Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0974324324017
Volume 31, Number 3, 2024, Pages 317-333 **P-ISSN: 0974-3243**

SUFFICIENCY-TYPE CONDITIONS FOR A TYPE OF STRICTLY DECREASING SOLUTIONS OF LINEAR CONTINUOUS-TIME DIFFERENTIAL SYSTEMS WITH BOUNDED POINT TIME-VARYING DELAYS

Manuel De la Sen

Institute of Research and Development of Processes
Faculty of Science and Technology
University of the Basque Country (UPV/EHU)
Leioa 48940, Bizkaia, Spain
e-mail: manuel.delasen@ehu.eus

Abstract

This paper investigates sufficiency-type conditions for strictly decreasing solutions of linear time-delay differential systems subject to a finite number of time-varying bounded point delays. The delay functions are not required to be time-differentiable nor even continuous but simply piecewise bounded continuous. It is not also required for the delay functions at any time instant to be upper-bounded. It is not necessary to have the knowledge of either the delay

Received: March 6, 2024; Accepted: April 22, 2024
2020 Mathematics Subject Classification: 34K06, 34K20, 34K27.
Keywords and phrases: time-delay system, time-delay function, bounded time-varying delays, zero-delay system.

How to cite this article: Manuel De la Sen, Sufficiency-type conditions for a type of strictly decreasing solutions of linear continuous-time differential systems with bounded point time-varying delays, Advances in Differential Equations and Control Processes 31(3) (2024), 317-333. https://doi.org/10.17654/0974324324017

Published Online: May 15, 2024

functions or their lower and upper bounds. It is proved that the supremum of any vector norm of the solution trajectory on consecutive time intervals of finite lengths is strictly decreasing under either stability conditions on the matrix which describes the delay-free dynamics, or on the one which describes the zero-delay auxiliary system, provided in both cases the contribution of the delayed dynamics is sufficiently small related to the convergence abscissas of the above matrices.

1. Introduction

Time-delay systems are very common in some real-life problems like, for instance, in war and peace problems, in some biological and epidemic models, in diffusion and teleoperation problems, etc. Delays can be either point constant or time-varying delays, or distributed delays or mixed combined point and distributed, and either commensurate, i.e., being integer multiples of a basic delay, or incommensurate, which do not obey the above rule, and also can be either be internal, i.e., in the state, or external, i.e., in the inputs and/or in the outputs which can also appear in a combined way. The internal delays make the system to be infinite-dimensional with infinitely many characteristic zeros, however, infinitely many of them are always stable with infinite modulus. Also, the number of characteristic zeros of a linear time-delay system is finite within each vertical band of any finite real part size allocated within the complex plane [1]. The background literature on those systems is very abundant to the levels of solution characterization, stability properties and applications. See, for instance, [1] for basic related bibliography. If the point delays are time-varying functions, it is usually assumed that they are known and that they are, for each time instant, less than the current time instant or that they have bounded time-derivatives.

In [2], a robust adaptive control approach is developed and analyzed in detail for the case of a continuous-time system multiple time-varying delays subject to uncertainties eventually included the presence of unmodeled dynamics. The system parameters are not known and their on-line estimation is used from registered input/output values to evaluate them and to update the time-varying adaptive controller. A switching scheme for multiestimation under different initial conditions for the various estimation algorithms, which are organized in a parallel disposal, is proposed so that only one of them is available along a certain time interval to update the controller parameters. The switching time instants to activate each new estimation algorithm which estimates the controlled system parameters, while correspondingly updates too the controller parameters along a certain time interval, are selected in such a way that the global stability of the whole closed-loop scheme is guaranteed. In [3], a *t* robust stability study is given in detail for the description of a fractional Caputo-type linear dynamic system with time delays. Its solution and its global stability for any given admissible bounded function of initial condition are formulated through the use of fixed point theory. In [4], several structures of potential stabilizing control laws are discussed for linear and time-invariant controlled systems which involve time-varying bounded point delays and unmeasurable states which are not directly available for measurement. The case of time-varying point delays is overviewed in [5] through the use of "ad hoc" Lyapunov-Krasovskii functionals for global asymptotic stability studies of linear continuous-time systems. A parallel study under integral inequality approaches is also described. The upper-bound of the delay functions is involved. On the other hand, l-infinity and L-infinity approaches are used in [6] in the discrete-time time and continuous-time cases, respectively, for the study of the global stability of positive linear systems with bounded delays by involving a comparison of the systems with their nominal counterparts which are defined as being subject to constant point delays of some arbitrary given finite sizes. In [7], a design of a position/force control scheme is performed for bilateral teleoperators with time-varying delays, which is based on estimations, under some relaxed assumptions, of both velocities and forces through the use of simply joint position measurements. It is shown that position and force tracking errors become ultimately bounded with arbitrarily small ultimate bounds in finite time. On the other hand, an estimation procedure of the reachable set for switched singular systems with time-varying delay and state jumps is discussed in [8]. In [9], time-varying delays belonging to a known interval being also time-differentiable with bounded time-derivative of known bound are considered.

