**Advances in Differential Equations and Control Processes**
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0974324324015
Volume 31, Number 2, 2024, Pages 285-297 **P-ISSN: 0974-3243**

# PERIODIC SOLUTIONS OF A SECOND-ORDER NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

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## Abstract

The article considers the problem of constructing a $2\pi$-periodic solution of a quasilinear second-order integro-differential equation. Using the Green's function of bounded solutions on the number line,

Received: February 18, 2024; Revised: April 29, 2024; Accepted: May 9, 2024

2020 Mathematics Subject Classification: 35Q35.

Keywords and phrases: periodic solutions, quasilinear second-order integro-differential equations, Green's function, integral equations on the number axis, exact and approximate solutions, method of successive approximations.

How to cite this article: A. T. Alymbaev, A. Bapa Kyzy and F. K. Sharshembieva, Periodic solutions of a second-order nonlinear Volterra integro-differential equation, Advances in Differential Equations and Control Processes 31(2) (2024), 285-297.
https://doi.org/10.17654/0974324324015



Published Online: May 14, 2024

the integro-differential equation is reduced to an integral equation. A $2\pi$-periodic solution to the integral equation is found using the projection-iteration method. A $2\pi$-periodic solution is sought as the limit of successive $2\pi$-periodic functions representable as a Fourier series. An estimate of the error of the difference between the exact and approximate solutions is obtained.

## 1. Introduction

In many problems of science and technology, there are phenomena that describe oscillatory processes, the mathematical models of which are differential and integro-differential equations and their systems. In this regard, one of the important issues in the study of quasilinear, nonlinear differential and integro-differential equations is the study of periodic solutions and the construction of an algorithm for finding them. Various methods are used to study periodic solutions. Among the existing methods, there are methods along with the proof of theorems for the existence of periodic solutions that make it possible to construct these solutions. Such methods for studying periodic solutions include the Galerkin method, a projection-iteration method combining the ideas of the Galerkin method and the method of successive approximations.

Issues of constructing periodic solutions according to the Galerkin method for non-autonomous systems of differential equations, systems of differential equations with delay and various types of non-autonomous integro-differential equations were studied in the works of [1, 2, 4, 5].

The works of differential research are devoted to the study of periodic solutions by projection-iteration methods of a system of differential equations, a system of differential equations with delay, and integro-differential equations. See [3, 6-10].

This work is devoted to the study of periodic solutions of a quasilinear second-order integro-differential equation.

## 2. Statement of the Problem

Consider a second-order integro-differential equation of the form:

$$\frac{d^2x(t)}{dt^2} = Ax(t) + f\left(t,\ x(t),\ \int_{t-\tau}^{t} \varphi(t,\ s,\ x(s))ds\right), \tag{1}$$

where $A$ is a positive real number, $f$, $\varphi$ are continuously-differentiable $2\pi$-periodic functions of $t$, $s$ and $\tau$ is a constant.

Denote by $C'(T \times D \times D)$ the space of $r$-times continuously differentiable functions $f(t,\ x,\ u)$ with respect to $(t,\ x,\ u) \in (T \times D \times D)$. The function $\varphi(t,\ s,\ x)$ is periodic in $t$, $s$ with period $2\pi$, where $T = [0,\ 2\pi]$, $D \subset R = (-\infty,\ +\infty)$.

We introduce norms as:

$$|\ f\ | = \max|\ Df\ |_0,\ \ |\ f\ |_0 = \max_{T\times D\times D} \|\ f(t,\ x,\ u)\|,\ \ |\ f\ |_0 = \left|\frac{1}{2\pi}\int_0^{2\pi} \|\ f\ \|^2 dt\right|^{\frac{1}{2}},$$

where $Df$ is the first-order partial derivative with respect to its variables, and $\|\cdot\|$ is the Euclidean norm.

Consider a second-order differential equation:

$$\frac{d^2x(t)}{dt^2} = Ax(t) + f(t), \tag{2}$$

where $f(t)$ is a continuous $2\pi$-periodic function, representable as a Fourier series:

$$f(t) = a_0 + \sqrt{2}\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{3}$$

Denote by $P_m f(t)$ the partial sum of Fourier series (3):

$$P_m f(t) = a_0 + \sqrt{2}\sum_{n=1}^{m}(a_n \cos nt + b_n \sin nt).$$

**Lemma.** *Let* $x = x(t) - 2\pi$ *be a periodic solution of equation* (2) *and let* $G(t, s)$ *be a Green's function*

$$G(t, s) = \begin{cases} \frac{1}{2\sqrt{A}} e^{\sqrt{A}(s-t)}, & -\infty < t \le s, \\ \frac{1}{2\sqrt{A}} e^{\sqrt{A}(t-s)}, & s < t < +\infty, \end{cases} \tag{4}$$

