Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0974324324020
Volume 31, Number 3, 2024, Pages 357-377 **P-ISSN: 0974-3243**

THE SOME BLAISE ABBO (SBA) PLUS METHOD APPLIED TO FRACTIONAL NONLINEAR TIME SCHRÖDINGER EQUATIONS IN d DIMENSION ($d = 1$, 2 or 3) IN THE SENSE OF CAPUTO

Oumar MADAI[1,*], Germain KABORE[2], Bakari Abbo[1], Ousséni SO[3] and Blaise SOME[2]

[1]University of N'Djamena
Tchad
e-mail: madaioumar9@gmail.com
bakariabbo@yahoo.fr

[2]Laboratoire d'Analyse Numérique, Informatique et de Biomathématiques (L.A.N.I.BIO)
Université Joseph KI-ZERBO
Burkina Faso
e-mail: germainkabore982@gmail.com
blaisesomeouaga1@gmail.com

Received: April 3, 2024; Accepted: June 6, 2024
2020 Mathematics Subject Classification: 65Nxx, 65Lxx, 65Mxx, 65Qxx.
Keywords and phrases: Some Blaise Abbo (SBA) plus method, fractional Schrödinger equations, Caputo derivation.
[*]Corresponding author

How to cite this article: Oumar MADAI, Germain KABORE, Bakari Abbo, Ousséni SO and Blaise SOME, The Some Blaise Abbo (SBA) plus method applied to fractional nonlinear time Schrödinger equations in d dimension (d = 1, 2 or 3) in the sense of Caputo, Advances in Differential Equations and Control Processes 31(3) (2024), 357-377.
https://doi.org/10.17654/0974324324020

Published Online: June 25, 2024

[3]Laboratoire d'Analyse Numérique, Informatique et de Biomathématiques (L.A.N.I.BIO)
École Normale Supérieure
Burkina Faso
e-mail: sousseni@yahoo.fr

Abstract

In this paper, we have solved some time fractional Schrödinger equations of order α with $0 < \alpha \leq 1$ in dimension 1, 2 or 3 in the sense of Caputo by the SBA plus method. This method is based on two principles (successive approximations, and Picard) and the Adomian method. Secondly, it uses a process of rapid convergence in the functional space of the problem posed towards the exact solution, if it exists.

1. Introduction

The nonlinear fractional-time Schrödinger equations describe numerous physical systems in fields as varied as quantum fluids and solids, compressible flows, the propagation of light in optical fibres and the propagation of surface waves on thin liquid films. The general form of these equations in dimension d is

$$i \frac{\partial^{\alpha} \psi(r, t)}{\partial t} = -\frac{1}{2} \Delta \psi(r, t) + \sigma |\psi(r, t)|^{2} \psi(r, t), \tag{1.1}$$

where $t \geq 0$; $0 < \alpha \leq 1$; $\psi \in C^{2}([0; t])$; $\frac{\partial^{\alpha}}{\partial t}(\cdot)$ is the fractional derivative in time; $r = \sum_{i=1}^{d} x_i e_i$; $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, and the parameter σ characterizes the nonlinearity of the problem.

Many researchers have studied the Cauchy problem for such equations. Aftalion et al. [16] studied the nonlinear Schrödinger equation from Bose-Einstein condensates to super solids in 2010. In 2013, Duboscq [21] analyzed and simulated the deterministic and stochastic Schrödinger equations and applied them to rotating Bose-Einstein condensates. Audiard [20] obtained some results on the Schrödinger equation in 2020. Rahma [18] studied the Schrödinger equation with fractional derivative in 2020.

Recently, in 2023, Higelin [17] generated numerical solutions for certain types of linear and nonlinear equations of fractional order.

In this paper, we are interested in the solutions of some Schrödinger equations in d dimension $(d = 1, 2 \text{ or } 3)$ of fractional order in the sense of Caputo by the SBA plus method. The work is organized as follows: Section 2 is for the preliminaries, Section 3 is for the description and convergence of the SBA plus method, and Section 4 is for the applications. Section 5 provides the conclusion.

2. Preliminaries

In this section, we give a few definitions and one property. For other definitions, properties and theorems concerning fractional calculations, we refer to the following references: [18, 23, 24].