In this brief paper, sufficiency-type conditions for the solution to be strictly decreasing in a certain sense, which, as a result, imply also in turn the global asymptotic stability of the differential system, are derived for the case of bounded time-varying piecewise-continuous, and in general multiple, delays. It is proved that the supremum of the vector norm of the solution trajectory on consecutive time intervals of finite lengths is strictly decreasing under such sufficiency-type conditions. The derivations of those conditions are based on the analysis of the solution in terms of norms. It is not required for the interval of validity of the delay functions to be known and it is not required either for the delays to time time-differentiable with bounded time-derivative. It is assumed that either the matrix of undelayed dynamics or the one of the particularized auxiliary zero-delay system are stability matrices. The main results of the paper are given in the next two sections and then conclusions end the paper.

1.1. Notation

$\boldsymbol{R}_+ = \{r \in \boldsymbol{R} : r > 0\}$; $\boldsymbol{R}_{0+} = \{r \in \boldsymbol{R} : r \geq 0\}$; $\boldsymbol{R}_- = \{r \in \boldsymbol{R} : r < 0\}$ and $\boldsymbol{R}_{-0} = \{r \in \boldsymbol{R} : r \leq 0\}$ are subsets of the set $\boldsymbol{R}$ of the real numbers.

$\boldsymbol{Z}_+ = \{r \in \boldsymbol{Z} : r > 0\}$; $\boldsymbol{Z}_{0+} = \{r \in \boldsymbol{Z} : r \geq 0\}$; $\boldsymbol{Z}_- = \{r \in \boldsymbol{Z} : r < 0\}$ and $\boldsymbol{Z}_{-0} = \{r \in \boldsymbol{Z} : r \leq 0\}$ are subsets of the set $\boldsymbol{Z}$ of the integer numbers.

$\boldsymbol{C}_+ = \{z \in \boldsymbol{C} : \operatorname{Re} z > 0\}$; $\boldsymbol{C}_{0+} = \{z \in \boldsymbol{C} : \operatorname{Re} z \geq 0\}$; $\boldsymbol{C}_- = \{z \in \boldsymbol{C} : \operatorname{Re} z < 0\}$ and $\boldsymbol{C}_{-0} = z \in \boldsymbol{C} : \operatorname{Re} z \leq 0$ are subsets of the set $\boldsymbol{C}$ of the complex numbers.

We write $\bar{n} = \{1, 2, ..., n\}$.

2. Main Results

Consider the nth differential system with a time-varying bounded delay:

$$\dot{x}(t) = Ax(t) + A_0 x(t - h(t)) \tag{1}$$

subject to a bounded real piecewise continuous function $\varphi : [-h_2, 0] \to \mathbf{R}^n$ of initial conditions, with $\varphi(\tau) = 0$ for $-h_2 \leq \tau < -h(0)$, provided that $h(0) \neq h_2$, $\overline{\varphi} = \sup_{-h_2 \leq 0} \| \varphi(\tau) \|$, $\varphi(0) = x(0) = x_0$ and $x(t) \equiv \varphi(t)$ for $t \leq 0$, where $x(t) \in \mathbf{R}^n$ is the state vector, A, $A_0 \in \mathbf{R}^{n \times n}$ are the delay-free and delayed matrices of dynamics and $h : \mathbf{R}_{0+} \to [h_1, h_2]$ is a piecewise continuous delay function for some finite $h_1 \geq 0$ and some finite $h_2 \geq h_1$.

The state-trajectory solution of (1) is unique for $t \geq 0$ for each given $\varphi : [-h_2, 0] \to \mathbf{R}^n$ defined by

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} A_0 x(\tau - h(\tau)) d\tau; \; t \geq 0. \tag{2}$$

It turns out that for each matrix A, there exist a norm-dependent real constant of minimum value $K = K(A) \geq 1$ and a real constant $\rho = \rho(A)$ such that $\| e^{At} \| \leq K e^{-\rho t}$ for all $t \geq 0$ with $\rho > 0$ if A is a stability matrix, that is, if $sp(A) \subset \mathbf{C}_-$, $\rho = 0$ if it is critically stable, that is, if $sp(A) \subset \mathbf{C}_{-0}$ with at least one eigenvalue of A being on the complex imaginary axis and $\rho < 0$ if A is unstable, that is, if at least one eigenvalue is in $\mathbf{C}_+$. If A is a stability matrix, then $-\rho < 0$ is its stability (or convergence) abscissa, that is, $0 > -\rho \geq \max \operatorname{Re}(\lambda_i(A))$, where $\lambda_i(A) \in sp(A)$; $i \in \overline{n}$, are the eigenvalues of A if they are distinct and $0 > -\rho > \max \operatorname{Re}(\lambda_i(A))$, if at least one of the eigenvalues has a multiplicity larger than one.

