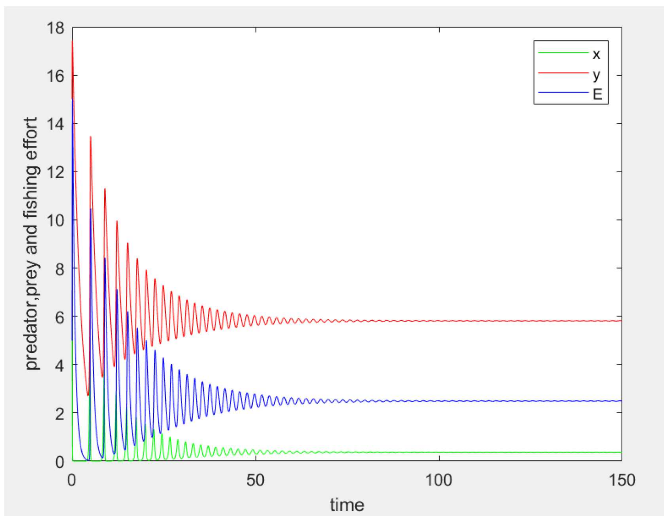

Figure 1. Convergence of the solution towards the equilibrium point P^* for $m = 0.4$.

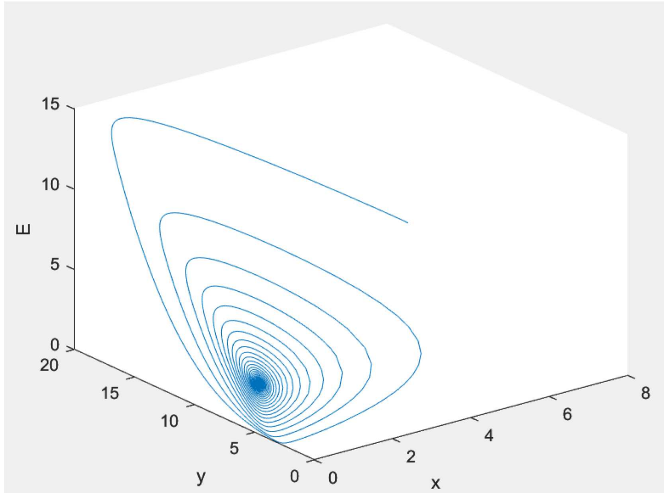

Figure 2. Phase portrait for $m = 0.4$.

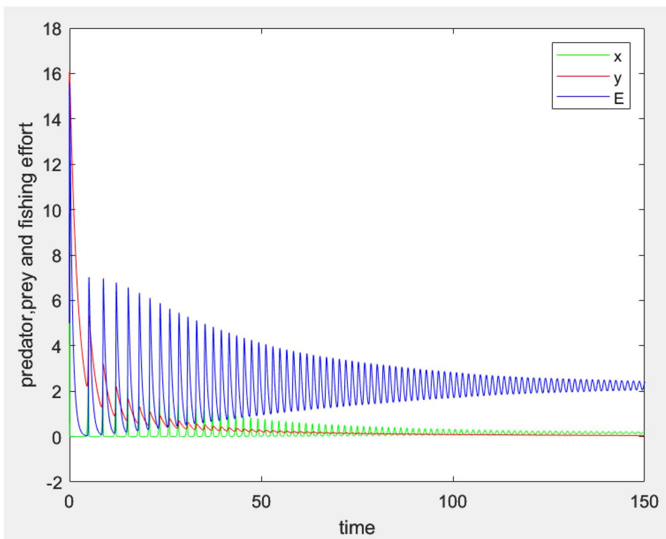

Figure 3. Convergence of the solution towards the equilibrium point P_2 for $m = 0.88$.

Theorem 2.3. *If the existence conditions* (H_5) *and* (H_6) *hold*, *then* P^* *is globally asymptotically stable*.

Proof. Let V be a function defined by

$$V(x, y, E) = \alpha_1\left[(x - x^*) - x^* \ln\left(\frac{x}{x^*}\right)\right] + \alpha_2\left[(y - y^*) - y^* \ln\left(\frac{y}{y^*}\right)\right]$$

$$+ \alpha_3\left[(E - E^*) - E^* \ln\left(\frac{E}{E^*}\right)\right],$$

where α_1, α_2 and α_3 are positive constants to be determined.

- We have $V(x^*, y^*, E^*) = 0$ and $V(x, y, E) > 0$, $\forall (x, y, E) \neq (x^*, y^*, E^*)$.