Definition 2.1. Consider $z \in \mathbb{C}$ such that $Re(z) > 0.$

The function defined by the following integral is called the *Gamma function*, denoted $\Gamma(z)$:

$$\Gamma(z) = \int_0^{+\infty} e^{-x} x^{z-1} dx. \tag{2.1}$$

Definition 2.2. For a complex number z, the *Mittag-Leffler function* $\mathbb{E}_\alpha$ is defined by

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0. \tag{2.2}$$

Property 2.1. Let α and t be two real numbers such that $0 < \alpha \leq 1.$ In [18], we have

(1) $\mathbb{E}_\alpha(it^\alpha) = \cos_\alpha(t^\alpha) + i\sin_\alpha(t^\alpha)$, where

$$\cos_\alpha(t^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{(t)^{2\alpha k}}{\Gamma(1+(2k)\alpha)},$$

$$\sin_\alpha(t^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{(t)^{2\alpha k+\alpha}}{\Gamma(1+(2k+1)\alpha)}.$$

(2) $|\mathbb{E}_\alpha(it^\alpha)| = 1.$

3. Description and Convergence of the SBA Plus Method

3.1. Description of the SBA plus method

For the description, we adopt the approach proposed by the inventor of the method in his book [4]. Thus it is done on the nonlinear Schrödinger equation of fractional order in dimension 1 and the technique adapts to nonlinear Schrödinger equations of fractional order in higher dimension.

In a Banach space E, consider the following nonlinear Schrödinger equation of fractional order in dimension 1:

$$i\frac{\partial^\alpha \psi(x, t)}{\partial t} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x, t) + \sigma|\psi(x, t)|^2\psi(x, t), \tag{3.1}$$

where $t \geq 0$; $0 < \alpha \leq 1$; $\psi \in C^2([0; t])$; $\frac{\partial^\alpha}{\partial t}(\cdot)$ the fractional derivative in time.

Multiplying by the conjugate of i, equation (3.1) becomes

$$\frac{\partial^\alpha \psi(x, t)}{\partial t} = \frac{i}{2}\frac{\partial^2}{\partial x^2}\psi(x, t) - i\sigma|\psi(x, t)|^2\psi(x, t). \tag{3.2}$$

Set

$$\begin{cases} L\psi = \dfrac{\partial^{\alpha}}{\partial t}\psi, \\ R\psi = \dfrac{i}{2}\dfrac{\partial^2}{\partial x^2}\psi, \\ N\psi = -i\sigma|\psi|^2\psi, \\ L^{-1}\psi = \mathcal{I}^{\alpha}\psi, \end{cases}$$

where L is an Adomian invertible linear operator, R is the remainder of the linear operator, N is a nonlinear operator, $\mathcal{D}^{\alpha}(\cdot) = \dfrac{\partial^{\alpha}(\cdot)}{\partial t}$ is the derivative in the Caputo sense and $\mathcal{I}^{\alpha}(\cdot)$ is the integral in the Riemann-Liouville sense.

We obtain

$$L\psi = R\psi + N\psi. \tag{3.3}$$

By applying L^{-1} to (3.3), we obtain the canonical Adomian form:

$$\psi = \beta + L^{-1}(R(\psi)) + L^{-1}(N(\psi)), \tag{3.4}$$

where β is such that $L\beta = 0$ and $L^{-1}R$ is a contracting operator. Applying the method of successive approximations to (3.4), we obtain

$$\psi^k = \psi^k(0) + L^{-1}(R(\psi^{k-1})) + L^{-1}(N(\psi^{k-1})); \quad k \geq 1. \tag{3.5}$$

Solving (3.5) by the method of approximations consists in determining at each iteration $(k = 1; 2; ...)$ approximate solutions $\psi^1, \psi^2, ..., \psi^k$, which form a series.

Posing

$$\psi^k = \sum_{n=0}^{+\infty} \psi_n^k, \tag{3.6}$$

we derive the following algorithm SBA plus:

$$\begin{cases} \psi_0^k = \psi^k(0)(t) + L^{-1}(N(\psi^{k-1})), \; k \geq 1, \\ \psi_{n+1}^k = L^{-1}(R(\psi_n^k)), \; n \geq 0. \end{cases} \tag{3.7}$$

Explicitly, the development of the algorithm (3.7) consists in first calculating the terms of the sequence $(\psi_n^k)_n$ for $k \geq 1$, and deduce ψ^k if the series $\psi^k = \sum_{n=0}^{+\infty} \psi_n^k$ converges in E.

First iteration

For $k = 1$, we calculate ψ^1 using the algorithm

$$\begin{cases} \psi_0^1 = \psi^1(0) + L^{-1}(N(\psi^0)), \\ \psi_{n+1}^1 = L^{-1}(R(\psi_n^1)); \; n \geq 0. \end{cases} \tag{3.8}$$

By applying $N(\psi^0) = 0$ to (3.8), we obtain

$$\begin{cases} \psi_0^1 = \psi^1(0), \\ \psi_{n+1}^1 = L^{-1}(R(\psi_n^1)); \; n \geq 0. \end{cases} \tag{3.9}$$

If the series $\left(\sum_{n=0}^{+\infty} \psi_n^1\right)$ is convergent, then we obtain: $\psi^1 = \sum_{n=0}^{+\infty} \psi_n^1$ which is an approximate solution of equation (3.1) in step 1.