The subsequent result holds concerning the boundedness and convergence to zero of the state trajectory solution provided that the matrix of delay-free dynamics A is a stability matrix and A_0 has a sufficiently small norm, related to the absolute stability abscissa of A:

Theorem 1. *Assume that* $\rho = \rho(A) > 0$, $\| A_0 \| < \rho/K$ *and that* (1) *is subject to any given bounded piecewise continuous function of initial conditions* $\varphi : [-h_2, 0] \to \boldsymbol{R}^n$. *Then, the following properties hold*:

(i) *The state trajectory solution is bounded according to*

$$\sup_{0 \le \tau \le t} \| x(\tau) \| \le K \| A_0 \| \overline{\varphi}/(\rho - K \| A_0 \|) < +\infty; \ \forall t \in \boldsymbol{R}_{0+}.$$

(ii) *If, furthermore*, $\| A_0 \| < \rho/(2K)$, *then the sequence*

$$\left\{ \sup_{t_n \le \tau \le t} \| x(\tau) \| : t(\ge t_n) \in \boldsymbol{R} \right\}_{n \in Z_{0+}}^{\infty}$$

is strictly decreasing and there exists the limit $\lim_{n \to \infty} \sup_{t_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{t_n\}_{n=0}^{\infty}$ *satisfying* $t_0 = 0$ *and* $t_n \ge t_{n-1} + h_2$ *for* $n \in \mathbf{Z}_{0+}$ *what implies, in turn, that* $\lim_{t \to \infty} x(t) = 0$.

Proof. From (2), we have

$$\| x(t) \| \le K e^{-\rho t} \left(\| x_0 \| + \int_0^t e^{\rho \tau} \| A_0 \| \| x(\tau - h(\tau)) \| d\tau \right)$$

$$\le K e^{-\rho t} \left(\| x_0 \| + \frac{e^{\rho t} - 1}{\rho} \| A_0 \| \int_0^t e^{\rho \tau} \| x(\tau - h(\tau)) \| d\tau \right)$$

$$= K e^{-\rho t} \left(\| x_0 \| + \frac{1 - e^{-\rho t}}{\rho} \| A_0 \| \int_0^t e^{\rho \tau} \| x(\tau - h(\tau)) \| d\tau \right)$$

$$\le K \left(e^{-\rho t} \| x_0 \| + \rho^{-1} \| A_0 \| \sup_{-h_2 \le \tau \le t - h_1} \| x(\tau) \| \right); \quad t \in \boldsymbol{R}_{0+} \tag{3}$$

which implies also that

$$\| x(t) \| \leq K\left(e^{-\rho t} \| x_0 \| + \rho^{-1} \| A_0 \| \sup_{-h_2 \leq \tau \leq t-h_1} \| x(\tau) \| \right)$$

$$\leq K\left(\| x_0 \| + \rho^{-1} \| A_0 \| \sup_{-h_2 \leq \tau \leq t} \| x(\tau) \| \right)$$

$$= K\left(\| x_0 \| + \rho^{-1} \| A_0 \| \max\left(\sup_{-h_2 \leq \tau \leq 0} \| \varphi(\tau) \| + \sup_{0 < \tau \leq t} \| x(\tau) \| \right)\right)$$

$$= K\left(\| x_0 \| + \rho^{-1} \| A_0 \| \max\left(\overline{\varphi} + \sup_{0 \leq \tau \leq t} \| x(\tau) \| \right)\right)$$

$$\leq K\left(M + \rho^{-1} \| A_0 \| \sup_{0 \leq \tau \leq t} \| x(\tau) \| \right); \quad t \in \boldsymbol{R}_{0+}, \tag{4}$$

where, for the initial time instant $t_0 = 0$, $M = M(t_0, \rho, \overline{\varphi}) = M(0, \rho, \overline{\varphi}) = \rho^{-1} \| A_0 \| \overline{\varphi} < +\infty$ is a positive constant which depends on the function of initial conditions. Equation (4) implies that

$$\sup_{0 \leq \tau \leq t} \| x(\tau) \| \leq K\left(M + \rho^{-1} \| A_0 \| \sup_{0 \leq \tau \leq t} \| x(\tau) \| \right); \quad t \in \boldsymbol{R}_{0+}. \tag{5}$$

If $\rho > K$, then from (5), it follows that

$$\sup_{0 \leq \tau \leq t} \| x(\tau) \| \leq \frac{KM\rho}{\rho - K \| A_0 \|} < +\infty; \quad t \in \boldsymbol{R}_{0+} \tag{6}$$

and property (i) is thus proved.