*satisfying the conditions*

$$G(t, t+0) - G(t, t-0) = 0,$$

$$G_t(t, t+0) - G_t(t, t-0) = 1. \tag{5}$$

*Then* $2\pi$ *periodic solution of equation* (2) *is represented as*

$$x(t) = \int_{-\infty}^{+\infty} G(t, s) f(s) ds \tag{6}$$

*and for the difference* $x(t) - P_m x(t)$, *the estimate* $|x(t) - P_m x(t)|_0 \le \gamma(m) \| f \|_0$, *at* $n^2 \ne A$, *for some integer* $n = n_0$,

$$\gamma(m) = \left[ \frac{2}{((m+1)^2 - n)^2} + \frac{2}{((m+2)^2 - n)^2} + \cdots \right]^{\frac{1}{2}}.$$

**Proof.** We show that the function $x(t)$ represented in the form (6) satisfies equation (2).

Represent (6) in the form

$$x(t) = \int_{-\infty}^{t-0} G(t, s) f(s) ds - \int_{+\infty}^{t+0} G(t, s) f(s) ds.$$

Taking into account the property (5) of the Green's function (4), we obtain

$$\frac{dx(t)}{dt} = G(t, t-0)f(t-0) - G(t, t+0)f(t+0) + \int_{-\infty}^{+\infty} G_t(t, s)f(s)ds$$

$$= (G(t, t-0) - G(t, t+0))f(t) + \int_{-\infty}^{+\infty} G(t, s)f(s)ds$$

$$= \int_{-\infty}^{t-0} G_t(t, s)f(s)ds - \int_{t+0}^{+\infty} G_t(t, s)f(s)ds,$$

$$\frac{d^2x(t)}{dt^2} = G_t(t, t-0)f(t-0) - G_t(t, t+0)f(t+0) + \int_{-\infty}^{+\infty} G_{tt}(t, s)f(s)ds$$

$$= (G_t(t, t-0) - G_t(t, t+0))f(t) + \int_{-\infty}^{+\infty} G_{tt}(t, s)f(s)ds$$

$$= f(t) + \int_{-\infty}^{+\infty} G_{tt}(t, s)f(s)ds. \tag{7}$$

Putting (6) and (7) into equation (2), we obtain

$$f(t) + \int_{-\infty}^{+\infty} [G_{tt}(t, s) - AG(t, s)]f(s)ds = f(t). \tag{8}$$

Given that

$$G_{tt}(t, s) = \begin{cases} \frac{\sqrt{A}}{2} e^{\sqrt{A}(s-t)}, & -\infty < t \le s, \\ \frac{\sqrt{A}}{2} e^{\sqrt{A}(t-s)}, & s < t < +\infty. \end{cases}$$

Taking into account (4), we obtain

$$G_{tt}(t, s) - AG(t, s) = \begin{cases} \frac{\sqrt{A}}{2} e^{\sqrt{A}(s-t)} - \frac{\sqrt{A}}{2} e^{\sqrt{A}(s-t)} = 0, & -\infty < t \le s, \\ \frac{\sqrt{A}}{2} e^{\sqrt{A}(t-s)} - \frac{\sqrt{A}}{2} e^{\sqrt{A}(t-s)} = 0, & s < t < +\infty. \end{cases}$$

Hence, $G_{tt}(t, s) - AG(t, s) = 0$. It follows from (8) that equality (2) is satisfied.

Further, since

$$\frac{d^2 P_m x(t)}{dt^2} = AP_m x(t) + P_m f(t),$$

$$\frac{d^2(x(t) - P_m x(t))}{dt^2} = A(x(t) - P_m x(t)) + f(t) - P_m f(t),$$

we obtain

$$\begin{aligned} & x(t) - P_m x(t) \\ &= \int_{-\infty}^{+\infty} G(t,\ s)\sqrt{2} \sum_{n=m+1}^{\infty} (f(s) - P_m f(s)) \\ &= \sqrt{2} \sum_{n=m+1}^{\infty} \int_{-\infty}^{+\infty} G(t,\ s)(a_n \cos ns + b_n \sin ns) ds \\ &= \frac{e^{-\sqrt{A}t}}{\sqrt{2A}} \sum_{n=m+1}^{\infty} \left( a_n \int_{-\infty}^{t} e^{\sqrt{A}s} \cos nsds + b_n \int_{-\infty}^{t} e^{\sqrt{A}s} \sin nsds \right) \\ &\quad + \frac{e^{\sqrt{A}t}}{\sqrt{2A}} \sum_{n=m+1}^{\infty} \left( a_n \int_{t}^{+\infty} e^{-\sqrt{A}s} \cos nsds + b_n \int_{t}^{+\infty} e^{-\sqrt{A}s} \sin nsds \right). \end{aligned} \tag{9}$$

