Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0974324324021
Volume 31, Number 3, 2024, Pages 379-395 **P-ISSN: 0974-3243**

HÖLDER REGULARITY OF SOLUTIONS FOR CERTAIN DEGENERATE PARABOLIC INTEGRO-DIFFERENTIAL EQUATION

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Abstract

In this paper, we consider the nonlinear parabolic equation with an integro-differential term. By using classical inequalities and the Moser iteration technique, we establish the estimates for u and ∇u. Then we prove an inequality of Poincaré type. As a byproduct of our proof, we derive a Campanato type growth estimate for u which follows from L^{∞} estimates of ∇u. Besides, the Hölder continuity of solution is presented by the isomorphism theorem.

Received: May 19, 2024; Accepted: June 21, 2024
2020 Mathematics Subject Classification: 35B65, 35K59, 35K65.
Keywords and phrases: integro-differential equation, Hölder continuity, Moser iteration.
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How to cite this article: Zongqing Yang and Junhui Xie, Hölder regularity of solutions for certain degenerate parabolic integro-differential equation, Advances in Differential Equations and Control Processes 31(3) (2024), 379-395. https://doi.org/10.17654/0974324324021

Published Online: July 2, 2024

1. Introduction

In this paper, we are interested in the degenerate parabolic integro-differential equation:

$$u_t - div(|\nabla u|^{p-2}\nabla u) - \int_0^t \frac{\partial}{\partial x_i}\left(b_{ij}(x, t, \tau)\frac{\partial u(x, t)}{\partial x_j}\right)d\tau = 0 \tag{1.1}$$

with $p > 2$, where $u = u(x, t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $\nabla = grad_x$, b_{ij} and $(b_{ij})_{x_i}$ are measurable.

The equation arises from the following mathematical model [3]:

$$u_t - div(|\nabla u|^{p-2}\nabla u) - B(x, t, u, \nabla u) = 0, \tag{1.2}$$

where $u = u(x, t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ with the function $B(x, t, u, \nabla u) \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times M^{NN} \to \mathbb{R}^N)$ satisfying a controllable growth condition $|B(x, t, u, \nabla u)| \le (1 + |\nabla u|)^{p-1}$. A local Hölder continuity for the weak solutions is obtained when $1 < p < \infty$ by Moser iteration and an inequality of Poincaré type. Besides, DiBenedetto [4] proved that solutions for the system (1.2) with a natural growth condition on B are Hölder continuous when $p \ge 2$ by a truncation idea of De Giorgi and a scaling approach. Subsequently, DiBenedetto and Chen investigated the case $1 < p < 2$ for the system (1.2) and proved Hölder continuity of solutions in [5]. Furthermore, in [6], DiBenedetto and Chen proved Hölder continuity for solutions of a parabolic system up to boundary when $p > \frac{2n}{n+2}$. Accordingly, Hölder continuity of ∇u is derived for different cases for p [7, 9, 10].

Involving integro-differential term in our equation is inspired by parabolic Volterra integro-differential equation (PVIDE) [1]:

$$u_t = div\,\vec{A}(x, t, u, u_x) + a(x, t, u, u_x) + \int_0^t div\,\vec{B}(x, t, \tau, u, u_x)d\tau, \tag{1.3}$$

where the integral term represents the effect of memory term in the material [14]. Under some structure condition on $\vec{A}$, $\vec{B}$ similar to the case of parabolic equations, the existence of unique weak solution is proved by using the Galerkin method. Moreover, regularity of the weak solution is also investigated. It is worth mentioning that similar problems drew a considerable attention of many scholars [2, 11-13, and the references therein].

Taking inspiration from the above results, we consider the local Hölder continuity for the weak solution of a parabolic equation with integro-differential term in this paper, based on the Moser iteration technique. It is particularly important to point out that the solution of the problem (1.1) is Hölder continuous in the interior of the domain, no information is needed on initial and boundary values.