Second iteration

For $k = 2$, we calculate ψ^2 using the algorithm

$$\begin{cases} \psi_0^2 = \psi^2(0) + L^{-1}(N(\psi^1)), \\ \psi_{n+1}^2 = L^{-1}(R(\psi_n^2)); \; n \geq 0. \end{cases} \tag{3.10}$$

We then evaluate $N\psi^1$. If $N\psi^1 = 0$, then ψ^1 is the general solution of the problem (3.1). Otherwise, if possible, we replace the initial problem by an equivalent transformation, with *overlineN* the new nonlinear term, so that by repeating the algorithm, we can obtain $\overline{N}\psi^1 = 0$. ψ being a complex number, we write

$$\psi(x, t) = u(x, t) + iv(x, t). \tag{3.11}$$

Replacing (3.11) in (3.2), we obtain

$$\mathcal{D}^{\alpha}(u + iv) = \frac{i}{2}\left(\frac{\partial^2(u + iv)}{\partial x^2}\right) - i\sigma|(u + iv)|^2(u + iv). \tag{3.12}$$

By transforming (3.12), we obtain

$$\begin{cases} \mathcal{D}^{\alpha}v = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} - u\sigma(u^2 + v^2), \\ \mathcal{D}^{\alpha}u = -\dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + v\sigma(u^2 + v^2). \end{cases} \tag{3.13}$$

Posing

$$\begin{cases} L(\cdot) = \mathcal{D}^{\alpha}(\cdot), \\ L^{-1}(\cdot) = I^{\alpha}(\cdot), \\ R_1(u, v) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}, \\ R_2(u, v) = -\dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}, \\ N_1(u, v) = -u\sigma(u^2 + v^2), \\ N_2(u, v) = v\sigma(u^2 + v^2), \end{cases}$$

equation (3.13) becomes

$$\begin{cases} Lv = R_1(u, v) + N_1(u, v), \\ Lu = R_2(u, v) + N_2(u, v). \end{cases} \tag{3.14}$$

Composing equation (3.14) by L^{-1}, we have

$$\begin{cases} L^{-1}Lv = L^{-1}R_1(u, v) + L^{-1}N_1(u, v), \\ L^{-1}Lu = L^{-1}R_2(u, v) + L^{-1}N_2(u, v), \end{cases} \tag{3.15}$$

else

$$L^{-1}Lu = u - u(0) \quad \text{and} \quad L^{-1}Lv = v - v(0),$$

$$\begin{cases} v = v(0) + L^{-1}R_1(u, v) + L^{-1}N_1(u, v), \\ u = u(0) + L^{-1}R_2(u, v) + L^{-1}N_2(u, v). \end{cases} \tag{3.16}$$

We have the following approximation:

$$\begin{cases} v^k = v^k(0) + L^{-1}R_1(u^k, v^k) + L^{-1}N_1(u^{k-1}, v^{k-1}), \\ u^k = u^k(0) + L^{-1}R_2(u^k, v^k) + L^{-1}N_2(u^{k-1}, v^{k-1}), \end{cases} \tag{3.17}$$

$$u^k = \sum_{n=0}^{+\infty} u_n^k \quad \text{and} \quad v^k = \sum_{n=0}^{+\infty} v_n^k.$$

We, therefore, determine the auxiliary unknowns u_n^k and v_n^k using the following SBA plus algorithm:

$$\begin{cases} v_0^k = v^k(0) + L^{-1}N_1(u^{k-1}, v^{k-1}), \\ u_0^k = u^k(0) + L^{-1}N_2(u^{k-1}, v^{k-1}), \\ v_{n+1}^k = L^{-1}R_1(u_n^k, v_n^k), \\ u_{n+1}^k = L^{-1}R_2(u_n^k, v_n^k), \end{cases} \quad k \geq 1 \quad \text{and} \quad n \geq 0. \tag{3.18}$$

The main unknowns are $\psi_n^k = u_n^k + iv_n^k$.

3.2. Convergence

For the convergence, we refer the reader to [4, 5, 10].

4. Applications

Example 4.1. Consider the following Schrödinger model in dimension 1:

$$\begin{cases} i\mathcal{D}^\alpha\psi + \dfrac{1}{2}\dfrac{\partial^2\psi}{\partial x^2} + (\sin^2 x)\psi - |\psi|^2\psi = 0, \\ \psi(x, 0) = \sin x, \end{cases} \tag{4.1}$$

where $t \geq 0$; $0 < \alpha \leq 1$; $\psi \in C^2([0; t])$; $\mathcal{D}^{\alpha}(\cdot)$ the derivative in the Caputo sense; and $\mathcal{I}^{\alpha}(\cdot)$ the integral in the Riemann-Liouville sense.