- Note that

$$\frac{dV}{dt} = \alpha_1 \frac{(x - x^*)}{x} \times \frac{dx}{dt} + \alpha_2 \frac{(y - y^*)}{y} \times \frac{dy}{dt} + \alpha_3 \frac{(E - E^*)}{E} \times \frac{dE}{dt}$$

$$= \alpha_1 (x - x^*)\left[r - \frac{rx}{K} - \frac{ay}{y + D} - mqE\right] + \alpha_2 (y - y^*)\left[\frac{eax}{y + D} - \mu\right]$$

$$+ \alpha_3 (E - E^*)[\beta(pmqx - c)].$$

Using the following system

$$\begin{cases} r - \dfrac{rx^*}{K} - \dfrac{ay^*}{y^* + D} - mqE = 0, \\ \dfrac{eax^*}{y^* + D} - \mu = 0, \\ \beta(pmqx^* - c) = 0, \end{cases}$$

we have

$$\frac{dV}{dt} = \alpha_1 (x - x^*)\left[\frac{-r}{K}(x - x^*) - \frac{aD(y - y^*)}{(y^* + D)(y + D)} - mq(E - E^*)\right]$$

$$+ \alpha_2 (y - y^*)\left[\frac{eay^* x + eaDx - eax^* y - eaDx^*}{(y^* + D)(y + D)}\right]$$

$$+ \alpha_3\beta pmq(x - x^*)(E - E^*)$$

$$= \alpha_1(x - x^*)\left[\frac{-r}{K}(x - x^*) - \frac{aD(y - y^*)}{(y^* + D)(y + D)} - mq(E - E^*)\right]$$

$$+ \alpha_2(y - y^*)\left[\frac{eay^*x + eaDx - eax^*y - eaDx^* + eax^*y^* - eax^*y^*}{(y^* + D)(y + D)}\right]$$

$$+ \alpha_3\beta pmq(x - x^*)(E - E^*)$$

$$= \frac{-\alpha_1 r}{K}(x - x^*)^2 - \frac{\alpha_2 eax^*(y - y^*)^2}{(y^* + D)(y + D)}$$

$$+ (x - x^*)(y - y^*)\left[\frac{-\alpha_1 aD + \alpha_2 ea(D + y^*)}{(y^* + D)(y + D)}\right]$$

$$+ pmq(x - x^*)(E - E^*)(\alpha_3\beta p - \alpha_1).$$

By setting $\alpha_1 = 1$, $\alpha_2 = \dfrac{D}{e(y^* + D)}$ and $\alpha_3 = \dfrac{1}{\beta p}$, we deduce that

$$\frac{dV}{dt} = \frac{-r}{K}(x - x^*)^2 - \frac{aDx^*(y - y^*)^2}{(y^* + D)^2(y + D)} < 0, \quad \forall (x,\ y,\ E) \neq (x^*,\ y^*,\ E^*).$$

So according to Lyapunov's theorem [19], the positive equilibrium P^* is globally asymptotically stable. □

3. Optimum Harvesting Strategy

Determining an optimal fishing strategy is always a problem for policy makers. The goal is to find a compromise between current and future harvests while ensuring economic and political interests. In our work, using the Pontryagin maximum principle, we determine an optimal harvesting strategy by considering a function that presents the continuous time flow of income.

Let U_{ad} be a non-empty set defined by

$$U_{ad} = \{m : [0, T] \to [m_{\min}, m_{\max}], \text{ Lebesgue measurable}\}.$$

Consider J a cost function defined by

$$J(m) = \int_0^{+\infty} e^{-\delta t}(pmqx - c)E dt,$$

where δ is the instantaneous annual discount rate, $m \in U_{ad}$ is the control function, and x represents prey density associated to m.

Our objective is to maximize the functional J, that is, to find the optimal control $\tilde{m} \in U_{ad}$ such that

$$J(\tilde{m}) = \max\{J(m), m \in U_{ad}\}.$$

Since J is convex and U_{ad} is compact, we can show that an optimal control exists [22].

Let $M(\tilde{x}, \tilde{y}, \tilde{E})$ be the optimal equilibrium point associated with the optimal control $\tilde{m}$. Set

$$J(m) = \int_0^T L(t, x_m(t), m(t))dt + \varphi(x_m(T)), \quad \forall m \in U_{ad}$$

with $L(t, x_m(t), m(t)) = e^{-\delta t}(pmqx - c)E$ and $\varphi(x_m(T)) = 0$.