Let Ω be an open set in $\mathbb{R}^N$, and for $T > 0$, let Ω_T denote the cylindrical domain $\Omega \times (0, T]$. A function

$$u(x, t) \in C^0[0, T; L^2(\Omega)] \cap L^p[0, T; W^{1,p}(\Omega)] \tag{1.4}$$

is a local weak solution to (1.1) if for every compact set $K \subset \Omega$ and every subinterval $[t_1, t_2] \subset [0, T]$,

$$\left[\int_K u\eta dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K -u\eta_t + |\nabla u|^{p-2}\nabla u \nabla \eta dxdt$$
$$+ \int_{t_1}^{t_2}\int_K \int_0^t \frac{\partial}{\partial x_i}\left(b_{ij}(x, t, \tau)\frac{\partial u}{\partial x_j}\right)\eta d\tau dxdt = 0 \tag{1.5}$$

for all bounded functions $\eta \in C^1(0, T, C_0^\infty(\Omega))$.

Now, we provide the main conclusions of this paper.

Theorem 1.1. *Let* $p > 2$, *and suppose that* $u \in L^p_{loc}(\Omega_T)$, *then* $u \in L^\infty_{loc}(\Omega_T)$. *Moreover, for all* $Q_{2R_0} \subset \Omega_T$,

$$\sup_{Q_{\frac{R_0}{2}}} |u| \le C\left[\left(⨍_{Q_{R_0}} |u|^p dz\right)^{\frac{1}{2}} + 1\right],$$

where C is a constant independent of R_0.

Furthermore, we have

Theorem 1.2. *Suppose* $S_{2R_0} \subset \Omega_T$ *and* $\nabla(\nabla u) \in L^2(\Omega_T)$, *then there exists a constant C independent of* R_0 *such that*

$$\sup_{S_{\frac{R_0}{2}}} |\nabla u| \le C\left[\left(⨍_{S_{R_0}} |\nabla u|^p dz\right)^{\frac{1}{2}} + 1\right].$$

Once we get Campanato type growth estimate for *u*, we have the following result:

Theorem 1.3. *Let* $p > 2$, *and suppose that* $u \in L^p_{loc}(\Omega_T)$ *for the problem* (1.1), *then* $u \in C^{0,\alpha}_{loc}(\Omega_T)$ *for some* $\alpha > 0$.

2. Proof of Theorem 1.1 and Theorem 1.2

In this section, we first prove that *u* and ∇u are bounded by the Moser iteration technique. More specifically, weak Harnack inequality holds for $|u|$ and $|\nabla u|$. Let $Q_R = \{(x, t) : |x - x_0| < R, t_0 - R^p < t < t_0\}$, where $(x_0, t_0) \in \Omega_T$ is arbitrary. Define $B_R = \{x : |x - x_0| < R\}$, $T_R = \{t : t_0 - R^p < t < t_0\}$.

Proof of Theorem 1.1. Let $r < R$ and ϕ be the standard cutoff function such that

$$\phi = 1 \text{ in } Q_r$$

$\phi = 0$ in a neighborhood of parabolic boundary of Q_R

$$0 \le \phi \le 1, \quad |\phi_t| \le \frac{C}{(R-r)^p}, \quad |\nabla\phi| \le \frac{C}{R-r}, \tag{2.1}$$

where C is a positive constant. Initially, we choose $\eta = u|u|^{\alpha}\phi^p$ as a test function to (1.1), where $\alpha > 0$. Then multiplying by η on both the sides of equation (1.1), and integrating by parts, we have

$$\int_{Q_R}\left(u_t\eta + |\nabla u|^{p-2}\nabla u \cdot \nabla\eta + \int_0^t\left(b_{ij}(x, t, \tau)\frac{\partial u(x, t)}{\partial x_j}d\tau\right)\eta_{x_i}\right)dz = 0. \tag{2.2}$$

Taking $\nabla\eta = (\alpha+1)\nabla u|u|^{\alpha}\phi^p + pu|u|^{\alpha}\phi^{p-1}\nabla\phi$ into account, we deduce that

$$\begin{aligned} 0 = &\int_{Q_R} u_t\eta dz + (\alpha+1)\int_{Q_R}|\nabla u|^p|u|^{\alpha}\phi^p dz \\ &+ p\int_{Q_R} u|u|^{\alpha}\nabla u|\nabla u|^{p-2}\phi^{p-1}\nabla\phi dz \\ &+ \int_{Q_R}\int_0^t b_{ij}(x, t, \tau)u_{x_j}\eta_{x_i}d\tau dz. \end{aligned} \tag{2.3}$$