The calculation shows that

$$\int_{-\infty}^{t} e^{\sqrt{A}s} \cos nsds = \frac{ne^{\sqrt{A}t}}{n^2 - A}\left[\sin nt + \frac{\sqrt{A}}{n}\cos nt\right],$$

$$\int_{-\infty}^{t} e^{\sqrt{A}s} \sin nsds = \frac{ne^{\sqrt{A}t}}{n^2 + A}\left[-\cos nt + \frac{\sqrt{A}}{n}\sin nt\right],$$

$$\int_{t}^{+\infty} e^{-\sqrt{A}s} \cos nsds = \frac{ne^{-\sqrt{A}t}}{n^2 - A}\left[-\sin nt + \frac{\sqrt{A}}{n}\cos nt\right],$$

$$\int_{t}^{+\infty} e^{-\sqrt{A}s} \sin nsds = \frac{ne^{-\sqrt{A}t}}{n^2 + A}\left[\cos nt + \frac{\sqrt{A}}{n}\sin nt\right].$$

Taking these calculations into account, from (9), we obtain

$$x(t) - P_m x(t)$$

$$= \frac{1}{\sqrt{2A}} \sum_{n=m+1}^{\infty} \left( \frac{na_n}{n^2 - A} \sin nt + \frac{a_n \sqrt{A}}{n^2 - A} \cos nt - \frac{nb_n}{n^2 + A} \cos nt + \frac{b_n \sqrt{A}}{n^2 + A} \sin nt \right.$$

$$\left. - \frac{na_n}{n^2 - A} \sin nt + \frac{a_n \sqrt{A}}{n^2 - A} \cos nt + \frac{nb_n}{n^2 + A} \cos nt + \frac{b_n \sqrt{A}}{n^2 + A} \sin nt \right)$$

$$= \frac{1}{\sqrt{2A}} \sum_{n=m+1}^{\infty} \left( \frac{2\sqrt{A} a_n}{n^2 - A} \cos nt + \frac{2\sqrt{A} b_n}{n^2 + A} \sin nt \right)$$

$$= \sqrt{2} \sum_{n=m+1}^{\infty} \left( \frac{a_n}{n^2 - A} \cos nt + \frac{b_n}{n^2 + A} \sin nt \right). \tag{10}$$

Estimating the difference $x(t) - P_m x(t)$ from (10), we obtain

$$\| x(t) - P_m x(t) \|_0 = \left\| \sum_{n=m+1}^{\infty} \left( \frac{\sqrt{2} a_n}{n^2 - A} \cos nt + \frac{\sqrt{2} b_n}{n^2 + A} \sin nt \right) \right\|_0 .$$

Hence, by the Bunyakovsky-Schwartz inequality, we have

$$\| x(t) - P_m x(t) \|_0^2 \le \left\| \sum_{n=m+1}^{\infty} \frac{\sqrt{2}}{n^2 - A} (a_n \cos nt + b_n \sin nt) \right\|_0^2$$

$$\le \sum_{n=m+1}^{\infty} \frac{\sqrt{2}}{(n^2 - A)^2} \sum_{n=m+1}^{\infty} (\| a_n \|^2 + \| b_n \|^2)$$

$$\le \| f \|_0^2 \sum_{n=m+1}^{\infty} \frac{\sqrt{2}}{(n^2 - A)^2}, \text{ when } n^2 \ne A.$$

Thus

$$\| x(t) - P_m x(t) \|_0 \le \gamma(m) \| f \|_0,$$

where

$$\gamma(m) = \left[\frac{2}{((m+1)^2 - n)^2} + \frac{2}{((m+2)^2 - n)^2} + \cdots\right]^{\frac{1}{2}}.$$

When $m \to \infty$, $\gamma(m) \to 0$. Hence $\| x(t) - P_m x(t) \|_0 \to 0$, $m \to \infty$.

