Advances in Differential Equations and Control Processes
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https://pphmjopenaccess.com
https://doi.org/10.17654/0974324324019
Volume 31, Number 3, 2024, Pages 347-355 **P-ISSN: 0974-3243**

HOMOTOPY PERTURBATION METHOD FOR SOLVING A NONLINEAR SYSTEM FOR AN EPIDEMIC

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Received: December 8, 2023; Accepted: March 2, 2024
2020 Mathematics Subject Classification: 34A99.
Keywords and phrases: ordinary differential equation, homotopy perturbation method, initial value problem, series solution.

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How to cite this article: Nada A. M. Alshomrani, Weam G. Alharbi, Ibtisam M. A. Alanazi, Lujain S. M. Alyasi, Ghadi N. M. Alrefaei, Seada A. Al'amri and Asmaa H. Q. Alanzi, Homotopy perturbation method for solving a nonlinear system for an epidemic, Advances in Differential Equations and Control Processes 31(3) (2024), 347-355.
https://doi.org/10.17654/0974324324019

Published Online: June 11, 2024

Abstract

This paper solves the SIR-epidemic model utilizing the homotopy perturbation method (HPM). The HPM is applied in a different way in contrast to the HPM in the literature. The current approach uses a new canonical form for the system of the SIR-epidemic. The analytic solution is obtained and compared with the published one, in addition, to the Runge-Kutta numerical method. The results show better accuracy than the corresponding ones.

1. Introduction

During the spread of COVID-19 in the beginning of 2020, a considerable attention was given to modeling the infectious diseases [1-3]. So, various classical and fractional mathematical models have been formulated [4-8] to study the dispersion of COVID-19. The susceptible-infected-recovered (SIR) model is a famous model which has been used to describe the mechanism of numerous diseases. This model becomes fundamental to describe a number of epidemics by means of linear/nonlinear ordinary differential equations (ODEs). The simplest SIR-model has been formulated in [5], given by the nonlinear system:

$$\frac{dR}{d\tau} = I(\tau), \tag{1}$$

$$\frac{dI}{d\tau} = \sigma[1 - R(\tau) - I(\tau)]I(\tau) - I(\tau), \tag{2}$$

where $\tau = t/T$, t is the time in days and T is the time of transmission of the virus. The infected and the recovered individuals are represented by $I(t)$ and $R(t)$, respectively, where $S(t)$ denotes the susceptible individuals: $S(t) = 1 - R(t) - I(t)$ while σ is the transmission rate (physical contact number between susceptible and infected individuals). The model is subjected to the initial conditions (ICs) [6]:

$$R(0) = A, \quad I(0) = B. \tag{3}$$

Adomian's approach [9-18] and the homotopy perturbation method (HPM) [19-22] are familiar methods to find approximate solutions of linear/nonlinear ODEs. The two series solutions obtained by these methods are identical when applied on the same canonical form of the given ODE. Although, the HPM has been implemented in [6] to solve the SIR-model, we use it a different way. The present approach is based on setting the problem (1)-(2) in a new canonical form. Such canonical form allows us to obtain different approximations for the present system. The accuracy of the new approximations is validated through comparison with the previous approximations using HPM in [6], and the also with the Runge-Kutta numerical method. It is concluded that the present approach is much accurate than the previous one [6] for a certain range of the transmission rate σ.

2. Application of the HPM

The system can be rewritten as

$$R'(\tau) = I(\tau), \tag{4}$$

$$I(\tau) = (\sigma - 1)I(\tau) - I(\tau)[R(\tau) + I(\tau)]. \tag{5}$$

Implementing the canonical form:

$$R'(\tau) = qI(\tau), \tag{6}$$

$$I(\tau) = (\sigma - 1)I(\tau) - qI(\tau)[R(\tau) + I(\tau)], \tag{7}$$

where q $(0 < q \leq 1)$ is the auxiliary parameter of the HPM and

$$R(\tau) = \sum_{n=0}^{\infty} q^n R_n(\tau), \quad I(\tau) = \sum_{n=0}^{\infty} q^n I_n(\tau). \tag{8}$$