Noting that $\sum u_{x_j} \le C|\nabla u|$, from (2.3), we deduce that

$$\begin{aligned} &\int_{Q_R} u_t\eta dz + (\alpha+1)\int_{Q_R}|\nabla u|^p|u|^{\alpha}\phi^p dz \\ \le\ & p\int_{Q_R}|u|^{\alpha+1}|\nabla u|^{p-1}\phi^{p-1}|\nabla\phi|dz \\ &+ C\int_{Q_R}\int_0^t|b_{ij}(x, t, \tau)||\nabla u||\nabla\eta|d\tau dz \\ \le\ & p\int_{Q_R}|u|^{\alpha+1}|\nabla u|^{p-1}\phi^{p-1}|\nabla\phi|dz \end{aligned}$$

$$+ C(\alpha + 1)\int_{Q_R}\int_0^t | b_{ij}(x, t, \tau) || \nabla u |^2 | u |^\alpha \phi^p d\tau dz$$

$$+ Cp\int_{Q_R}\int_0^t | b_{ij}(x, t, \tau) || \nabla u || u |^{\alpha+1} \phi^{p-1} | \nabla\phi | d\tau dz. \quad (2.4)$$

Considering that b_{ij} is measurable and $\int_0^t | b_{ij}(x, t, \tau) | d\tau < C,$ by Young's inequality with $\varepsilon,$ we have

$$\int_{Q_R} u_t \eta dz + (\alpha + 1)\int_{Q_R} | \nabla u |^p | u |^\alpha \phi^p dz$$

$$\leq \varepsilon p\int_{Q_R} | \nabla u |^p | u |^\alpha \phi^p dz + C(\varepsilon) p\int_{Q_R} | u |^{\alpha+p} | \nabla\phi |^p dz$$

$$+ C[(\alpha + 1)\varepsilon + p\varepsilon]\int_{Q_R} | \nabla u |^p | u |^\alpha \phi^p dz + CC(\varepsilon)(\alpha + 1)\int_{Q_R} | u |^\alpha \phi^p dz$$

$$+ CpC(\varepsilon)\int_{Q_R} | u |^{\alpha+\frac{p}{p-1}} \phi^{(p-2)\frac{p}{p-1}} | \nabla\phi |^{\frac{p}{p-1}} dz$$

$$\leq [C(\alpha + 1)\varepsilon + Cp\varepsilon + p\varepsilon]\int_{Q_R} | \nabla u |^p | u |^\alpha \phi^p dz$$

$$+ C(\varepsilon) p\int_{Q_R} | u |^{\alpha+p} | \nabla\phi |^p dz$$

$$+ C\varepsilon C(\varepsilon)(\alpha + 1)\int_{Q_R} | u |^{\alpha+p} dz + CC(\varepsilon)(\alpha + 1) | Q_R |$$

$$+ C\varepsilon pC(\varepsilon)\int_{Q_R} | u |^{\alpha+p} dz + \frac{CC(\varepsilon)}{(R - r)^{\frac{p}{p-1}}} | Q_R |, \quad (2.5)$$

where C is a constant and $C(\varepsilon)$ is a constant depending on $\varepsilon.$ Furthermore, owing to $\frac{d}{dt}(u\eta) = (\alpha + 2) | u |^\alpha u u_t \phi^p + p | u |^{\alpha+2} \phi^{p-1} \phi_t,$ we infer that

$$\int_0^t\int_{B_R} [(\alpha + 2) | u |^\alpha u u_t \phi^p + p | u |^{\alpha+2} \phi^{p-1} \phi_t] dxds = \int_{B_R} u\eta dx \,|_{s=0}^{s=t}, \quad (2.6)$$

which means that

$$\frac{1}{\alpha+2}\sup_t\int_{B_R}u\eta dx-\frac{p}{\alpha+2}\int_{Q_R}|u|^{\alpha+2}\phi^{p-1}\phi_t dz\le\int_{Q_R}u_t\eta dz. \tag{2.7}$$

