Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0974324324018
Volume 31, Number 3, 2024, Pages 335-346 **P-ISSN: 0974-3243**

ANALYZING AND SIMULATING THE OSCILLATION IN MULTIDIMENSIONAL ODD COMPETITIVE SYSTEMS

Yury I. Brodsky

Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences
40 Vavilova str., Moscow 119333
Russia

Abstract

We consider a multidimensional odd competitive model, which is a generalization to a multidimensional (more than two dimensions) case of both the P. Verhulst logistic model and the A. Lotka and V. Volterra competition model.

1. Introduction

It is known that the trajectory evolution ends in a stable node, in one- (Verhulst [1]) and two-dimensional (Lotka – Volterra, [2]) models of competition. Starting from dimension 3, a stationary point can have the type

Received: April 19, 2024; Accepted: May 11, 2024
2020 Mathematics Subject Classification: 00A71.
Keywords and phrases: population models, multidimensional competition, oscillation, modeling, simulation.
How to cite this article: Yury I. Brodsky, Analyzing and simulating the oscillation in multidimensional odd competitive systems, Advances in Differential Equations and Control Processes 31(3) (2024), 335-346. https://doi.org/10.17654/0974324324018

Published Online: May 28, 2024

of center or focus, there may be oscillations of the trajectory around the stationary point. This happens if the coefficients of the double standard matrix are related by a non-transitive order, as in the game “rock, paper, scissors.” For example, in the three-dimensional case, there are populations *A*, *B* and *C*, *A* as tolerant for *B* and intolerant for *C*, *B* as tolerant for *C* and intolerant for *A*, and finally, *C* as tolerant for *A* and intolerant for *B*.

In this paper, we investigate multidimensional $(n \geq 3,\ n \text{ is odd})$ competition systems for the oscillation occurrence.

2. A Bit of Linear Algebra and Matrix Theory

Let $1 > d > 0$. We will consider square $n \times n$ matrices:

$$\begin{Vmatrix} 1 & 1+d & \cdots & 1-d \\ 1-d & 1 & \cdots & 1+d \\ \cdots & \cdots & \cdots & \cdots \\ 1+d & 1-d & \cdots & 1 \end{Vmatrix}, \tag{1}$$

where n is an odd, and device-like even matrix:

$$\begin{Vmatrix} 1 & 1+d & \cdots & 1+d \\ 1-d & 1 & \cdots & 1-d \\ \cdots & \cdots & \cdots & \cdots \\ 1-d & 1+d & \cdots & 1 \end{Vmatrix}, \tag{2}$$

where n is even. These seemingly similar matrices actually have very different properties, as we will see further.

The determinant of the matrix (2) is not difficult to calculate, it is equal to d^n. The determinant of the matrix (1) is also calculated directly, although more cumbersome, it is equal to $n^2 d^{n-1}$, [3].

We now calculate the eigenvalues of the matrix (1). Note that this matrix is a circulant which is a special kind of Toeplitz matrix that is fully determined by its first row. Subsequent rows are obtained from the previous ones by a cyclic shift to the right by one position. From the linear algebra course (for example, [4]) it is known that the circulant is diagonalized by the discrete Fourier transform.

In this case, the eigenvalues of the matrix (1) can be determined as:

$$\lambda_j = 1 + (1+d)\varepsilon_j + (1-d)\varepsilon_j^2 + \cdots + (1-d)\varepsilon_j^{n-1},$$

where

$$\varepsilon_j = \cos\frac{2\pi j}{n} + i\sin\frac{2\pi j}{n}, \quad 0 \le j \le n-1.$$

After some transformations, we finally get: $\lambda_0 = n$, $\lambda_{k_{1,2}} = \pm id\tan\left(\frac{\pi k}{n}\right)$, $1 \le k \le \frac{n-1}{2}$.

Further, on the one hand, as mentioned above, the determinant of the matrix (1) is $n^2 d^{n-1}$, while on the other hand, being the product of all eigenvalues, it is $nd^{n-1}\prod_{k=1}^{\frac{n-1}{2}}\tan^2\left(\frac{\pi k}{n}\right)$. Equating these expressions, we get a curious identity: $n = \prod_{k=1}^{\frac{n-1}{2}}\tan^2\left(\frac{\pi k}{n}\right)$ or $2n+1 = \prod_{k=1}^{n}\tan^2\left(\frac{\pi k}{2n+1}\right)$, $1 \le k \le n$.

This equality is a special case of Eulerian decomposition of a sine into a product [5]. When $n = 1$, it is well-known: $\tan\frac{\pi}{3} = \sqrt{3}$.