Note that

$$\psi(x, t) = u(x, t) + iv(x, t). \tag{4.2}$$

By replacing (4.2) in (4.1) and after the transformation, we obtain

$$\begin{cases} \mathcal{D}^{\alpha} v = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} - u(u^2 + v^2 - \sin^2 x), \\ \mathcal{D}^{\alpha} u = -\dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + v(u^2 + v^2 - \sin^2 x). \end{cases} \tag{4.3}$$

Posing

$$\begin{cases} \begin{cases} L\psi = \dfrac{\partial^{\alpha}}{\partial t}\psi, \\ R\psi = \dfrac{i}{2}\dfrac{\partial^2}{\partial x^2}\psi, \\ N\psi = (\sin^2 x)\psi - |\psi|^2\psi, \end{cases} \\ \begin{cases} L(\cdot) = D^{\alpha}(\cdot), \\ L^{-1}(\cdot) = I^{\alpha}(\cdot), \\ R_1(u, v) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}, \\ R_2(u, v) = -\dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}, \\ N_1(u, v) = -u(u^2 + v^2 - \sin^2 x), \\ N_2(u, v) = v(u^2 + v^2 - \sin^2 x), \end{cases} \end{cases}$$

we obtain the following algorithm SBA plus:

$$\begin{cases} \psi_0^k = \psi^k(0) + L^{-1}(N(\psi^{k-1})), \\ \psi_{n+1}^k = L^{-1}(R(\psi_n^k)), \end{cases} \quad k \geq 1, n \geq 0, \tag{4.4}$$

$$\begin{cases} v_0^k = v^k(0) + L^{-1}N_1(u^{k-1}, v^{k-1}), \\ u_0^k = u^k(0) + L^{-1}N_2(u^{k-1}, v^{k-1}), \\ v_{n+1}^k = L^{-1}R_1(u_n^k, v_n^k), \\ u_{n+1}^k = L^{-1}R_2(u_n^k, v_n^k), \end{cases} \quad k \geq 1,\ n \geq 0. \tag{4.5}$$

For $k = 1$,

$$\begin{cases} \psi_0^1 = \psi^1(0) + L^{-1}(N(\psi^0)), \\ \psi_{n+1}^1 = L^{-1}(R(\psi_n^1)), \end{cases} \tag{4.6}$$

$$\begin{cases} v_0^1 = v^1(0) + L^{-1}N_1(u^0, v^0), \\ u_0^1 = u^1(0) + L^{-1}N_2(u^0, v^0), \\ v_{n+1}^1 = L^{-1}R_1(u_n^1, v_n^1), \\ u_{n+1}^1 = L^{-1}R_2(u_n^1, v_n^1). \end{cases} \tag{4.7}$$

Using Picard's principle,

$$N(\psi^0) = 0, \quad N_1(u^0, v^0) = 0 \quad \text{and} \quad N_2(u^0, v^0) = 0.$$

We obtain

$$\begin{cases} v_0^1 = v^1(0) = 0, \\ u_0^1 = u^1(0) = \sin x, \end{cases} \tag{4.8}$$

$$\begin{cases} \begin{cases} v_1^1 = I^\alpha R_1(u_0^1, v_0^1) \\ u_1^1 = I^\alpha R_2(u_0^1, v_0^1) \end{cases} \\ \begin{cases} v_2^1 = I^\alpha R_1(u_1^1, v_1^1) \\ u_2^1 = I^\alpha R_2(u_1^1, v_1^1) \end{cases} \\ \quad\vdots \\ \begin{cases} v_n^1 = I^\alpha R_1(u_{n-1}^1, v_{n-1}^1) \\ u_n^1 = I^\alpha R_2(u_{n-1}^1, v_{n-1}^1), \end{cases} \end{cases} \tag{4.9}$$

$$\begin{cases}\begin{cases} v_1^1 = \dfrac{-\sin xt^{\alpha}}{2\Gamma(\alpha+1)} \\ u_1^1 = 0 \end{cases} \\ \begin{cases} v_2^1 = 0 \\ u_2^1 = \dfrac{-\sin xt^{2\alpha}}{4\Gamma(2\alpha+1)} \end{cases} \\ \quad\vdots \\ \begin{cases} v_{2n+1}^1 = (-1)^{n+1} \dfrac{\sin xt^{(2n+1)\alpha}}{2^{2n+1}\Gamma((2n+1)\alpha+1)};\ n \geq 0 \\ u_{2n+1}^1 = 0 \end{cases} \\ \begin{cases} v_{2n}^1 = 0 \\ u_{2n}^1 = (-1)^n \dfrac{\sin xt^{2n\alpha}}{2^{2n}\Gamma(2n\alpha+1)};\ n \geq 0. \end{cases} \end{cases} \tag{4.10}$$

For any natural number n, we have

$$\psi_n^1 = u_n^1 + iv_n^1,$$

$$= \sin x \frac{\left(-\frac{1}{2}i\right)^n t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Therefore, the approximate solution to the problem at the first iteration is

$$\psi^1 = \sum_{n=0}^{+\infty} \psi_n^1 = \sum_{n=0}^{+\infty} \sin x \frac{\left(-\frac{1}{2}i\right)^n t^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$= \sin x \mathbb{E}\left(-\frac{1}{2}it^{\alpha}\right). \tag{4.11}$$