Combining (2.5) and (2.7), it is not hard to obtain

$$\begin{aligned}&\sup_t\int_{B_R}u\eta dx+(\alpha+2)[(\alpha+1)-C(\alpha+1)\varepsilon-Cp\varepsilon-p\varepsilon]\\&\times\left(\int_{Q_R}|\nabla u|^p|u|^\alpha\phi^p dz+\int_{Q_R}|u|^{\alpha+p}|\nabla\phi|^p dz\right)\\&\le p\int_{B_R}|u|^{\alpha+2}\phi^{p-1}|\phi_t|dz+(\alpha+2)\\&\times[(\alpha+1)-C(\alpha+1)\varepsilon-Cp\varepsilon-p\varepsilon+C(\varepsilon)p]\int_{Q_R}|u|^{\alpha+p}|\nabla\phi|^p dz\\&+C\varepsilon C(\varepsilon)(\alpha+1)(\alpha+2)\int_{Q_R}|u|^{\alpha+p}dz+CC(\varepsilon)(\alpha+1)(\alpha+2)|Q_R|\\&+C\varepsilon pC(\varepsilon)(\alpha+2)\int_{Q_R}|u|^{\alpha+p}dz+(\alpha+2)\frac{CC(\varepsilon)}{(R-r)^{\frac{p}{p-1}}}|Q_R|,\end{aligned} \tag{2.8}$$

where ε is to be determined to satisfy $((\alpha+1)-C(\alpha+1)\varepsilon-Cp\varepsilon-p\varepsilon)>0$. Besides, $C'\gg 1$ is chosen to satisfy that

$$\begin{aligned}&\int_{Q_R}\left|\nabla\left(\phi|u|^{\frac{\alpha+p}{p}}\right)\right|^p dz\\&\le C'(\alpha+2)[(\alpha+1)-C(\alpha+1)\varepsilon-Cp\varepsilon-p\varepsilon]\\&\times\left(\int_{Q_R}|\nabla u|^p|u|^\alpha\phi^p dz+\int_{Q_R}|u|^{\alpha+p}|\nabla\phi|^p dz\right).\end{aligned} \tag{2.9}$$

Obviously, with a constant $C(\alpha, C', \varepsilon, p)$, (2.8) and (2.9) lead to the following estimate:

$$\sup_t \int_{B_R} |u|^{\alpha+2}\phi^p dx + \int_{Q_R} \left|\nabla\left(\phi|u|^{\frac{\alpha+p}{p}}\right)\right|^p dz$$

$$\leq C\int_{Q_R} |u|^{\alpha+2}\phi^{p-1}|\phi_t|dz + \frac{C}{(R-r)^p}\int_{Q_R}|u|^{\alpha+p}dz$$

$$+ C\int_{Q_R}|u|^{\alpha+p}dz + C|Q_R| + \frac{C}{(R-r)^{\frac{p}{p-1}}}|Q_R|$$

$$\leq \frac{C}{(R-r)^p}\int_{Q_R}|u|^{\alpha+p}dz + \frac{C}{(R-r)^p}|Q_R|, \tag{2.10}$$

emphasizing that the last inequality utilizes Hölder inequality. Besides, by Sobolev inequality, we have

$$\sup_{t_0-r^p\leq t\leq t_0}\int_{B_r}|u|^{\alpha+2}dx + \int_{T_r}\left[\int_{B_r}|u|^{(\alpha+p)\frac{n}{n-p}}dx\right]^{\frac{n-p}{n}}dt$$

$$\leq \sup_t \int_{B_R}|u|^{\alpha+2}\phi^p dx + \int_{Q_R}\left|\nabla\left(\phi|u|^{\frac{\alpha+p}{p}}\right)\right|^p dz$$

$$\leq \frac{C}{(R-r)^p}\int_{Q_R}|u|^{\alpha+p} + \frac{C}{(R-r)^p}|Q_R|. \tag{2.11}$$

Applying Hölder inequality again, it is not difficult to have

$$\int_{Q_r}|u|^{(\alpha+p)\left(1+\frac{p}{n}\frac{\alpha+2}{\alpha+p}\right)}$$

$$\leq \left[\sup_t \int_{B_r}|u|^{\alpha+2}dx\right]^{\frac{p}{n}}\left[\int_{T_r}\left(\int_{B_r}|u|^{\frac{(\alpha+p)n}{n-p}}dx\right)^{\frac{n-p}{n}}dt\right]$$

$$\leq C\left[\frac{1}{(R-r)^p}\int_{Q_R}|u|^{\alpha+p}dz + \frac{|Q_R|}{(R-r)^p}\right]^{1+\frac{p}{n}}. \tag{2.12}$$