The eigenvector e, corresponding to $\lambda_0 = n$, has all the components equal to 1. Invariant two-dimensional subspaces, corresponding to pairs of imaginary eigenvalues $\lambda_{k_{1,2}}$, are linear shells of the following vectors pairs:

$$\begin{cases} \left(1,\ \cos\left(\frac{2\pi k}{n}\right),\ \ldots,\ \cos\left(\frac{2\pi(n-1)k}{n}\right)\right), \\ \left(1,\ \sin\left(\frac{2\pi k}{n}\right),\ \ldots,\ \sin\left(\frac{2\pi(n-1)k}{n}\right)\right), \end{cases} \quad 1 \le k \le \frac{n-1}{2}.$$

3. Odd Competitive System Local Analysis

We will consider the n-dimensional system of competition equations $(n\text{-odd},\ n > 1)$:

$$\frac{dx_i}{dt} = \alpha_i x_i \left(1 - \sum_{j=1}^{n} m_{i,j} \frac{x_j}{x_j^*}\right),\ m_{k,k} = 1. \quad (3)$$

Here α_i is the Malthusian factor of the population i, the environment capacity x_i^*, that is the maximum population size, which would be established in this model if there were no other populations in it. A matrix $M = \| M_{i,j} \|_{i,j=1}^{n}$ is a competition matrix. Its components $m_{i,j}$ are the double standard factors [6], their meaning in the subject area of the model – comparison of inter-population competition with the intra-population one. So $m_{i,j}$ shows how many times the competition of population j with population i is stronger $(m_{i,j} > 1,\ j$ is intolerant for $i)$, or vice versa, – weaker $(m_{i,j} < 1,\ j$ is tolerant for $i)$, than the competition within the population j itself. Since the competition within populations acts here as a measurement standard, so all diagonal elements are equal to 1.

Let us make a few assumptions for simplicity:

- Get rid of environment capacities in (3) by moving to the relative numbers of populations $X_i = \frac{x_i}{x_i^*}$. This substitution is reversible, so if we get any results, then by making a reverse substitution, we will interpret them in terms of the initial system (3) with environment capacities.

- Assume the same Malthusian factors: $\alpha_i = \alpha, 1 \le i \le n.$ This is a much more restrictive assumption; however, the author is not able to get any meaningful results without it. Having made this assumption, we can get rid of α in the equations by replacing time $t = \alpha\tau.$

- We assume that the matrix $M = \| m_{i,j} \|_{i,j=1}^{n}$ is of the form (1), because we showed above that it produces the oscillation. So, $M = \| d_{i,j} \|_{i,j}^{n},$ where

$$d_{i,j} = \begin{cases} 1 - (-1)^{j-i}\, d, & i < j; \\ 1, & i = j; \\ 1 + (-1)^{i-j}\, d, & i > j. \end{cases}$$

Here $1 > d > 0.$

As a result of the assumptions and transformations made, we get the following system of equations:

$$\dot{X}_i = X_i \left(1 - \sum_{j=1}^{n} d_{i,j} X_j \right). \tag{4}$$

In the previous section it was mentioned, that the determinant of the matrix $D = \| d_{i,j} \|_{i,j=1}^{n}$ is $n^2 d^{n-1} > 0,$ so the solution of the linear system

$$1 - \sum_{j=1}^{n} d_{i,j} X_j = 0$$

exists, is unique, and equals $\overline{X} = \left(\frac{1}{n}, ..., \frac{1}{n} \right).$ The vector $\overline{X}$ is the stationary point of the system (4). We examine the system (4) in a small vicinity of this stationary point. Let $X_i = \overline{X}_i + x_i,$ where x_i are small. Then, neglecting the higher orders of smallness, we get

$$\dot{x}_i = -\overline{X}_i \sum_{j=1}^{n} d_{i,j} x_j = -\frac{1}{n} \sum_{j=1}^{n} d_{i,j} x_j,$$

and after another time replacement $\tau = \dfrac{1}{n}t$:

$$\dot{x}_i = -\sum_{j=1}^{n} d_{i,\,j} x_j .$$

The resulting system is a linear homogeneous system of differential equations with constant coefficients. To solve this system, we are to find the eigenvalues and eigenvectors of the matrix $-D$.

We found eigenvalues and invariant subspaces for the matrix (1), $D = \| d_{i,\,j} \|_{i,\,j=1}^{n}$ in the previous section. In our equations the same matrix appears, but with a minus sign. Everything remains true to it with the following correction: the only real eigennumber $-n$ and the determinant $-n^2 d^{n-1}$ now become negative; the rest, purely imaginary eigenvalues, just switch places in pairs. Also remain invariant for $-D$, all found for D two-dimensional invariant subspaces.