The lemma is thus proved.

## 3. The Main Result

Consider the integro-differential equation (1).

$2\pi$ periodic solution of the integro-differential equation (1) is found by the method of successive approximations:

$$\frac{d^2 x_i(t)}{dt^2} = Ax_i(t) + f\left(t,\ x_{i-1}(t),\ \int_{t-\tau}^{t} \varphi(t,\ s,\ x_{i-1}(s))\right)ds,\ \ i = 1, 2, 3, .... \quad (11)$$

We take the $2\pi$ periodic function $x_0(t) \in D \subset R$ as the initial approximation.

Equation (1) for each fixed value $i = k$ represents a differential equation of the form (2) with a continuous $2\pi$-periodic function

$$f_{i-1}(t) = f\left(t,\ x_{i-1}(t),\ \int_{t-\tau}^{t} \varphi(t,\ s,\ x_{k-1}(s))\right)ds,$$

represented in Fourier series of the form

$$f_{k-1}(t) = a_{0k-1} + \sqrt{2}\sum_{n=1}^{\infty}(a_{nk-1}\cos nt + b_n k - 1\sin nt). \quad (12)$$

Then according to the lemma, the differential equation

$$\frac{d^2 x_i(t)}{dt^2} = Ax_k(t) + f_{k-1}(t)$$

has a $2\pi$-periodic solution represented as

$$x_k(t) = \int_{-\infty}^{+\infty} G(t, s) f_{k-1}(s) ds \tag{13}$$

and for the difference $x_k(t) - P_m x_k(t)$, the following estimate is valid:

$$\| x_k(t) - P_m x_k(t) \|_0 \leq \gamma(m) \| f \|_0.$$

Substituting (12) into (13), we obtain

$$x_k(t) = \frac{a_{0k-1}}{2A} + \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{a_{nk-1}}{n^2 - A} \cos nt + \frac{b_{nk-1}}{n^2 + A} \sin nt \right). \tag{14}$$

Note that

$$a_{0k-1} = \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} f\left( t, x_{k-1}(t), \int_{t-\tau}^{t} \varphi(t, s, x_{k-1}(s)) ds \right) dt,$$

$$a_{nk-1} = \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} f\left( t, x_{k-1}(t), \int_{t-\tau}^{t} \varphi(t, s, x_{k-1}(s)) ds \right) \cos ntdt,$$

$$b_{nk-1} = \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} f\left( t, x_{k-1}(t), \int_{t-\tau}^{t} \varphi(t, s, x_{k-1}(s)) ds \right) \sin nt\, dt,$$

$$n = 1, 2, 3, \ldots.$$

We prove the convergence of the sequence $\{x_i(t)\}$ to the exact solution $x(t)$ of the integro-differential equation (1).

Consider a sequence of $2\pi$-periodic functions of the form

$$x_i(t) = \int_{-\infty}^{+\infty} G(t, s) f\left( s, x_{i-1}(s), \int_{s-\tau}^{s} \varphi(s, v, x_{i-1}(v)) dv \right) ds. \tag{15}$$

We estimate the difference $x_{i+1}(t) - x_i(t)$:

$$x_{i+1}(t) - x_i(t)$$

$$= \int_{-\infty}^{+\infty} G(t, s)\left[ f\left(s, x_i(s), \int_{s-\tau}^{s} \varphi(s, v, x_i(v))dv\right)\right.$$

$$\left. - f\left((s, x_{i-1}(s), \int_{s-\tau}^{s} \varphi(s, v, x_{i-1}(v))dv\right)\right]ds$$

$$= \int_{-\infty}^{+\infty} G(t, s)\left[\frac{\partial f}{\partial x}(x_i(s) - x_{i-1}(s)) + \frac{\partial f}{\partial u}\int_{s-\tau}^{s}\frac{\partial \varphi}{\partial x}(x_i(v) - x_{i-1}(v))dv\right]ds.$$

Hence, we obtain

$$| x_{i+1}(t) - x_i(t) | \leq \int_{-\infty}^{+\infty} | G(t, s) | (| f |_1 + | f |_1 | \varphi |_1 \tau) | x_i(s) - x_{i-1}(s) |_0$$

$$\leq \frac{1}{2\sqrt{A}} \int_{-\infty}^{+\infty} e^{\sqrt{A}s-t} | f |_1 (1 + | \varphi |_1 \tau) | x_i(s) - x_{i-1}(s) |_0 ds$$

$$= \frac{1}{2A} | f |_1 (1 + | \varphi |_1 \tau) | x_i(t) - x_{i-1}(t) |_0. \tag{16}$$