To prove property (ii), note from (1) that

$$x(t) = e^{At} x(t_1) + \int_{t_1}^{t} e^{A(t-\tau)} A_0 x(\tau - h(\tau)) d\tau; \quad t \geq t_1 \tag{7}$$

which leads, in a similar way as (5) has been obtained for $t_0 = 0,$ to

$$M\left(t_1,\, \rho,\, \sup_{t_1 - h_2 \le \tau \le t_1} \| x(\tau) \|\right) = \rho^{-1} \| A_0 \| \sup_{t_1 - h_2 \le \tau \le t_1} \| x(\tau) \|$$

$$\le \rho^{-1} \| A_0 \| \sup_{0 \le \tau \le \infty} \| x(\tau) \|$$

$$\le \rho^{-1} \| A_0 \| K \| A_0 \| \overline{\varphi} / (\rho - K \| A_0 \|) < +\infty \quad (8)$$

which replaced into (7) yields:

$$\sup_{t_1 \le \tau \le t} \| x(\tau) \| \le \left(\frac{K \| A_0 \|}{\rho - K \| A_0 \|} \right)^2 \overline{\varphi}; \quad t(\ge t_1) \in \boldsymbol{R}_{0+}. \quad (9)$$

Proceeding recursively, we have

$$\sup_{t_n \le \tau \le t} \| x(\tau) \| \le \left(\frac{K \| A_0 \|}{\rho - K \| A_0 \|} \right)^{n+1} \overline{\varphi} \text{ for } n \in \boldsymbol{Z}_{0+},\ t(\ge t_n) \in \boldsymbol{R}_{0+} \quad (10)$$

and taking into account that $\| A_0 \| < \rho/(2K)$ implies that $K \| A_0 \| / (\rho - K \| A_0 \|) < 1,$ from (11) follows that, for any given function of initial conditions:

$$\lim_{n \to \infty} \sup_{t_n \le \tau \le t} \| x(\tau) \| = \lim_{t \to \infty} \| x(t) \| = 0. \quad (11)$$

This establishes property (ii). □

Remark 1. The specific value of the constant $K = K(A) \ge 1$ is not very relevant in the boundedness and convergence analysis, that is, it can be taken equal to unity as it is now seen. First note that for any prefixed real constant $\rho_0 = \rho_0(A) \in (0, \rho(A)),$ there exists a finite real constant $t_0 = t_0(\rho_0, \rho, K)$ such that $Ke^{-\rho t} \le e^{-\rho_0 t}$ for $t \ge t_0$ since for given $K \ge 1,$ such a non-unique t_0 exists fulfilling $K \le e^{(\rho - \rho_0) t}$ for all $t \ge t_0.$

Thus, from (2), we get as an alternative to (3), after replacing $Ke^{-\rho t} \le e^{-\rho_0 t}$ for any $t \ge t_0$, that

$$\| x(t) \| = Ke^{-\rho_0(t-t_0)}\left(\| x(t_0) \| + \frac{1-e^{-\rho_0(t-t_0)}}{\rho_0} \| A_0 \| \int_{t_0}^{t} e^{\rho_0 \tau} \| x(\tau - h(\tau)) \| d\tau \right)$$

$$\le e^{-\rho_0(t-t_0)} \| x(t_0) \| + \rho_0^{-1} \| A_0 \| \sup_{t_0 - h_2 \le \tau \le t - h_1} \| x(\tau) \|; \quad t(\ge t_0) \in \boldsymbol{R}_{0+}. \tag{12}$$

Then, instead of (4), we get

$$\| x(t) \| \le M_0 + \rho_0^{-1} \| A_0 \| \sup_{t_0 \le \tau \le t} \| x(\tau) \|; \quad t(\ge t_0) \in \boldsymbol{R}_{0+} \tag{13}$$

with

$$M_0 = M_0\left(t_0, \rho_0 \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \| \right)$$

$$= \rho_0^{-1} \| A_0 \| \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \|$$

$$\le \overline{M}_0\left(t_0, \rho_0, \max\left(\overline{\varphi}, \sup_{0 < \tau \le t_0} \| x(\tau) \| \right) \right) < +\infty, \tag{14}$$