For $k = 2$,

$$\begin{cases} \psi_0^2 = \psi^2(0) + L^{-1}(N(\psi^1)), \\ \psi_{n+1}^2 = L^{-1}(R(\psi_n^2)), \end{cases} \tag{4.12}$$

$$\begin{cases} v_0^2 = v^2(0) + L^{-1}N_1(u^1, v^1), \\ u_0^2 = u^2(0) + L^{-1}N_2(u^1, v^1), \\ v_{n+1}^2 = L^{-1}R_1(u_n^2, v_n^2), \\ u_{n+1}^2 = L^{-1}R_2(u_n^2, v_n^2). \end{cases} \tag{4.13}$$

Now, we evaluate $N\psi^1$:

$$N\psi^1 = (\sin^2 x)\psi^1 - |\psi^1|^2\psi^1,$$

$$= (\sin^3 x)\mathbb{E}\left(-\frac{1}{2}it^\alpha\right) - \sin^3 x\,\mathbb{E}\left(-\frac{1}{2}it^\alpha\right),$$

$$N\psi^1 = 0 \text{ implies } N_1(u^1, v^1) = 0 \text{ and } N_2(u^1, v^1) = 0.$$

Hence the exact solution to the problem is

$$\psi(x, t) = \sin x\,\mathbb{E}\left(-\frac{1}{2}it^\alpha\right).$$

Example 4.2. Consider the following Schrödinger model in dimension 2:

$$\begin{cases} i\mathcal{D}^\alpha\psi + \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - (1 - \sin^2 x \sin^2 y)\psi - |\psi|^2\psi = 0, \\ \psi(x, y, 0) = \sin x \sin y, \end{cases} \tag{4.14}$$

where $t \geq 0$; $0 < \alpha \leq 1$; $\psi \in C^2([0; t])$; $\mathcal{D}^\alpha(\cdot)$ the derivative in the Caputo sense; and $\mathcal{I}^\alpha(\cdot)$ the integral in the Riemann-Liouville sense.

Consider

$$\psi(x, y, t) = u(x, y, t) + iv(x, y, t). \tag{4.15}$$

Replacing (4.15) in (4.14), and after transformation, we obtain

$$\begin{cases} \mathcal{D}^{\alpha} v = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} - u - u(u^2 + v^2 - \sin^2 x \sin^2 y), \\ \mathcal{D}^{\alpha} u = -\frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\frac{\partial^2 v}{\partial y^2} + v + v(u^2 + v^2 - \sin^2 x \sin^2 y). \end{cases} \tag{4.16}$$

Posing

$$\begin{cases} \begin{cases} L\psi = \frac{\partial^{\alpha}}{\partial t}\psi, \\ R\psi = \frac{i}{2}\left(\frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi\right) - i\psi, \\ N\psi = (\sin^2 x \sin^2 y)\psi - |\psi|^2\psi, \end{cases} \\ \begin{cases} L(\cdot) = D^{\alpha}(\cdot), \\ L^{-1}(\cdot) = I^{\alpha}(\cdot), \\ R_1(u, v) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} - u, \\ R_2(u, v) = -\frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\frac{\partial^2 v}{\partial y^2} + v, \\ N_1(u, v) = u(u^2 + v^2 - \sin^2 x \sin^2 y), \\ N_2(u, v) = v(u^2 + v^2 - \sin^2 x \sin^2 y), \end{cases} \end{cases}$$

we obtain the following algorithm SBA plus:

$$\begin{cases} \psi_0^k = \psi^k(0) + L^{-1}(N(\psi^{k-1})), \\ \psi_{n+1}^k = L^{-1}(R(\psi_n^k)), \end{cases} \quad k \geq 1,\ n \geq 0, \tag{4.17}$$

$$\begin{cases} v_0^k = v^k(0) + L^{-1}N_1(u^{k-1}, v^{k-1}), \\ u_0^k = u^k(0) + L^{-1}N_2(u^{k-1}, v^{k-1}), \\ v_{n+1}^k = L^{-1}R_1(u_n^k, v_n^k), \\ u_{n+1}^k = L^{-1}R_2(u_n^k, v_n^k), \end{cases} \quad k \geq 1,\ n \geq 0. \tag{4.18}$$