Let

$$\lambda = \frac{1}{2A} | f |_1 (1 + | \varphi |_1 \tau) < 1. \tag{17}$$

Then from (16), we obtain

$$| x_{i+1}(t) - x_i(t) |_0 \leq \lambda | x_i(t) - x_{i-1}(t) |_0 \leq \lambda^2 | x_{i-1}(t) - x_{i-2}(t) |_0$$

$$\leq \cdots \leq \lambda^i | x_1(t) - x_0(t) |_0.$$

Now, we estimate the difference $x_1(t) - x_0(t)$:

$$x_1(t) - x_0(t) = x_1(t) - P_m x_1(t) + P_m x_1(t) - x_0(t).$$

By the lemma, we obtain

$$
\begin{aligned}
&| x_1(t) - x_0(t) |_0 \\
&\le | x_1(t) - P_m x_1(t) | + | P_m x_1(t) - x_0(t) | \\
&\le \gamma(m) \| f \|_0 + \left| P_m \left[ \int_{-\infty}^{+\infty} G(t, s) f\left( s, x_0(s), \int_{s-\tau}^{s} \varphi(s, v, x_0(v)) dv \right) ds \right] \right|_0 \\
&\quad + | x_0(t) |_0 \\
&\le \gamma(m) | f |_0 + \int_{-\infty}^{+\infty} G(t, s) | f |_0 ds + | x_0(t) |_0 \\
&= \gamma(m) | f |_0 + \frac{| f |_0}{2A} + | x_0(t) |_0 \\
&= \frac{1}{2A} (1 + 2A\gamma(m)) | f |_0 + | x_0(t) |_0 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
| x_{i+1}(t) - x_i(t) |_0 &\le \lambda^i | x_1(t) - x_0(t) |_0 \\
&\le \frac{\lambda^i}{2A} (1 + 2A\gamma(m)) + 2A | x_0(t) |_0 .
\end{aligned} \tag{18}
$$

Further,

$$
\begin{aligned}
&| x_{i+p}(t) - x_i(t) |_0 \\
&\le | x_{i+p}(t) - x_{i+p-1}(t) + x_{i+p-1}(t) - x_{i+p-2}(t) + \cdots + x_{i+1}(t) - x_i(t) |_0 \\
&\le | x_{i+p}(t) - x_{i+p-1}(t) |_0 + | x_{i+p-1}(t) - x_{i+p-2}(t) |_0 + \cdots + | x_{i+1}(t) - x_i(t) |_0 .
\end{aligned}
$$

Therefore, by (18), we obtain

$$
\begin{aligned}
&| x_{i+p}(t) - x_i(t) |_0 \\
&\le \frac{1}{2A} (\lambda^{i+p-1} + \lambda^{i+p-2} + \cdots + \lambda^i)(1 + 2A\gamma(m) + 2A | x_0(t) |_0) \\
&\le \frac{\lambda^i}{2A} (1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1} + \lambda^p + \cdots) [(1 + 2A\gamma(m) + 2A | x_0(t) |_0)].
\end{aligned}
$$

Hence, by the compression condition (17), we obtain the estimate as follows:

$$| x(t) - x_i(t) |_0 \leq \frac{\lambda^i}{2A\lambda(1-\lambda)} [(1+\gamma(m)) | f |_0 + | x_0(t) |_0], \quad i = 1, 2, 3, \ldots.$$

Thus, the theorem is proved.

**Theorem.** *Suppose that the integro-differential equation* (1) *satisfies*:

(a) *There exists a Green's function* (4) *boundedly solvable* $G(t, s)$ *on the numerical axis satisfying* (5).

(b) *The compression condition* (17) *is fulfilled for* $\tau > \tau_0$ *-const, and* $n^2 - A \neq 0$.

*Then the* $2\pi$ *-periodic solution* $x(t)$ *of the integro-differential equation* (1) *is defined as the limit of the function sequence obtained according to algorithm* (15).

Note that for a $2\pi$-periodic function $x = x(t)$, the case of resonance is possible, i.e., the difference $n^2 - A$ is small enough, so to avoid the phenomenon of resonance, we should choose the number $A$ so that $n^2 - A \neq 0$.

This option is always possible if the constant $A$ is not a positive integer.

## Acknowledgement

The authors thank the anonymous reviewers for their suggestions and comments which eventually improved the paper to a considerable extent.

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