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APPROXIMATED SOLUTIONS OF THE HOMOGENEOUS LINEAR FRACTIONAL DIFFUSION-CONVECTION-REACTION EQUATION

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Abstract

Our work focused on solving a homogeneous linear fractional diffusion, diffusion-convection and diffusion-convection-reaction

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model with various initial conditions and appropriate parameters. We used the Adomian decomposition method (ADM) to find exact or approximate solutions.

1. Introduction

For decades, mathematicians have paid particular attention to fractional calculus, because a fractional derivative is not a derivative at the local point, but takes into account history and non-local distributed effects. In addition, fractional mathematical models are more realistic and practical than other classical models. There are diffusion models, i.e., partial differential equations describing the behavior of the collective displacement of particles, convection-diffusion models, which are a combination of the diffusion and convection equations and describe physical phenomena, and also linear or non-linear homogeneous or non-homogeneous fractional reaction models. Fractional models take into account the history and distributed effects of the system under study. Any process modeled by fractional equations then has a memory effect. Models can include linear and non-linear partial differential equations (PDEs) or ordinary differential equations (ODEs). We took a mathematical model of linear fractional diffusion-convection-reaction with parameters whose problem is as follows:

$$\begin{cases} \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = \lambda \dfrac{\partial^2 u(x, t)}{\partial x^2} + \gamma \dfrac{\partial u(x, t)}{\partial x} + \beta u(x, t) \\ u(x, 0) = f(x) \end{cases} \tag{1}$$

with λ the diffusion coefficient, γ the convection coefficient, β the reaction coefficient all positive reals; $t \geq 0$, $x \in \mathbb{R}$ and $0 < \alpha \leq 1$ and $f(x)$ is the initial condition.

Our main objective is to use Adomian's decompositional method [2, 5, 8, 13, 15, 18-20], for solving the homogeneous linear fractional diffusion equation, the homogeneous linear fractional diffusion-convection equation and the homogeneous linear fractional diffusion-convection-reaction equation.

2. Definitions

2.1. Gamma function

Function $\Gamma(\alpha)$. It is defined by the following integral [1, 3, 4, 6, 7, 9-12, 14, 16, 17, 21]:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt, \tag{2}$$

where α is a complex number such that $\mathrm{Re}(\alpha) > 0$. The Gamma function Γ is decreasing on $[0, 1]$.

Gamma function $\Gamma(\alpha)$ satisfies [1, 3, 4, 7, 9, 11, 12, 14, 16, 17, 21]:

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \text{ where } \alpha > 0. \tag{3}$$

Euler's Gamma function generalizes the factorial.

2.2. Beta function

The beta function is defined by the Euler integral of the first kind

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt, \quad \forall p, q > 0. \tag{4}$$

2.3. Mittag-Leffler function

For $z \in \mathbb{C}$, the Mittag-Leffler function $E_\alpha(z)$ is defined as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \text{ where } \alpha > 0. \tag{5}$$

In particular,

$$E_1(z) = e^z.$$

This function can be generalized for two positive parameters α and β as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}. \tag{6}$$

3. Fractional Integral

A primitive of a continuous function on $[a; b]$ is given by the expression:

$$(I_0 h)(t) = \int_0^t h(x)dx. \tag{7}$$

For a primitive of order 2, we have

$$(I_0^2 h)(t) = \int_0^t \left(\int_0^x h(s)ds \right) dx = \int_0^t (t - x)h(x)dx. \tag{8}$$

If $h(t) = C$ with a constant C, then we have

$$I_a^{\alpha}(C) = \frac{Ct^{\alpha}}{\Gamma(\alpha + 1)}. \tag{9}$$

4. Convergence and Uniqueness of the Solution

Consider the general form of the fractional-order partial differential equation:

$$(S) : \begin{cases} \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = R(u(x, t)) \\ u(x, 0) = p(x) \end{cases} \tag{10}$$

with $0 < \alpha \leq 1$ and $R(u(x, t)) = \lambda \dfrac{\partial^2 u(x, t)}{\partial x^2} + \gamma \dfrac{\partial u(x, t)}{\partial x} + \beta u(x, t)$.

Setting $L_t u(x, t) = \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}$, we have

$$L_t u(x, t) = R(u(x, t)). \tag{11}$$

Applying $L_t^{-1}(\cdot) = I_0^{\alpha}(\cdot)$ to (11), we have

$$u(x, t) = p(x) + I_0^{\alpha}(R(u(x, t))). \tag{12}$$

The solution is sought in the form of a convergent series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad n \geq 0. \tag{13}$$

By introducing (13) into (12), we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x, t) = p(x) \\ u_{n+1}(x, t) = I_0^{\alpha}(R(u_n(x, t))), \quad n \geq 0. \end{cases} \tag{14}$$

Theorem. *We have* $\left|\frac{MT^{\alpha}}{\Gamma(\alpha+1)}\right| < 1,\ p \in C(\mathbb{R}^n),\ u(x, t) \in C(\Omega),$ *the p and u are respectively bounded by m and M such that* $\exists m = \sup_{x \in \mathbb{R}} |p(x)|$ *and* $\exists M = \sup_{(x, t) \in \Omega} |u(x, t)| > 0$ *or* $\Omega = \mathbb{R}^n \times [0; T]$; *then the Adomian algorithm is convergent and problem* (S) *has a unique solution*.