The trajectory of the system (4) in the small vicinity of the stationary point $\overline{X}$ behaves as follows: negative real eigennumber "pulls" it into the affine hyperplane $\{X : X = \overline{X} + x, (x, e) = 0\}$. This hyperplane passes through n points $(1.0, ..., 0), (0.1, ..., 0), ..., (0, ..., 1)$.

It is a direct sum of $(n-1)/2$ two-dimensional affine subspaces, in which oscillations occur with frequencies:

$$d \tan\left(\frac{\pi k}{n}\right),\ 1 \le k \le \frac{n-1}{2}.$$

Recall that we twice made a time replacement: $t = \alpha\tau$ and $\tau = \dfrac{1}{n}t.$ Returning to the original time, we get the oscillation frequencies:

$$\frac{\alpha d}{n} \tan\left(\frac{\pi k}{n}\right),\ 1 \le k \le \frac{n-1}{2}.$$

Note that as $n \to \infty,$ is true:

$$\frac{\alpha d}{n} \tan \frac{\pi(n-j)}{2n} \underset{n \to \infty}{\to} \frac{2\alpha d}{j\pi}, \quad j = 1, 2, \ldots << \frac{n-1}{2}.$$

The frequency of the highest harmonic of an odd n-dimensional system tends to $\frac{2\alpha d}{\pi}$, as $n \to \infty.$ The limit frequencies of the previous harmonics will be less in 2, 3, 4, 5, ... times.

Returning to the original variables $x_i = x_i^* X_i$, we get: the stationary point $\bar{x}$ of the original system has components $\bar{x}_i = \frac{x_i^*}{n}$, and the oscillation hyperplane passes through the points of environment capacities:

$$(x_1^*, 0, \ldots, 0), (0, x_2^*, 0, \ldots, 0), \ldots, (0, \ldots, 0, x_n^*).$$

This hyperplane is the boundary of tolerance that divides tolerant and intolerant regions [6] in the phase space of the system (3).

The equation $1 - \sum_{i=1}^{n} d_{i,j} \frac{x_i}{x_i^*} = 0$ defines the stationary point of the system (3). Its solution is $(\bar{x}_1, \ldots, \bar{x}_n)$, $\bar{x}_i = \frac{x_i^*}{n}$.

The invariant subspaces in the original variables will look like this: $(x_1^*, x_2^*, \ldots, x_n^*)$ – one-dimensional subspace corresponding to a single real eigenvalue, and two-dimensional subspaces corresponding to the conjugate imaginary pairs:

$$\begin{cases} \left(x_1^*, x_2^* \cos \frac{2\pi k}{n}, \ldots, x_n^* \cos \frac{2\pi(n-1)k}{n}\right), \\ \left(x_1^*, x_2^* \sin \frac{2\pi k}{n}, \ldots, x_n^* \sin \frac{2\pi(n-1)k}{n}\right), \end{cases} 1 \le k \le \frac{n-1}{2}. \tag{5}$$

4. Simulation Experiments

The 7-dimensional competition model was built in the AnyLogic [7] simulation system using its system dynamics tools.

According to the theory described above, three types of oscillations with different frequencies are possible in such a system: $3 = (7 - 1)/2.$ Each of these types can be distinguished separately by setting initial conditions in the corresponding invariant two-dimensional subspace.

By the analogy with the work [8], it is interesting to consider the potentials of individual populations

$$C(x_i) = \frac{1}{\alpha_i}\left(\ln x_i - \frac{nx_i}{x_i^*}\right), \tag{6}$$

where $n = 7,$ and the potential of the entire system, which is their sum

$$C = \sum_{i=1}^{n} \frac{1}{\alpha_i}\left(\ln x_i - \frac{nx_i}{x_i^*}\right). \tag{7}$$

It is also interesting to check, based on the simulation experiments, whether the potential of the entire system is constant or not.

Let us start by highlighting the different types of oscillations. As initial conditions, we take the coordinates of a stationary point

$$(\bar{x}_1, \ldots, \bar{x}_n), \quad \bar{x}_i = \frac{x_i^*}{n},$$

to which we add vectors of one of the invariant subspaces from (5) that are small in value.

As a result, we get three types of oscillations with different frequencies, which are shown in Figure 1. It can be seen that in the small neighborhood of the stationary point, the oscillations are indistinguishable from harmonic ones.