For $k = 1$,

$$\begin{cases} \begin{cases} v_0^1 = 0 \\ u_0^1 = \sin x \sin y \end{cases} \\ \begin{cases} v_1^1 = \dfrac{-2\sin x \sin yt^{\alpha}}{\Gamma(\alpha+1)} \\ u_1^1 = 0 \end{cases} \\ \begin{cases} v_2^1 = 0 \\ u_2^1 = \dfrac{-4\sin x \sin yt^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases} \\ \quad \vdots \\ \begin{cases} v_{2n+1}^1 = (-1)^{n+1} \dfrac{2^{2n+1}\sin x \sin yt^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}; \ n \geq 0 \\ u_{2n+1}^1 = 0 \end{cases} \\ \begin{cases} v_{2n}^1 = 0 \\ u_{2n}^1 = (-1)^n \dfrac{2^{2n}\sin x \sin yt^{2n\alpha}}{\Gamma(2n\alpha+1)}; \ n \geq 0. \end{cases} \end{cases} \tag{4.19}$$

For any natural number n, we have

$$\psi_n^1 = u_n^1 + iv_n^1,$$

$$= \sin x \sin y \frac{(-2i)^n t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Therefore, the approximate solution to the problem at the first iteration is

$$\psi^1 = \sum_{n=0}^{+\infty} \psi_n^1 = \sum_{n=0}^{+\infty} \sin x \sin y \frac{(-2i)^n t^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$= \sin x \sin y \, \mathbb{E}(-2it^{\alpha}). \tag{4.20}$$

For $k = 2$,

$$\begin{cases} \psi_0^2 = \psi^2(0) + L^{-1}(N(\psi^1)), \\ \psi_{n+1}^2 = L^{-1}(R(\psi_n^2)), \end{cases} \tag{4.21}$$

$$\begin{cases} v_0^2 = v^2(0) + L^{-1}N_1(u^1, v^1), \\ u_0^2 = u^2(0) + L^{-1}N_2(u^1, v^1), \\ v_{n+1}^2 = L^{-1}R_1(u_n^2, v_n^2), \\ u_{n+1}^2 = L^{-1}R_2(u_n^2, v_n^2). \end{cases} \tag{4.22}$$

Now, we evaluate $N\psi^1$:

$$N\psi^1 = (\sin^2 x \sin^2 y)\psi^1 - |\psi^1|^2 \psi^1$$

$$= (\sin^3 x \sin^3 y)\mathbb{E}(-2it^{\alpha}) - \sin^3 x \sin^3 y \, \mathbb{E}(-2it^{\alpha})$$

$$N\psi^1 = 0 \text{ implies } N_1(u^1, v^1) = 0 \text{ and } N_2(u^1, v^1) = 0.$$

Hence the exact solution to the problem is

$$\psi(x, y, t) = \sin x \sin y \, \mathbb{E}(-2it^{\alpha}).$$

Example 4.3. Consider the following Schrödinger model in dimension 3:

$$\begin{cases} i\mathcal{D}^{\alpha}\psi + \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - (1 - \sin^2 x \sin^2 y \sin^2 z)\psi - |\psi|^2\psi = 0, \\ \psi(x, y, z, 0) = \sin x \sin y \sin z, \end{cases} \tag{4.23}$$

where $t \geq 0$; $0 < \alpha \leq 1$; $\psi \in C^2([0; t])$; $\mathcal{D}^\alpha(\cdot)$ the derivative in the Caputo sense; and $\mathcal{I}^\alpha(\cdot)$ the integral in the Riemann-Liouville sense.

Consider

$$\psi(x, y, z, t) = u(x, y, z, t) + iv(x, y, z, t). \tag{4.24}$$

Replacing (4.24) in (4.23), and after transformation, we obtain

$$\begin{cases} \mathcal{D}^\alpha v = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u}{\partial z^2} - u - u(u^2 + v^2 - \sin^2 x \sin^2 y \sin^2 z), \\ \mathcal{D}^\alpha u = -\frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}\frac{\partial^2 v}{\partial z^2} + v + v(u^2 + v^2 - \sin^2 x \sin^2 y \sin^2 z). \end{cases} \tag{4.25}$$

Posing

$$\begin{cases} \begin{cases} L\psi = \frac{\partial^\alpha}{\partial t}\psi, \\ R\psi = \frac{i}{2}\left(\frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi\right) - i\psi, \\ N\psi = (\sin^2 x \sin^2 y \sin^2 z)\psi - |\psi|^2\psi, \end{cases} \\ \begin{cases} L(\cdot) = \mathcal{D}^\alpha(\cdot), \\ L^{-1}(\cdot) = I^\alpha(\cdot), \\ R_1(u, v) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u}{\partial z^2} - u, \\ R_2(u, v) = -\frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + v, \\ N_1(u, v) = u(u^2 + v^2 - \sin^2 x \sin^2 y \sin^2 z), \\ N_2(u, v) = v(u^2 + v^2 - \sin^2 x \sin^2 y \sin^2 z), \end{cases} \end{cases}$$

we obtain the following algorithm SBA plus:

$$\begin{cases} \psi_0^k = \psi^k(0) + L^{-1}(N(\psi^{k-1})), \\ \psi_{n+1}^k = L^{-1}(R(\psi_n^k)), \end{cases} \quad k \geq 1,\ n \geq 0, \tag{4.26}$$

$$\begin{cases} v_0^k = v^k(0) + L^{-1}N_1(u^{k-1}, v^{k-1}), \\ u_0^k = u^k(0) + L^{-1}N_2(u^{k-1}, v^{k-1}), \\ v_{n+1}^k = L^{-1}R_1(u_n^k, v_n^k), \\ u_{n+1}^k = L^{-1}R_2(u_n^k, v_n^k), \end{cases} \quad k \geq 1,\ n \geq 0. \tag{4.27}$$