Denote $r_v = \frac{R}{2}(1 + 2^{-v})$, $v = 0, 1, 2, ...,$ and $Q_v = Q_{r_v}$. From (2.12), we have

$$\fint_{Q_v} |u|^{(\alpha+p)\left(1+\frac{p}{n}\frac{\alpha+2}{\alpha+p}\right)} dz$$

$$\leq C \frac{|Q_R|^{1+\frac{p}{n}}}{|R-r|^{p\left(1+\frac{p}{n}\right)}|Q_r|}\left[\fint_{Q_R} |u|^{(\alpha+p)} dz + 1\right]^{1+\frac{p}{n}}. \tag{2.13}$$

Define α_v inductively by

$$\alpha_{v+1} = \left(1 + \frac{p}{n}\right)\alpha_v + \frac{2p}{n}, \quad \alpha_0 = 0.$$

Then we see that $\alpha_v = 2(\theta^v - 1)$, where $\theta = 1 + \frac{p}{n}$. Also, note that

$$\lim_{v\to\infty} \frac{\alpha_v + p}{\theta^v} = 2.$$

Defining ϕ_v by

$$\phi_v = \fint_{Q_v} |u|^{\alpha_v + p} dz,$$

(2.12) can be written in the form:

$$\phi_{v+1} \leq C\phi_v^{\theta} + C, \tag{2.14}$$

where C is independent of R. Iterating (2.14), we prove the desired conclusion of Theorem 1.1, when $p > 2$.

For the convenience of the proof of Theorem 1.2, we define a new cylinder S_R by $S_R = B_R(x_0) \times (t_0 - R^2, t_0)$. Though, it is similar to the proof of Theorem 1.1, the calculation is more complicated, by which we deduce that $\nabla^2 u \in L^2(\Omega_T)$.

Proof of Theorem 1.2. By differentiating (1.1) with respect to x_l, we have

$$(u_{x_l})_t - (a_{ij}|\nabla u|^{p-2}u_{x_l x_i})_{x_j} - \left(\int_0^t \frac{\partial}{\partial x_i} b_{ij}(x, t, \tau)\frac{\partial u}{\partial x_j} d\tau\right)_{x_l} = 0, \quad (2.15)$$

where $a_{ij} = \delta_{ij} + (p-2)\frac{u_{x_i}u_{x_j}}{|\nabla u|^2}$, δ_{ij} is the Kronecker delta function. For $r < R$, we introduce a cutoff function ψ satisfying

$$\psi = 1 \text{ in } S_r,$$

$$\psi = 0 \text{ in a neighborhood of parabolic boundary of } S_R,$$

$$0 \le \psi \le 1, \ |\psi_t| \le \frac{C}{(R-r)^2}, \ |\nabla\psi| \le \frac{C}{R-r}. \quad (2.16)$$

Suppose $\alpha > 0$, and choose $\eta = u_{x_l}|\nabla u|^{\alpha}\psi^2$ as a test function to (2.15). Then multiplying by η on both the sides of equation (2.15), we get

$$\int_{S_R}(u_{x_l})_t\eta dz + \int_{S_R} a_{ij}|\nabla u|^{p-2}u_{x_l x_i}\eta_{x_j}dz$$
$$+ \int_{S_R}\int_0^t \frac{\partial}{\partial x_i}\left(b_{ij}(x, t, \tau)\frac{\partial u}{\partial x_j}\right)\eta_{x_l} d\tau dz = 0. \quad (2.17)$$

Recalling Theorem 4 in [4], we deduce that

$$\sup_t \int_{B_R}|\nabla u|^{\alpha+2}\psi^2 dx + \int_{S_R}\left|\nabla\left(\psi|\nabla u|^{\frac{\alpha+p}{2}}\right)\right|^2 dz$$
$$\le C\int_{S_R}|\nabla u|^{\alpha+2}\psi|\psi_t|dz + C\int_{S_R}|\nabla u|^{\alpha+p}|\nabla\psi|^2 dz$$
$$+ \int_{S_R}\int_0^t \frac{\partial}{\partial x_i}\left(b_{ij}(x, t, \tau)\frac{\partial u}{\partial x_j}\right)\eta_{x_l} d\tau dz. \quad (2.18)$$