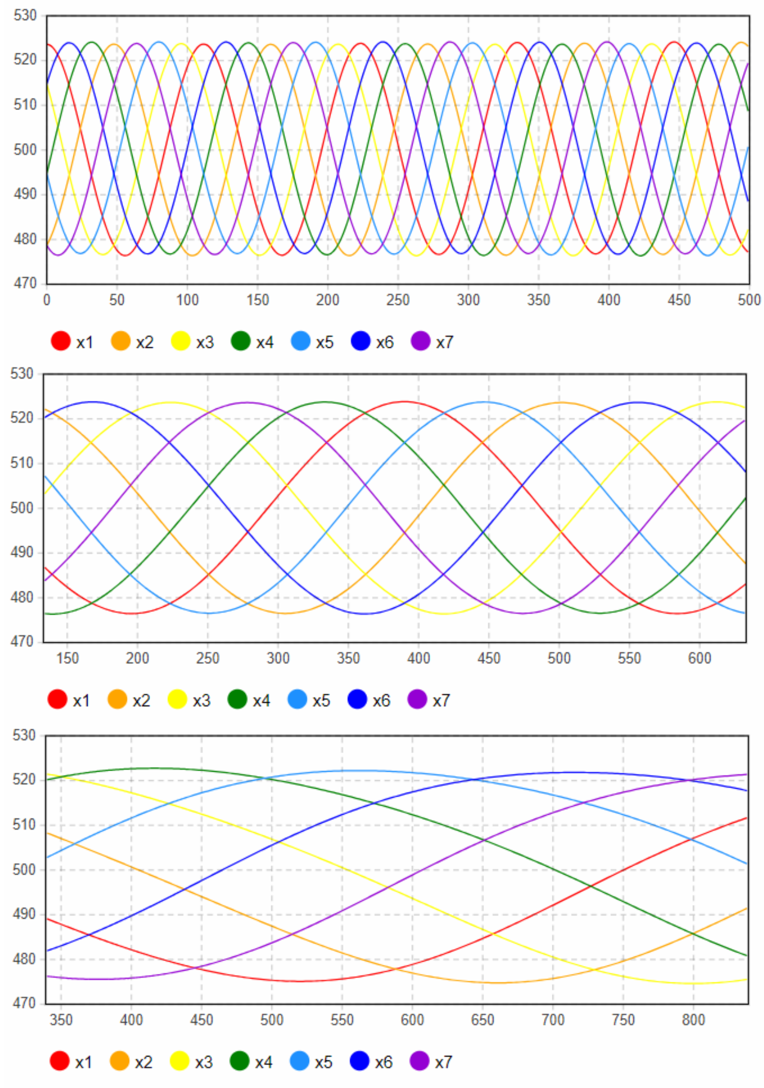

Figure 1. Oscillations of three different frequencies in the small neighborhood of the stationary point.

Further, we show the results of experiments with large deviations of the trajectory from the stationary point.

In this case, the two-dimensional subspaces lose their invariance; even if we take the initial data in such a space, the remaining oscillation frequencies still contribute to the trajectory. The harmonic nature of the oscillations is also disrupted (Figure 2).