For $k = 1$,

$$\begin{cases} \begin{cases} v_0^1 = v^1(0) = 0 \\ u_0^1 = u^1(0) = \sin x \sin y \sin z \end{cases} \\ \begin{cases} v_1^1 = -\dfrac{5}{2}\dfrac{\sin x \sin y \sin z t^{\alpha}}{\Gamma(\alpha+1)} \\ u_1^1 = 0 \end{cases} \\ \begin{cases} v_2^1 = 0 \\ u_2^1 = \dfrac{-25 \sin x \sin y \sin z t^{2\alpha}}{4\Gamma(2\alpha+1)} \end{cases} \\ \quad \vdots \\ \begin{cases} v_{2n+1}^1 = (-1)^{n+1} \dfrac{\left(\frac{5}{2}\right)^{2n+1} \sin x \sin y \sin z t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)};\ n \geq 0 \\ u_{2n+1}^1 = 0 \end{cases} \\ \begin{cases} v_{2n}^1 = 0 \\ u_{2n}^1 = (-1)^n \dfrac{\left(\frac{5}{2}\right)^{2n} \sin x \sin y \sin z t^{2n\alpha}}{\Gamma(2n\alpha+1)};\ n \geq 0. \end{cases} \end{cases} \tag{4.28}$$

For any natural number n,

$$\psi_n^1 = u_n^1 + iv_n^1,$$

$$= \sin x \sin y \sin z \frac{\left(-\frac{5}{2}i\right)^n t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Consequently, the approximate solution of the problem at the first iteration is as follows:

$$\psi^1 = \sum_{n=0}^{+\infty} \psi_n^1 = \sum_{n=0}^{+\infty} \sin x \sin y \sin z \frac{\left(-\frac{5}{2}i\right)^n t^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$= \sin x \sin y \sin z \mathbb{E}\left(-\frac{5}{2}it^{\alpha}\right). \tag{4.29}$$

For $k = 2$,

$$\begin{cases} \psi_0^2 = \psi^2(0) + L^{-1}(N(\psi^1)), \\ \psi_{n+1}^2 = L^{-1}(R(\psi_n^2)), \end{cases} \tag{4.30}$$

$$\begin{cases} v_0^2 = v^2(0) + L^{-1}N_1(u^1, v^1), \\ u_0^2 = u^2(0) + L^{-1}N_2(u^1, v^1), \\ v_{n+1}^2 = L^{-1}R_1(u_n^2, v_n^2), \\ u_{n+1}^2 = L^{-1}R_2(u_n^2, v_n^2). \end{cases} \tag{4.31}$$

Now, we evaluate $N\psi^1$:

$$N\psi^1 = (\sin^2 x \sin^2 y \sin^2 z)\psi^1 - |\psi^1|^2\psi^1$$

$$= (\sin^3 x \sin^3 y \sin^3 z)\mathbb{E}\left(-\frac{5}{2}it^{\alpha}\right) - \sin^3 x \sin^3 y \sin^3 z\mathbb{E}\left(-\frac{5}{2}it^{\alpha}\right)$$

$N\psi^1 = 0$ implies $N_1(u^1, v^1) = 0$ and $N_2(u^1, v^1) = 0.$

So the exact solution to the problem is as follows:

$$\psi(x, y, z, t) = \sin x \sin y \sin z \mathbb{E}\left(-\frac{5}{2} it^{\alpha}\right).$$

5. Conclusion

The SBA plus method has enabled us to obtain exact solutions to fractional nonlinear Schrödinger equations in dimension 1, 2 or 3. This technique is used to overcome the difficulties associated with the calculation of Adomian polynomials.

References

[1] A. Kadem and D. Baleanu, Analytical method based on Walsh function combined with orthogonal polynomial for fractional transport equation, Commun. Nonlinear Sci. Numer. Simul. 15(3) (2010), 491-501.

[2] A. Kadem and D. Baleanu, On fractional coupled Whitham-Broer-Kaup equations, Rom. J. Phys. 56(5) (2011), 629-635.

[3] A. R. Nabulsi, The fractional white dwarf hydrodynamical nonlinear differential and emergence of quark stars, Appl. Math. Comput. 218(6) (2011), 2837-2849.