Inserting equations (8) into (6-7) and using the ICs (3), we obtain

$$R_0'(\tau) = 0, \quad I_0'(\tau) - (\sigma - 1)I_0(\tau) = 0, \quad R_0(0) = A, \quad I_0(0) = B, \tag{9}$$

and

$$R'_{n+1}(\tau) = I_n(\tau), \quad R_{n+1}(0) = 0, \tag{10}$$

$$I'_{n+1}(\tau) - (\sigma - 1)I_{n+1}(\tau) = -\sigma\sum_{k=0}^{n} I_{n-k}(\tau)[I_k(\tau) + R_k(\tau)],$$

$$I_{n+1}(0) = 0. \tag{11}$$

Integrating (10) and (11), we have

$$R_{n+1}(\tau) = \int_0^\tau I_n(\mu)d\mu, \tag{12}$$

$$I_{n+1}(\tau) = -\sigma e^{(\sigma-1)t}\int_0^\tau e^{-(\sigma-1)\mu}\sum_{k=0}^{n} I_{n-k}(\mu)[I_k(\mu) + R_k(\mu)]d\mu, \; n \geq 0. \tag{13}$$

From (9), we can obtain $R_0(\tau)$ and $I_0(\tau)$ as

$$R_0(\tau) = A, \quad I_0(\tau) = Be^{(\sigma-1)\tau}. \tag{14}$$

Employing (10) and (11) when $n = 0$, we get

$$R_1(\tau) = \frac{B}{\sigma - 1}[e^{(\sigma-1)\tau} - 1], \tag{15}$$

$$I_1(\tau) = \frac{B^2\sigma}{\sigma - 1}[e^{2(\sigma-1)\tau} - e^{(\sigma-1)\tau}] - \sigma AB\tau e^{(\sigma-1)\tau}. \tag{16}$$

For $n = 1$, we have

$$R_2(\tau) = -\frac{B\sigma(2A + B)}{2(\sigma-1)^2} - \frac{B^2\sigma}{2(\sigma-1)^2}e^{2(\sigma-1)\tau}$$

$$- \frac{B\sigma}{2(\sigma-1)^2}(-2A - 2B - 2A\tau + 2A\sigma\tau)e^{2(\sigma-1)\tau}, \tag{17}$$

$$I_2(\tau) = -\frac{B^3\sigma^2}{(\sigma-1)^2}e^{3(\sigma-1)\tau} + \frac{B\sigma}{2(\sigma-1)^2}(-2B - 2AB\sigma - 4B^2\sigma$$

$$+ 4AB\sigma(\sigma-1)\tau)e^{2(\sigma-1)\tau} + \frac{B\sigma}{2(\sigma-1)^2}$$

$$\times(2B - 2B\tau + 2AB\sigma + 2B^2\sigma + 2B\sigma\tau + 2AB\sigma\tau$$

$$+ A^2\tau^2\sigma - 2AB\sigma^2\tau - 2A^2\tau^2\sigma^2 + A^2\sigma^3\tau^2)e^{(\sigma-1)\tau}. \tag{18}$$

Similarly, we can obtain higher-order components. Hence, the approximate solution is given, as $q \to 1$, by

$$R(\tau) = A + \frac{B}{\sigma-1}[e^{(\sigma-1)\tau} - 1] - \frac{B\sigma(2A+B)}{2(\sigma-1)^2} - \frac{B^2\sigma}{2(\sigma-1)^2}e^{2(\sigma-1)\tau}$$

$$- \frac{B\sigma}{2(\sigma-1)^2}(-2A - 2B - 2A\tau + 2A\sigma\tau)e^{2(\sigma-1)\tau} + \cdots, \tag{19}$$

and

$$I(\tau) = Be^{(\sigma-1)\tau} + \frac{B^2\sigma}{\sigma-1}[e^{2(\sigma-1)\tau} - e^{(\sigma-1)\tau}]$$

$$- \sigma AB\tau e^{(\sigma-1)\tau} - \frac{B^3\sigma^2}{(\sigma-1)^2}e^{3(\sigma-1)\tau} + \frac{B\sigma}{2(\sigma-1)^2}$$

$$\times(-2B - 2AB\sigma - 4B^2\sigma + 4AB\sigma(\sigma-1)\tau)e^{2(\sigma-1)\tau}$$

$$+ \frac{B\sigma}{2(\sigma-1)^2}e^{(\sigma-1)\tau}\times(2B - 2B\tau + 2AB\sigma + 2B^2\sigma + 2B\sigma\tau$$

$$+ 2AB\sigma\tau + A^2\tau^2\sigma - 2AB\sigma^2\tau - 2A^2\tau^2\sigma^2 + A^2\sigma^3\tau^2) + \cdots. \tag{20}$$