since t_0 is finite. Thus,

$$\sup_{t_0 \le \tau \le t} \| x(\tau) \| \le \frac{\| A_0 \| \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \|}{\rho_0 - \| A_0 \|} < +\infty; \quad t(\ge t_0) \in \boldsymbol{R}_{0+} \tag{15}$$

instead of (6). Also, instead of (10), we get

$$\sup_{t_n \le \tau \le t} \| x(\tau) \| \le \left(\frac{\| A_0 \|}{\rho_0 - \| A_0 \|} \right)^{n+1} \left(\sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \| \right)$$

$$\text{for } n \in \boldsymbol{Z}_{0+}, \; t(\ge t_n) \in \boldsymbol{R}_{0+} \tag{16}$$

for any given real sequence $\{t_n\}_{n=0}^{\infty}$ satisfying $t_n \ge t_{n-1} + h_2$ for $n \in \boldsymbol{Z}_+$.

As a result, an alternative result to Theorem 1 follows under easier test conditions.

Corollary 1. *Assume any given prefixed real constant* $\rho_0 \in (0, \rho)$, $\| A_0 \| < \rho_0$ *and that* (1) *is subject to any given bounded piecewise continuous function of initial conditions* $\varphi : [-h_2, 0] \to \boldsymbol{R}^n$. *Then, the following properties hold*:

(i) *The state trajectory solution is bounded according to* $\sup_{t_0 \le \tau \le t} \| x(\tau) \| \le \| A_0 \| \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \| / (\rho_0 - \| A_0 \|) < +\infty;\ \forall t (\ge t_0) \in \boldsymbol{R}_{0+}$, *where* $t_0 = t_0(\rho_0, \rho, K)$ *is finite subject to the constraint* $t_0 \ge \ln K / (\rho - \rho_0)$.

(ii) *If*, *furthermore*, $\| A_0 \| < \rho_0 / 2$, *then the sequence*

$$\left\{ \sup_{t_n \le \tau \le t} \| x(\tau) \| : t(\ge t_n) \in \boldsymbol{R} \right\}_{n \in \mathbf{Z}_{0+}}^{\infty}$$

is strictly decreasing and there exists the limit $\lim_{n \to \infty} \sup_{t_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{t_n\}_{n=0}^{\infty}$ *satisfying* $t_n \ge t_{n-1} + h_2$ *for* $n \in \mathbf{Z}_+$ *what implies*, *in turn*, *that* $\lim_{t \to \infty} x(t) = 0$.

Proposition 1. *Under the hypotheses of Theorem* 1, *or those of Corollary* 1, *the matrix* $(A + A_0)$ *is a stability matrix*.

Proof. Note that Theorem 1 and Corollary 1 provide the properties irrespective of the interval domain $[h_1, h_2]$ provided that it is a non-negative bounded interval of finite measure or of measure zero. The last case corresponds to a constant delay of any finite value $h(t) = h_1 = h_2 \ge 0$. The particular case of zero constant delay $h_1 = h_2 = 0$ leads that the delay-free differential system $\dot{x}(t) = (A + A_0) x(t)$ with initial conditions $x_0 = \varphi(0)$

and $\varphi(t) = 0$ for $t < 0$ is globally asymptotically stable irrespective of the finite initial condition. □

Remark 2. In the same way that there are real constants $K = K(A) \geq 1$, $\rho = \rho(A) > 0$ and any $\rho_0 \in (0, \rho)$, such that $\| e^{At} \| \leq Ke^{-\rho t}$ and $\| e^{A\tau} \| \leq e^{-\rho_0 \tau}$ for all $t \geq 0$, $\tau \geq t_0$ and some $t_0 = t_0(\rho_0) > 0$, employed to derive sufficient conditions for boundedness and convergence to zero of the solution of (1) by considering that the matrix of dynamics for zero-delay (i.e., for $h(t) \equiv 0$), that is, $(A + A_0)$, is a stability matrix (see Proposition 1) since, in particular, the delay-free system is globally asymptotically stable for zero delay under Theorem 1 or Corollary 1. For that purpose, we redefine the real constants $K = K(A + A_0) \geq 1$ and $\rho = \rho(A + A_0) > 0$ such that $\| e^{(A+A_0)t} \| \leq Ke^{-\rho t}$ and $\| e^{(A+A_0)\tau} \| \leq e^{-\rho_0 \tau}$ for all $t \geq 0$, $\tau \geq t_0$, some $t_0 = t_0(\rho_0) > 0$. Now, write equation (1) equivalently as

$$\dot{x}(t) = (A + A_0) + A_0(x(t - h(t)) - x(t)). \tag{17}$$

Based on (17), the solution (2) can be expressed in an equivalent form, as follows:

$$x(t) = e^{(A+A_0)t} x_0 + \int_0^t e^{(A+A_0)(t-\tau)} A_0(x(\tau - h(\tau)) - x(\tau)) d\tau; \quad t \geq 0. \tag{18}$$