[4] Blaise SOME, Méthode SBA de résolution des modèles mathématiques en environnement, Éditions Universitaires Européennes, 2018.

[5] B. Abbo, Nouvel algorithme numérique de résolution des équations différentielles ordinaries (EDO) et des équations aux dérivées partielles (EDP) non linéaires, Thèse de Doctorat unique, Université de Ouagadougou, UFR/SEA, Département Mathématique et Informatique, Burkina Faso, 2007.

[6] B. ABBO, O. SO, G. BARRO and B. SOME, A new numerical algorithm for solving nonlinear partial differential equations with initial and boundary conditions, Far East J. Appl. Math. 28(1) (2007), 37-52.

[7] B. Zheng, (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics, Commun. Theor. Phys. 58 (2012), 623-630.

[8] Fatemeh NOROUZI and Gaston M. N'GUÉRÉKATA, A new study of fractional order financial system via homotopy analysis, Analele Universitatii Oradea, Fasc. Matematica, Tom XXVII(1) (2020), 141-152.

[9] G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, Math. Comput. Modelling 13(7) (1990), 17-43.

[10] Germain KABORE, KÉRÉ Moumini, Windjiré SOME, Ousséni SO and Blaise SOME, Solving some fractional equations, in the sense of Riemann-Liouville, of Navier-Stokes by the numerical method SBA plus, International Journal of Numerical Methods and Applications 23(2) (2023), 209-228. http://dx.doi.org/10.17654/0975045223012.

[11] G. KABORE, W. SOME, M. KÉRÉ, O. SO and B. SOME, Solving some fractional ordinary differential equations by SBA method, Journal of Mathematics Research 15(1) (2023), 47-56. doi:10.5539/jmr.v15n1p47.

[12] H. Jafari, A. Kadem, D. Baleanu and T. Yilmaz, Solutions of the fractional Davey-Stewartson equations with variational iteration method, Rom. Rep. Phys. 64(2) (2012), 337-346.

[13] H. Jafari, A. Kadem, D. Baleanu and T. Yilmaz, Variational iteration method for a fractional-order Brusselator system, Abstr. Appl. Anal. (2014), 1-6. Article ID 496323.

[14] J. Singh, D. Kumar and A. Kilicman, Numerical solutions of nonlinear, fractional partial differential equations arising in spatial diffusion of biological populations, Abstr. Appl. Anal. (2014), 1-12. Article ID 535793.

[15] J. T. Katsikadelis, Nonlinear dynamic analysis of viscoelastic membranes described with fractional differential models, J. Theoret. Appl. Mech. 50(3) (2012), 743-753.

[16] Amandine Aftalion, Jean Dalibard and Christophe Josserand, Equation de Schrödinger non lineaire: des condensats de Bose Einstein aux supersolides, 2010.

[17] NSIMBA NDAMBA Higelin, Generate numerical solutions for certain types of linear and non-linear equations of fractional order, Marien Ngouabi University, Congo, Dissertation defended on 14/01/2023.

[18] RAMADANI RAHMA, Étude de l'équation de Schrödinger avec dérivée fractionnaire, Université AMAR TELIDJI LAGHOUAT, Algérie, Memoire de Master soutenu le 17/09/2020.

[19] KORICHI ZINEB, Étude de systèmes statistiques dans diverses dimensions d’espace basée sur mécanique quantique fractionnaire, Université de KASDI MERBAH OUARGLA, Algérie, Thèse soutenue le 30/11/2016.

[20] Corentin Audiard, Quelques résultats autour de l'équation de Schrödinger, Mémoire de synthèse en vue d'obtenir une habilitation à diriger des recherches, Université de Sorbonne, Soutenue le 08/10/2020.

[21] ROMAIN DUBOSCQ, Analyse et simulation d'équations de Schrödinger déterministes et Stochastiques, Applications aux Condensats de Bose-Einstein en rotation, Université de Lorraine, Thèse soutenue le 28/11/2013.

[22] Michael Ruzhansky, Niyaz Tokmagambetov and Berikbol T. Torebek, On a non-local problem for a multi-term fractional diffusion-wave equation, 2020. arxiv: 1812.01336V3(math.AP).

[23] KHALOUTA Ali, Résolution des équations aux dérivées partielles linéaires et non linéaires moyennant des approches analytiques, Extension aux cas d'EDP d'ordre fractionnaire, Université Ferhat ABBAS SETIF1, Algerie, Thèse de doctorat soutenue le 14/12/19.

[24] NEFNAF ASMA KERROUCHE IBTISSEM, Résolution des équations différentielles partielles d'ordre fractionnaire moyennant des approches semi-analytique, Université Mohamed El-Bachir El-Ibrahimi Bordj Bou Arreridj, Algérie, Mémoire de Master soutenu, 2021.