Thus the Hamiltonian function is defined by

$$\begin{aligned}
&H(t, x(t), \lambda(t), m(t)) \\
&= L(t, x(t), \lambda(t), m(t)) + \langle p, f(t, x(t), m(t))\rangle \\
&= e^{-\delta t}(pmqx - c)E + \lambda_1\left[rx - \frac{r}{K}x^2 - \frac{axy}{y + D} - qmxE\right] \\
&\quad + \lambda_2\left[\frac{eaxy}{y + D} - \mu y\right] + \lambda_3\beta(pmqx - c)E
\end{aligned}$$

with λ_1, λ_2 and λ_3 as adjoint variables.

Suppose that the optimal control is singular, i.e., the optimal solution would not be in $m_{\min}$ or in $m_{\max}$. Then

$$\frac{\partial H}{\partial m} = pqxEe^{-\delta t} - \lambda_1 qxE + \lambda_3 \beta pqxE = 0. \tag{3}$$

According to the Pontryagin maximum principle [20], we get

$$\frac{d\lambda_1}{dt} = -\left[e^{-\delta t} pmqE + \lambda_1\left(r - \frac{2r}{K} - \frac{ay}{y + D}\right) + \lambda_2 \frac{eay}{y + D} + \lambda_3 \beta pqmE\right], \tag{4}$$

$$\frac{d\lambda_2}{dt} = \lambda_1 \frac{aDx}{(y + D)^2} - \lambda_2\left(\frac{eaDx}{(y + D)^2} - \mu\right), \tag{5}$$

$$\frac{d\lambda_3}{dt} = -e^{-\delta t}(pmq - c) + \lambda_1 qmx - \lambda_3 \beta(pmqx - c). \tag{6}$$

From the relation (3), we have

$$\lambda_1 = pe^{-\delta t} + \beta p \lambda_3. \tag{7}$$

By replacing the expression of $\lambda_1(t)$ in (6), we get

$$\frac{d\lambda_3}{dt} = \beta c \lambda_3 + ce^{-\delta t}$$

whose solution is

$$\lambda_3(t) = \frac{-c}{\delta + \beta c} e^{-\delta t}. \tag{8}$$

Using the relations (7) and (8), we obtain

$$\lambda_1(t) = pe^{-\delta t}\left(1 - \frac{\beta c}{\delta + \beta c}\right). \tag{9}$$

The substitution of $\lambda_1(t)$ in equation (5) gives

$$\frac{d\lambda_2}{dt} = A_1 e^{-\delta t} + A_2 \lambda_2, \tag{10}$$

with

$$A_1 = p\left(1 - \frac{\beta c}{\delta + \beta c}\right) \times \frac{aDx}{(y + D)^2}$$

and

$$A_2 = -\left(\frac{eaDx}{(y + D)^2} - \mu\right) = \frac{eaxy}{(y + D)^2}.$$

The resolution of equation (10) gives us

$$\lambda_2(t) = \frac{-A_1}{\delta + A_2} e^{-\delta t}. \tag{11}$$

Using the equilibrium equations in the relation (4), we obtain

$$\frac{d\lambda_1}{dt} = -e^{-\delta t} pmqE + \lambda_1 \frac{r}{K} - \lambda_2 \frac{eay}{y + D} - \lambda_3 \beta c. \tag{12}$$

By replacing $\lambda_1(t)$, $\lambda_2(t)$ and λ_3 in (12) and by integration, we have

$$\lambda_1(t) = \frac{-e^{-\delta t}}{\delta}\left[-pmqE + p\left(1 - \frac{\beta c}{\delta + \beta c}\right) \times \frac{r}{K} x + \frac{A_1}{\delta + A_2} \times \frac{eay}{y + D} + \frac{\beta c^2}{\delta + \beta c}\right]. \tag{13}$$

The relations (7) and (13) allow us to deduce the optimal control:

$$\tilde{m} = \frac{1}{pq\tilde{E}}\left[\frac{p}{\delta}\left(1 - \frac{\beta c}{\delta + \beta c}\right) + p\left(1 - \frac{\beta c}{\delta + \beta c}\right) \times \frac{r}{K} \tilde{x} + \frac{A_1}{\delta + A_2} \times \frac{ea\tilde{y}}{\tilde{y} + D} + \frac{\beta c^2}{\delta + \beta c}\right].$$

The adjoint variables $\lambda_i(t)$, $i = 1, 2, 3$ satisfy the transversality conditions, i.e., $\lim_{t \to +\infty} \lambda_i(t) = 0$, $i = 1, 2, 3$.

4. Conclusion

In this paper, we studied a prey-predator model with harvesting on prey. We had taken into account the stock of prey available for harvesting and considered that the fishing effort is a function of time that satisfies a differential equation. The existence and uniqueness of the solution, the boundary conditions of the solution and the analysis of the points of equilibrium were studied. We finished by proposing an optimal harvesting strategy to maximize the resources of the fishery.

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