Instead of (3), from (17) follows that, for redefinitions of $K = K(A + A_0) \geq 1$, $\rho = \rho(A + A_0) > 0$ and any $\rho_0 = \rho_0(A + A_0)$ subject to $0 < \rho_0 < \rho$:

$$\| x(t) \| \leq Ke^{-\rho t} \left(\| x_0 \| + \int_0^t e^{\rho\tau} \| A_0 \| \| x(\tau - h(\tau)) - x(\tau) \| d\tau \right)$$

$$\leq K \left(e^{-\rho t} \| x_0 \| + 2\rho^{-1} \| A_0 \| \sup_{-h_2 \leq \tau \leq t - h_1} \| x(\tau) \| \right); \quad t \in \boldsymbol{R}_{0+} \tag{19}$$

and, instead of (13)-(14), according to (17), we have

$$\| x(t) \| \le M_0 + 2\rho_0^{-1} \| A_0 \| \sup_{t_0 \le \tau \le t} \| x(\tau) \|; \quad t(\ge t_0) \in \boldsymbol{R}_{0+} \tag{20}$$

with

$$M_0 = M_0\left(t_0, \rho_0, 2 \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \| \right)$$

$$= 2\rho_0^{-1} \| A_0 \| \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \|$$

$$\le \overline{M}_0\left(t_0, \rho_0, 2 \max\left(\overline{\varphi}, \sup_{0 < \tau \le t_0} \| x(\tau) \| \right) \right) < +\infty. \tag{21}$$

Under similar derivation of those used to prove Theorem 1 and Corollary 1, we have the following result based on the equivalent expression (17) to (1).

Theorem 2. *Assume that* $\rho = \rho(A + A_0) > 0$, $K = K(A + A_0) \ge 1$, $0 < \rho_0 = \rho_0(A + A_0) < \rho$ *while* (1) *is subject to any given bounded piecewise continuous function of initial conditions* $\varphi : [-h_2, 0] \to \boldsymbol{R}^n$. *Then, the following properties hold*:

(i) *If* $\| A_0 \| < \rho/(2K)$, *then the state trajectory solution is bounded according to* $\sup_{0 \le \tau \le t} \| x(\tau) \| \le 2K \| A_0 \| \overline{\varphi}/(\rho - 2K \| A_0 \|) < +\infty$; $\forall t \in \boldsymbol{R}_{0+}$. *If* $\| A_0 \| < \rho_0/2$, *then the state trajectory solution is bounded according to*

$$\sup_{t_0 \le \tau \le t} \| x(\tau) \| \le 2 \| A_0 \| \sup_{t_0 - h_2 < \tau \le t_0} \| x(\tau) \| / (\rho_0 - 2 \| A_0 \|) < +\infty;$$

$\forall t(\ge t_0) \in \boldsymbol{R}_{0+}$, *where* $t_0 = t_0(\rho_0, \rho, K)$ *is finite and subject to the constraint* $t_0 \ge \ln K/(\rho - \rho_0)$.

(ii) *If*, $\| A_0 \| < \rho/(4K)$, *then the sequence*

$$\left\{ \sup_{t_n \le \tau \le t} \| x(\tau) \| : t(\ge t_n) \in \boldsymbol{R} \right\}_{n \in \mathbf{Z}_{0+}}^{\infty}$$

is strictly decreasing and there exists the limit $\lim_{n \to \infty} \sup_{t_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{t_n\}_{n=0}^{\infty}$ *satisfying* $t_0 = 0$ *and* $t_n \ge t_{n-1} + h_2$ *for* $n \in \mathbf{Z}_{0+}$ *what implies*, *in turn*, *that* $\lim_{t \to \infty} x(t) = 0$. *If* $\| A_0 \| < \rho_0/4$, *then the sequence* $\left\{ \sup_{t_n \le \tau \le t} \| x(\tau) \| : t(\ge t_n) \in \boldsymbol{R} \right\}_{n \in \mathbf{Z}_{0+}}^{\infty}$ *is strictly decreasing and there exists the limit* $\lim_{n \to \infty} \sup_{t_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{t_n\}_{n=0}^{\infty}$ *satisfying* $t_n \ge t_{n-1} + h_2$ *for* $n \in \boldsymbol{Z}_{+}$ *what implies*, *in turn*, *that* $\lim_{t \to \infty} x(t) = 0$.