Proof. We have the following Adomian algorithm:

$$\begin{cases} u_0(x, t) = p(x) \\ u_{n+1}(x, t) = I_0^{\alpha}(R(u_n(x, t))), \quad n \geq 0, \end{cases} \tag{15}$$

$$\begin{cases} |u_0(x, t)| = |p(x)| \leq m \\ |u_1(x, t)| = |I_0^{\alpha}(R(u_0(x, t)))| \leq \dfrac{MT^{\alpha}}{\Gamma(\alpha+1)} \\ |u_2(x, t)| = |I_0^{\alpha}(R(u_1(x, t)))| \leq \left(\dfrac{MT^{\alpha}}{\Gamma(\alpha+1)}\right)^2 \\ |u_3(x, t)| = |I_0^{\alpha}(R(u_2(x, t)))| \leq \left(\dfrac{MT^{\alpha}}{\Gamma(\alpha+1)}\right)^3 \\ \vdots = \vdots \\ |u_n(x, t)| = |I_0^{\alpha}(R(u_{n-1}(x, t)))| \leq \left(\dfrac{MT^{\alpha}}{\Gamma(\alpha+1)}\right)^n; \quad n > 0. \end{cases} \tag{16}$$

Summing member by member (16), we obtain

$$\sum_{n=0}^{+\infty} |u_n(x, t)| = m + \frac{MT^{\alpha}}{\Gamma(\alpha+1) - MT^{\alpha}}; \; n > 0.$$

Hence

$$\sum_{n=0}^{+\infty} |u_n(x, t)|$$

is absolutely convergent.

Uniqueness of solution

Let $u_n(x, t)$, $v_n(x, t)$ be solutions of (10) with $u_n(x, t) \neq v_n(x, t)$. Then for *u* and *v*, we have the following algorithms:

$$\begin{cases} u_0(x, t) = p(x) \\ u_{n+1}(x, t) = I_0^{\alpha}(R(u_n(x, t))), \quad n \geq 0 \end{cases} \tag{17}$$

and

$$\begin{cases} v_0(x, t) = p(x) \\ v_{n+1}(x, t) = I_0^{\alpha}(R(v_n(x, t))), \quad n \geq 0. \end{cases} \tag{18}$$

Differentiating between (17) and (18) yields:

$$\begin{cases} u_0(x, t) - v_0(x, t) = p(x) - p(x) = 0 \\ \qquad \Rightarrow u_0(x, t) = v_0(x, t) \\ u_1(x, t) - v_1(x, t) = I_0^{\alpha}(R(u_0(x, t))) - I_0^{\alpha}(R(v_0(x, t))) = 0 \\ \qquad \Rightarrow u_1(x, t) = v_1(x, t) \\ u_2(x, t) - v_2(x, t) = I_0^{\alpha}(R(u_1(x, t))) - I_0^{\alpha}(R(v_1(x, t))) = 0 \\ \qquad \Rightarrow u_2(x, t) = v_2(x, t) \\ u_3(x, t) - v_3(x, t) = I_0^{\alpha}(R(u_2(x, t))) - I_0^{\alpha}(R(v_2(x, t))) = 0 \\ \qquad \Rightarrow u_3(x, t) = v_3(x, t) \\ \vdots = \vdots \\ u_n(x, t) - v_n(x, t) = I_0^{\alpha}(R(u_{n-1}(x, t))) - I_0^{\alpha}(R(v_{n-1}(x, t))) = 0 \\ \qquad \Rightarrow u_n(x, t) = v_n(x, t) \end{cases}$$

So $\forall n \geq 0$, we have $u_n(x, t) - v_n(x, t) = 0 \Rightarrow u_n(x, t) = v_n(x, t)$; or according to the hypothesis $u_n(x, t) \neq v_n(x, t)$; which is contradictory, so the problem has a unique solution.

5. Homogeneous Linear Fractional Diffusion Equation

$$\begin{cases} \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = \lambda \dfrac{\partial^2 u(x, t)}{\partial x^2} \\ u(x, 0) = A \sin(\omega x) + B \cos(\omega x) \end{cases} \tag{19}$$

with $\lambda \in \mathbb{R}_+$; $A, B \in \mathbb{R}$, $t \geq 0$; $\omega \in \mathbb{R}_+^*$, $x \in \mathbb{R}$ and $0 < \alpha \leq 1$.