Noting that $\sum u_{x_i x_j} \leq C|\nabla(\nabla u)|$, $\sum u_{x_i} \leq C|\nabla u|$, we can estimate $\int_{S_R}\int_0^t \frac{\partial}{\partial x_i}\left(b_{ij}(x, t, \tau)\frac{\partial u}{\partial x_j}\right)\eta_{x_l} d\tau dz$ by a simple calculation. Specifically, we have

$$\sup_t \int_{B_R} |\nabla u|^{\alpha+2}\psi^2 dx + \int_{S_R}\left|\nabla\left(\psi|\nabla u|^{\frac{\alpha+p}{2}}\right)\right|^2 dz$$

$$\leq C\int_{S_R} |\nabla u|^{\alpha+2}\psi|\psi_t|dz + C\int_{S_R}|\nabla u|^{\alpha+p}|\nabla\psi^2|dz$$

$$+ C(\alpha+1)\| b_{ij_{x_i}}\|_{L^\infty(S_R)}\int_{S_R}|\nabla(\nabla u)||\nabla u|^{\alpha+1}\psi^2 dz$$

$$+ C\| b_{ij_{x_i}}\|_{L^\infty(S_R)}\int_{S_R}|\nabla u|^{\alpha+2}\psi|\nabla\psi|dz$$

$$+ C(\alpha+1)\| b_{ij}\|_{L^\infty(S_R)}\int_{S_R}|\nabla(\nabla u)|^2|\nabla u|^{\alpha}\psi^2 dz$$

$$+ C\| b_{ij}\|_{L^\infty(S_R)}\int_{S_R}|\nabla(\nabla u)||\nabla u|^{\alpha+1}\psi|\nabla\psi|dz. \tag{2.19}$$

Furthermore, according to Young's inequality, we can obtain that

$$\sup_t \int_{B_R} |\nabla u|^{\alpha+2}\psi^2 dx + \int_{S_R}\left|\nabla\left(\psi|\nabla u|^{\frac{\alpha+p}{2}}\right)\right|^2 dz$$

$$\leq \frac{C}{(R-r)^2}\int_{S_R}|\nabla u|^{\alpha+p}dz + \frac{C}{(R-r)^2}|S_R|$$

$$+ C(\alpha+1)\| b_{ij_{x_i}}\|_{L^\infty(S_R)}\left(\varepsilon\int_{S_R}|\nabla|\nabla u||^2|\nabla u|^{\alpha+p-2}\psi^2 dz\right.$$

$$\left. + C(\varepsilon)\int_{S_R}|\nabla u|^{\alpha+p-2}\psi^2 dz + C(\varepsilon)\int_{S_R}|\nabla|\nabla u||\psi^2 dz\right)$$

$$+ C\| b_{ij_{x_i}} \|_{L^\infty(S_R)}\left(\varepsilon\int_{S_R} |\nabla u|^{\alpha+p}\psi|\nabla\psi|dz + C(\varepsilon)\int_{S_R}\psi|\nabla\psi|dz\right)$$

$$+ \varepsilon C(\alpha+1)\| b_{ij} \|_{L^\infty(S_R)}\int_{S_R} |\nabla|\nabla u||^2|\nabla u|^{\alpha+p-2}\psi^2 dz$$

$$+ C(\varepsilon)C(\alpha+1)\| b_{ij} \|_{L^\infty(S_R)}\int_{S_R} |\nabla|\nabla u||^2\psi^2 dz$$

$$+ \varepsilon C\| b_{ij} \|_{L^\infty(S_R)}\int_{S_R} |\nabla|\nabla u||^2|\nabla u|^{\alpha+1}\psi dz$$

$$+ C(\varepsilon)C\| b_{ij} \|_{L^\infty(S_R)}\int_{S_R} |\nabla u|^{\alpha+p}|\nabla\psi^2|dz$$

$$+ \varepsilon C\| b_{ij} \|_{L^\infty(S_R)}\int_{S_R} |\nabla\psi|^2 dz. \tag{2.20}$$