3. Some Direct Extensions for the Case of Multiple Delays

It is possible to proceed in the same way to derive more general results for the presence of r time-varying piecewise continuous delay functions in $[h_1, h_2]$. Consider the differential system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{r} A_i x(t - h_i(t)) \\ &= \left(A + \sum_{i=1}^{r} A_i \right) x(t) + \sum_{i=1}^{r} A_i (x(t - h_i(t) - x(t))), \end{aligned} \tag{22}$$

where $A, A_i \in \boldsymbol{R}^{n \times n}$ $(i \in \bar{r})$ are, respectively, the delay-free and delayed matrices of dynamics for the r bounded real piecewise continuous, in general, incommensurate delay functions $h_i : \boldsymbol{R}_{0+} \to [h_1, h_2]$; $i \in \bar{r}$, subject to any given bounded piecewise continuous function of initial conditions

$\varphi : [-h_2, 0] \to \boldsymbol{R}^n$, such that, if $\overline{A} = A + \sum_{i=1}^{r} A_i$ and $\overline{A}_0 = \overline{A} - A = \sum_{i=1}^{r} A_i$, then the solution of (22) is

$$x(t) = e^{At}x_0 + \sum_{i=1}^{r}\int_0^t e^{A(t-\tau)}A_i x(\tau - h_i(\tau))d\tau$$

$$= e^{\overline{A}t}x_0 + \sum_{i=1}^{r}\int_0^t e^{\overline{A}(t-\tau)}\overline{A}_0(x(\tau - h_i(\tau)) - x(\tau))d\tau; \quad t \geq 0 \tag{23}$$

so that

$$\| e^{\overline{A}t} \| \leq \overline{K}e^{-\rho t}, \quad \| e^{\overline{A}\tau} \| \leq e^{-\overline{\rho}_0 \tau}; \quad \forall t \geq 0, \ \forall \tau \geq \bar{t}_0 \tag{24}$$

for some real constants $\overline{\rho} = \overline{\rho}(\overline{A})$, $\overline{\rho}_0 = \overline{\rho}_0(\overline{A}) > 0$ and $\overline{K} = \overline{K}(\overline{A}) \geq 1$, with $\overline{\rho}_0 < \overline{\rho}$, and any finite $\bar{t}_0 = \bar{t}_0(\overline{\rho}_0, \overline{\rho}, \overline{K})$.

Thus, the following result holds, via (22)-(24), as a direct extension of Theorem 1, Corollary 1 and Theorem 2 for the case of r finite piecewise-continuous bounded delay functions.

Theorem 3. *The following properties hold*:

(i) *If* $\| \overline{A}_0 \| < \overline{\rho}/r\overline{K}$, *then the state trajectory solution is bounded according to* $\sup_{0 \leq \tau \leq t} \| x(\tau) \| = r\overline{K}\| \overline{A}_0 \|\overline{\varphi}/(\overline{\rho} - r\overline{K} \| \overline{A}_0 \|) < +\infty$; $\forall t \in \boldsymbol{R}_{0+}$. *If* $\| \overline{A}_0 \| < \overline{\rho}_0/r$, *then the state trajectory solution is bounded according to*

$$\sup_{0 \leq \tau \leq t} \| x(\tau) \| \leq r\| \overline{A}_0 \| \sup_{\bar{t}_0 - h_2 < \tau \leq \bar{t}_0} \| x(\tau) \|/(\overline{\rho}_0 - r\| \overline{A}_0 \|) < +\infty; \quad \forall t \in \boldsymbol{R}_{0+},$$

for any given finite $\bar{t}_0 \geq \ln \overline{K}/(\overline{\rho} - \overline{\rho}_0)$.

(ii) *If* $\| \overline{A}_0 \| < \overline{\rho}/(2r\overline{K})$, *then the sequence*

$$\left\{ \sup_{t_n \leq \tau \leq t} \| x(\tau) \| : t(\geq t_n) \in \boldsymbol{R} \right\}_{n \in \mathbf{Z}_{0+}}^{\infty}$$

is strictly decreasing and there exists the limit $\lim_{n\to\infty} \sup_{t_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{t_n\}_{n=0}^{\infty}$ *satisfying* $t_0 = 0$ *and* $t_n \ge t_{n-1} + h_2$ *for* $n \in \mathbf{Z}_{0+}$ *what implies, in turn, that* $\lim_{t\to\infty} x(t) = 0$. *If* $\| \overline{A}_0 \| < \overline{\rho}_0/(2r)$, *then the state trajectory solution is strictly decreasing and there exists the limit* $\lim_{n\to\infty} \sup_{\bar{t}_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{\bar{t}_n\}_{n=0}^{\infty}$ *satisfying* $\bar{t}_0 \ge \ln \overline{K}/(\overline{\rho} - \overline{\rho}_0)$ *and* $\bar{t}_n \ge \bar{t}_{n-1} + h_2$ *for* $n \in \mathbf{Z}_{0+}$ *what implies, in turn, that* $\lim_{t\to\infty} x(t) = 0$.