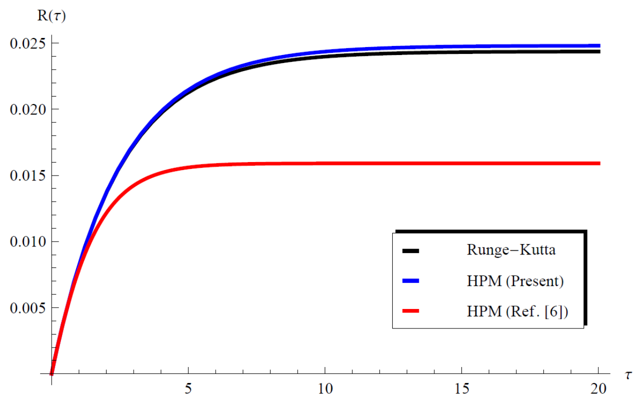

Figure 1. Comparison between the present HPM for $R(\tau)$ (equation (19)), the Runge-Kutta numerical solution, and the HPM in [6] for $A = 0$, $B = 0.01$, and $\sigma = 0.6$.

3. Validations

The objective of this section is to validate our results through comparison with the Runge-Kutta numerical method and the HPM in [6]. In [6], the authors used different canonical form of the HPM and accordingly they obtained the following approximations:

$$R(t) = A - Be^{-\tau} + B + \sigma B\left[-B(-\tau e^{-\tau} - e^{-\tau}) - \frac{1}{2}Be^{-2\tau} + Be^{-\tau} - \tau e^{-\tau} - e^{-\tau}\right] - \sigma B\left(\frac{3}{2}B - 1\right), \tag{21}$$

$$I(t) = Be^{-\tau} + e^{-\tau}[-\sigma B(B\tau - Be^{-\tau} - \tau) - \sigma B^2] - \frac{1}{2}\sigma Be^{-\tau}(4\sigma B^2 - 2\sigma B + 2B) - \frac{1}{2}\sigma Be^{-\tau}[4\sigma B^2\tau e^{-\tau} + 6\sigma B^2 e^{-\tau} - 2\sigma B^2 e^{-2\tau} - \sigma B^2\tau^2 - 2\sigma B^2\tau^{-2}\sigma Be^{-\tau} + 2\sigma B\tau + 2\sigma B\tau^2 - 4\sigma B\tau e^{-\tau} + 2B\tau - 2Be^{-\tau} - \sigma\tau^2]. \tag{22}$$

Figure 1 shows the comparison between the present HPM, the Runge-Kutta numerical solution, and the HPM [6] for $A = 0$, $B = 0.01$, and

$\sigma = 0.4$ for $R(\tau)$. Also, Figure 2 displays the comparison between the present HPM, the Runge-Kutta numerical solution, and the HPM for the same values of parameters for $I(\tau)$. The results show that our approach enjoys better accuracy in contrast to the HPM [6].

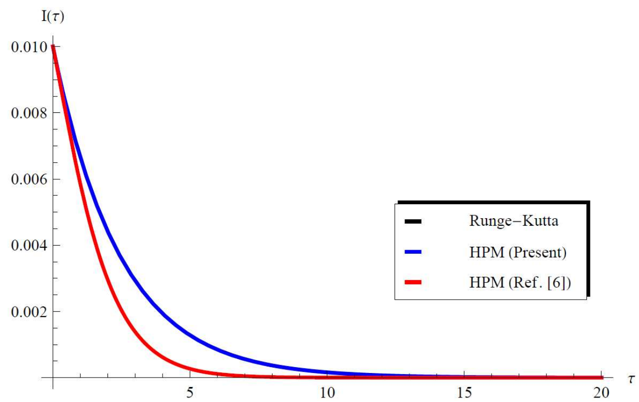

Figure 2. Comparison between the present HPM for $I(\tau)$ (equation (20)), the Runge-Kutta numerical solution, and the HPM in [6] for $A = 0$, $B = 0.01$, and $\sigma = 0.6$.

4. Conclusions

In this paper, the SIR-epidemic model was solved utilizing the homotopy perturbation method (HPM). The HPM was applied in a different way compared with the HPM in the literature. The current HPM uses a new canonical form for the system of the SIR-epidemic. The analytic solution was obtained and compared with the corresponding ones in [6] and also with the Runge-Kutta numerical method. The present results show better accuracy than the corresponding ones.

Acknowledgement

The authors extend their appreciation to the Deanship of Research and Graduate Studies at University of Tabuk for funding this work through Research Grant no.0028-1444-S.

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