Since $|\nabla(\nabla u)|$ belongs to L^2, it follows that

$$\sup_t \int_{B_R} |\nabla u|^{\alpha+2}\psi^2 dx + \int_{S_R}\left|\nabla\left(\psi|\nabla u|^{\frac{\alpha+p}{2}}\right)\right|^2 dz$$

$$\leq \int_{S_R}\frac{C}{(R-r)^2}|\nabla u|^{\alpha+p}dz + \frac{C}{(R-r)^2}|S_R| + C. \tag{2.21}$$

By Hölder inequality and Sobolev inequality, we have

$$\int_{S_r}|\nabla u|^{(\alpha+p)\left(1+\frac{2}{n}\frac{\alpha+2}{\alpha+p}\right)}$$

$$\leq \left[\sup_t\int_{B_r}|\nabla u|^{\alpha+2}dx\right]^{\frac{2}{n}}\left[\int_{T_r}\left(\int_{B_r}|\nabla u|^{\frac{(\alpha+p)n}{n-2}}dx\right)^{\frac{n-2}{n}}dt\right]$$

$$\leq C\left[\frac{1}{(R-r)^2}\int_{S_R}|\nabla u|^{\alpha+p}dz + \frac{|S_R|}{(R-r)^2}\right]^{1+\frac{2}{n}} + C. \tag{2.22}$$

From (2.22), we get

$$\fint_{S_r} |\nabla u|^{(\alpha+p)\left(1+\frac{2}{n}\frac{\alpha+2}{\alpha+p}\right)}$$

$$\leq C\frac{|S_R|^{1+\frac{2}{n}}}{(R-r)^{2\left(1+\frac{2}{n}\right)}|S_R|}\left[\fint_{S_R} |\nabla u|^{\alpha+p}dz+1\right]^{1+\frac{2}{n}}+\frac{C}{|S_r|}. \tag{2.23}$$

Set $k_v = \frac{R}{2}(1+2^{-v}),\ v = 0, 1, 2, \ldots.$

For $\mu = 1+\frac{2}{n}$, defining α_v inductively by

$$\alpha_{v+1}+p = \mu\alpha_v + p + \frac{4}{n}, \quad \alpha_0 = 0,$$

and $\phi(v) = \fint_{S_{k_v}} |u|^{\alpha_v+p}dz,\ r = k_v$, (2.22) can be written as follows:

$$\phi(v+1) \leq C\phi(v)^{\mu} + C \tag{2.24}$$

for some C depending only on n and p. Note that

$$\lim_{v\to\infty}\frac{\mu^v}{\alpha_v+p} = \frac{1}{2}.$$

Iterating (2.24), we prove the desired result of Theorem 1.2.

3. Hölder Continuity of u

In this section, we define $T_{\overline{R}} = (t_0 - 2R^p,\ t_0 - R^p)$ and $Q_{\overline{R}} = B_R \times T_{\overline{R}}$. We now introduce a cutoff function $\eta \in C_0^\infty(B_R)$ such that

$$\eta = 1 \text{ in } B_{\frac{R}{2}}$$

$$0 \leq \eta \leq 1, \quad \nabla\eta \leq \frac{c}{R}.$$

Also, we define

$$u_{R,t} = \frac{1}{|B_R|}\int_{B_R} u(x, t)dx$$

and

$$u_R = \frac{1}{|Q_R|}\int_{Q_R} udz.$$

First we give a Lemma which is essential for the Poincaré inequality for solution of a degenerate parabolic equation.

Lemma 3.1. *If* $Q_{2R} \subset \Omega_T$, *then u satisfies the following inequality*:

$$\sup_{t\in[s,t_0]}\int_{T_{\overline{R}}} ds\int_{B_R} \eta^p|u(x, t) - u_{R,s}|^2 dx$$

$$\leq CR^p\int_{Q_{2R}}|\nabla u|^p dz + C\int_{Q_{\overline{R}}}|u - u_{R,t}|^2 dz + CR^{n+2p}, \tag{3.1}$$

for all $s \in T_{\overline{R}}$, *where C depends only on n and p.*

Proof. Since $u \in C^0[0, T; L^2(\Omega)]$, there exists $\tilde{t}(s) \in [s, t_0]$ such that

$$\int_{B_R} \eta^p|u(x, \tilde{t}) - u_{R,s}|^2 dx = \sup_{t_0\geq t\geq s}\int_{B_R} \eta^p|u(x, t) - u_{R,s}|^2 dx.$$