(iii) *If* $\| \overline{A}_0 \| < \overline{\rho}/((r+1)\overline{K})$, *then the state trajectory solution is bounded according to*

$$\sup_{0 \le \tau \le t} \| x(\tau) \| \le (r+1)\overline{K} \| \overline{A}_0 \| \overline{\varphi}/(\overline{\rho} - (r+1)\overline{K} \| \overline{A}_0 \|) < +\infty; \ \forall t \in \mathbf{R}_{0+}. \quad (25)$$

If $\| \overline{A}_0 \| < \overline{\rho}_0/((r+1))$, *then the state trajectory solution is bounded according to*

$$\sup_{0 \le \tau \le t} \| x(\tau) \| \le 2(r+1)\overline{K} \| \overline{A}_0 \| \sup_{\bar{t}_0 - h_2 < \tau \le \bar{t}_0} \| x(\tau) \| /(\overline{\rho} - 2(r+1)\overline{K} \| \overline{A}_0 \|) < +\infty;$$

$$\forall t \in \mathbf{R}_{0+}, \quad (26)$$

for any given finite $\bar{t}_0 \ge \ln \overline{K}/(\overline{\rho} - \overline{\rho}_0)$.

(iv) *If* $\| \overline{A}_0 \| < \overline{\rho}/(2(r+1)\overline{K})$, *then the state trajectory solution is strictly decreasing and there exists the limit* $\lim_{n\to\infty} \sup_{\bar{t}_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{t_n\}_{n=0}^{\infty}$ *satisfying* $t_0 = 0$ *and* $t_n \ge t_{n-1} + h_2$ *for* $n \in \mathbf{Z}_{0+}$ *what implies, in turn, that* $\lim_{t\to\infty} x(t) = 0$. *If* $\| \overline{A}_0 \| < \overline{\rho}_0/(2(r+1))$, *then the sequence* $\left\{ \sup_{\bar{t}_n \le \tau \le t} \| x(\tau) \| : t(\ge \bar{t}_n) \in \mathbf{R} \right\}_{n \in \mathbf{Z}_{0+}}^{\infty}$ *is strictly decreasing* *and there exists the limit* $\lim_{n\to\infty} \sup_{\bar{t}_n \le \tau \le t} \| x(\tau) \| = 0$ *for any given real sequence* $\{\bar{t}_n\}_{n=0}^{\infty}$ *defined by any finite* $\bar{t}_0 \ge \ln \bar{K}/(\bar{\rho} - \bar{\rho}_0)$ *and* $\bar{t}_n \ge \bar{t}_{n-1} + h_2$ *for* $n \in \mathbf{Z}_{0+}$ *what implies*, *in turn*, *that* $\lim_{t\to\infty} x(t) = 0.$

Remark 3. Theorem 3 still holds by replacing $\| \bar{A}_0 \| \to \sup_{t \in \mathbf{R}_{0+}} \| \bar{A}_0(t) \|$ if $\bar{A}_0 : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$ is a piecewise bounded function matrix

$$\bar{A}_0(t) = \bar{A}(t) - A = \sum_{i=1}^{r} A_i(t); \quad \forall t \in \mathbf{R}_{0+}.$$

Remark 4. Alternative results to Theorem 3 may be obtained in the same way, under more involved derivations and weaker stability and convergence constraints for the solution, for the case when the variation interval for each delay is known, that is, $h_i : \mathbf{R}_{0+} \to [h_{i1}, h_{i2}] \subset [\underline{h}_1, \overline{h}_2]$ for some finite $h_{i1} \ge 0$ and some finite constants $h_{i2} \ge h_{i1}$, $i \in \bar{r}$, $\underline{h}_1 = \min(h_{i1} : i \in \bar{r})$ and $\overline{h}_2 = \max(h_{i2} : i \in \bar{r})$.

4. Conclusions

Linear time-delay systems with time-varying bounded delays have been considered in this article. Sufficiency-type conditions for the supremum on consecutive time intervals of finite lengths of the vector norm of the solution trajectory to be strictly decreasing have been obtained. Neither the knowledge of interval of validity of the delay functions, nor for the delays to time time-differentiable with bounded time-derivative is required. It is assumed that either the matrix of undelayed dynamics or that of dynamics of the auxiliary zero-delay particularized system is a stability matrix.

Acknowledgement

The author is grateful to Basque Government for Grant IT1555-22.

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