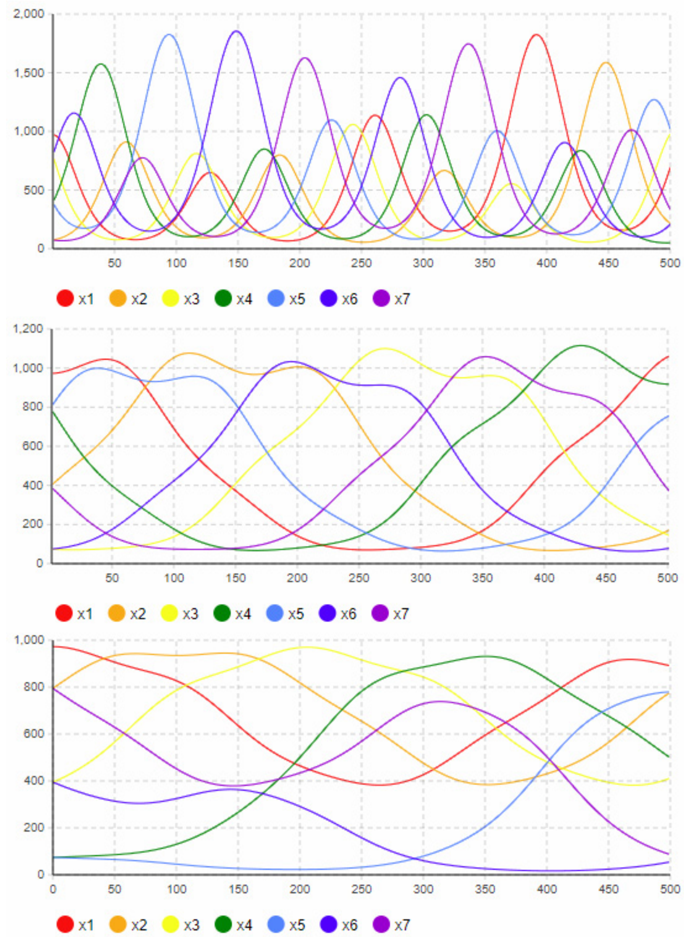

Figure 2. Oscillations of three different frequencies in the large neighborhood of the stationary point.

It is interesting to consider the dynamics of the populations' potentials (6) and the potential of the entire system (7), when the deviations from the stationary point are sufficiently large. Finally, let us take arbitrary initial data. In addition, we let the Malthusian factors be different.

We see that the potential of the entire system (7) remains constant even with unequal Malthusian coefficients and arbitrary (non-owned to invariant subspaces) initial data, whereas the potentials of individual populations fluctuate in Figure 3. The potentials of the populations differ in magnitude markedly. This is because the Malthusian factors differ, see (6).

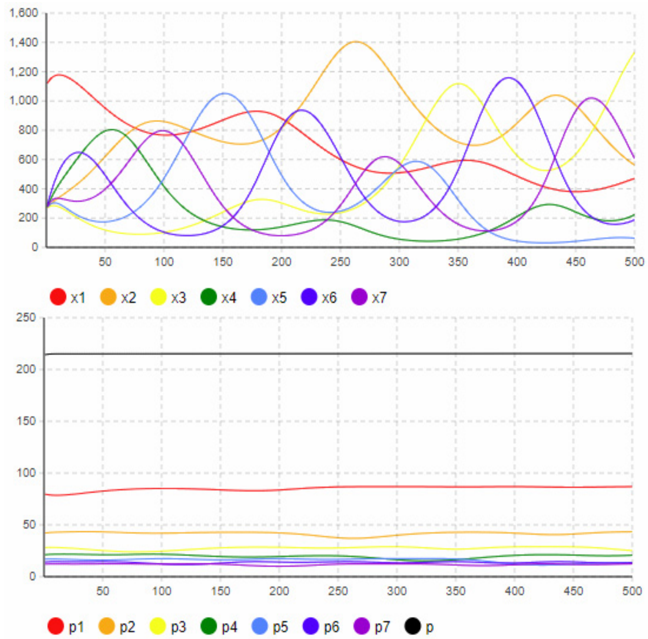

Figure 3. Oscillation and potentials in the large vicinity with different malthusian factors.

In the work [9] a description of the cell-automatic analog of the multidimensional competition model was given. It turns out that oscillations also occur in such discrete models and even the limiting frequency can be estimated using the formula $\frac{2\alpha d}{\pi}$.

References

[1] P. Verhulst, Notice sur la loi que la population suit dans son accroissement, Correspondance mathématique et physique 10 (1838), 113-121. Retrieved 06 December 2023.

[2] V. Volterra, Lecóns sur la théorie mathematique de la lutte pour la vie, Gauthier-Villars, Paris, 1931, p. 214.

[3] Y. I. Brodsky, Oscillations in multidimensional odd competitive systems, Modeling, Decomposition and Optimization of Complex Dynamic Processes 25(1) (2010), 7-24 (in Russian).

[4] R. Gray, Toeplitz and circulant matrices: a review, Foundations and Trends, 2006, p. 100. DOI:10.1561/0100000006.

[5] G. M. Fikhtengol'ts, The Fundamentals of Mathematical Analysis, Vol. 2, Pergamon, 1965, p. 540.

[6] Y. I. Brodsky, Tolerance, Intolerance, Identity: Simple math models of cultures' interaction, LAP LAMBERT Academic Publishing, 2011, p. 68.

[7] I. Grigoryev, AnyLogic 7 in Three Days, 2 ed., AnyLogic, 2015, p. 256.

[8] V. A. Bobrov and Yu. I. Brodsky, Simulating the soft power effect in two-dimensional models of A. Lotka – V. Volterra, WSEAS Transactions on Computers 22 (2023), 171-176. DOI:10.37394/23205.2023.22.20.

[9] V. A. Bobrov and Yu. I. Brodsky, Modeling of double standards and soft power in cellular automata competition systems, E3S Web of Conf. 405 (2023), 02016. DOI:10.1051/e3sconf/202340502016.