Take $(u - u_{R,s})\eta^p\xi_{[s,\tilde{t}]}$ as a test function to equation (1.1), where $\xi_{[s,\tilde{t}]} : R \to R$ is the characteristic function which means that $\xi_{[s,\tilde{t}]}(s) = 1$ for all $s \in [s, \tilde{t}]$ and $\xi_{[s,\tilde{t}]}(s) = 0$ for all $s \notin [s, \tilde{t}]$. Hence we show that

$$\int_{Q_R} u_t(u - u_{R,s})\eta^p\xi_{[s,\tilde{t}]}dz + \int_{Q_R}|\nabla u|^{p-2}\nabla u\cdot\nabla((u - u_{R,s})\eta^p)\xi_{[s,\tilde{t}]}dz$$

$$+\int_{Q_R}\int_0^t b_{ij}\frac{\partial u}{\partial x_j}\nabla((u - u_{R,s})\eta^p)\xi_{[s,\tilde{t}]}d\tau dz = 0. \tag{3.2}$$

Considering that $p > 2, \| b_{ij} \|_{L^\infty} < C,$ by Young's inequality, we have

$$\int_{B_R} | u(x, \tilde{t}) - u_{R,s} |^2 \eta^p dx + \int_{B_R \times [s, \tilde{t}]} | \nabla u |^p \eta^p dz$$

$$\leq \int_{B_R} | u - u_{R,s} |^2 \eta^p (x, s) dx + \frac{\varepsilon}{R^p} \int_{B_R \times [s, \tilde{t}]} | u - u_{R,s} |^p \eta^{p(p-1)} dz$$

$$+ C \int_{B_R \times [s, \tilde{t}]} | \nabla u |^p dz + \frac{C\varepsilon}{R^p} \int_{B_R \times [s, \tilde{t}]} | u - u_{R,s} |^p \eta^{p-1} dz + CR^{n+p} \tag{3.3}$$

for some C independent of R and $0 \leq \eta \leq 1.$ Integrating (3.3) with respect to s from $t_0 - 2R^p$ to $t_0 - R^p,$ we have

$$\int_{T_{\overline{R}}} ds \int_{B_R} | u(x, \tilde{t}) - u_{R,s} |^2 \eta^p dx$$

$$\leq \int_{Q_{\overline{R}}} | u - u_{R,t} |^2 \eta^p dz + \frac{C\varepsilon}{R^p} \int_{T_{\overline{R}}} ds \int_{B_R \times [s, \tilde{t}]} | u - u_{R,s} |^2 \eta^p dz$$

$$+ CR^p \int_{Q_{2R}} | \nabla u |^p dz + CR^{n+2p}. \tag{3.4}$$

By the choice of $\bar{t},$ we have that for small $\varepsilon > 0,$

$$\int_{T_{\overline{R}}} ds \int_{B_R} | u(x, \tilde{t}) - u_{R,s} |^2 \eta^p dx$$

$$\leq C \int_{Q_{\overline{R}}} | u - u_{R,t} |^2 \eta^p dz + CR^p \int_{Q_{2R}} | \nabla u |^p dz + CR^{n+2p} \tag{3.5}$$

for $p > 2.$ This completes the proof of Lemma 3.1. □

According to Lemma 3.1, we can have the following result:

Lemma 3.2. *If* $Q_{2R} \subset \Omega_T,$ *then*

$$\int_{Q_{\frac{R}{2}}} | u - u_{\frac{R}{2}} |^2 dz \leq CR^2 \int_{Q_{2R}} | \nabla u |^2 dz + CR^p \int_{Q_{2R}} | \nabla u |^p dz + CR^{n+2p}, \tag{3.6}$$

where C is independent of R.

This is the same as Theorem 4 in [3].

Proof of Theorem 1.3. Since ∇u is bounded, from Lemma 3.2, we can deduce the Campanato type growth estimate for u such that

$$\int_{Q_R} | u - u_R |^2 dz \leq CR^{n+p+2},$$

where C is independent of R. Hence by isomorphism theorem of Da Prato [8] follows the result.

Acknowledgement

This work is supported by National Natural Science Foundation of China (No. 11761030), Hubei Provincial Natural Science Foundation of China (No. 2022CFC016) and Cultivation Project for High-Level Scientific Research Achievements of Hubei Minzu University (No. PY20002).

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