Proposition 1. *The ADM method provides an exact solution to the diffusion problem*

$$u(x, t) = [A \sin(\omega x) + B \cos(\omega x)] \cdot E_{\alpha}(-\lambda \omega^2 t^{\alpha}).$$

Proof. Let us ask $Lu(x, t) = \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}$. We have

$$Lu(x, t) = \lambda \frac{\partial^2 u(x, t)}{\partial x^2}. \tag{20}$$

Applying $L_t^{-1}(\cdot) = I_0^{\alpha}(\cdot)$ to (20), we obtain

$$u(x, t) = A\sin(\omega x) + B\cos(\omega x) + \lambda I_0^{\alpha}\left(\frac{\partial^2 u(x, t)}{\partial x^2}\right). \tag{21}$$

The solution of the problem is sought in the form of a convergent series:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t); \quad n \geq 0. \tag{22}$$

Replacing (22) in (21), we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x, t) = A\sin(\omega x) + B\cos(\omega x) \\ u_{n+1}(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_n(x, t)}{\partial x^2}\right); \quad n \geq 0. \end{cases} \tag{23}$$

Calculating the following terms, we obtain:

for $n = 0$,

$$u_1(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_0(x, t)}{\partial x^2}\right)$$

$$= [A\sin(\omega x) + B\cos(\omega x)]\frac{-\lambda\omega^2 t^{\alpha}}{\Gamma(\alpha + 1)}$$

for $n = 1$,

$$u_2(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_1(x, t)}{\partial x^2}\right)$$

$$= [A\sin(\omega x) + B\cos(\omega x)]\frac{(-\lambda\omega^2 t^{\alpha})^2}{\Gamma(2\alpha + 1)}$$

for $n = 2$,

$$u_3(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_2(x, t)}{\partial x^2}\right)$$

$$= [A\sin(\omega x) + B\cos(\omega x)]\frac{(-\lambda\omega^2 t^{\alpha})^3}{\Gamma(3\alpha + 1)}.$$

Recursively, we have

$$\begin{cases} u_0(x, t) = [A\sin(\omega x) + B\cos(\omega x)] \\ u_1(x, t) = [A\sin(\omega x) + B\cos(\omega x)]\dfrac{(-\lambda\omega^2 t^{\alpha})}{\Gamma(\alpha + 1)} \\ u_2(x, t) = [A\sin(\omega x) + B\cos(\omega x)]\dfrac{(-\lambda\omega^2 t^{\alpha})^2}{\Gamma(2\alpha + 1)} \\ u_3(x, t) = [A\sin(\omega x) + B\cos(\omega x)]\dfrac{(-\lambda\omega^2 t^{\alpha})^3}{\Gamma(3\alpha + 1)} \\ \vdots = \vdots \\ u_n(x, t) = [A\sin(\omega x) + B\cos(\omega x)]\dfrac{(-\lambda\omega^2 t^{\alpha})^n}{\Gamma(n\alpha + 1)}. \end{cases}$$

The solution to the problem is

$$u(x, t) = [A\sin(\omega x) + B\cos(\omega x)]\sum_{n=0}^{\infty}\frac{(-\lambda\omega^2 t^{\alpha})^n}{\Gamma(n\alpha + 1)}$$

$$= [A\sin(\omega x) + B\cos(\omega x)]\cdot E_{\alpha}(-\lambda\omega^2 t^{\alpha})$$

with $E_{\alpha}(-\lambda\omega^2 t^{\alpha})$ the Mittag-Leffler function.

6. Homogeneous Linear Fractional Diffusion-convection Equation

$$\begin{cases} \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = \lambda\dfrac{\partial^2 u(x, t)}{\partial x^2} + \gamma\dfrac{\partial u(x, t)}{\partial x} \\ u(x, 0) = A\sin(\omega x) + B\cos(\omega x) \end{cases} \tag{24}$$

with $\lambda, \gamma \in \mathbb{R}_+$; $t \geq 0$; $\omega \in \mathbb{R}_+^*$, $A, B \in \mathbb{R}$; $x \in \mathbb{R}$ and $0 < \alpha \leq 1$.

Proposition 2. *The ADM method provides an approximate* 4*th-order solution to the diffusion-convection problem*:

$$u(x, t) \simeq A\sin(\omega x) + B\cos(\omega x) + [A_1 \sin(\omega x) + B_1 \cos(\omega x)]\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ [A_2 \sin(\omega x) + B_2 \cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+ [A_3 \sin(\omega x) + B_3 \cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$+ [A_4 \sin(\omega x) + B_4 \cos(\omega x)]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}.$$

Proof. Let us put $Lu(x, t) = \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}$. We have

$$Lu(x, t) = \lambda \frac{\partial^2 u(x, t)}{\partial x^2} + \gamma \frac{\partial u(x, t)}{\partial x}. \tag{25}$$

Let us apply $L_t^{-1}(\cdot) = I_0^{\alpha}(\cdot)$ fractional integral to (25), we obtain

$$u(x, t) = A\sin(\omega x) + B\cos(\omega x) + \lambda I_0^{\alpha}\left(\frac{\partial^2 u(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u(x, t)}{\partial x}\right). \tag{26}$$

The solution of the problem is sought in the form of a convergent series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t); \quad n \geq 0. \tag{27}$$

Replacing (27) in (26), we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x, t) = A\sin(\omega x) + B\cos(\omega x) \\ u_{n+1}(x, t) = \lambda I_0^{\alpha}\left(\dfrac{\partial^2 u_n(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\dfrac{\partial u_n(x, t)}{\partial x}\right); \ n \geq 0. \end{cases} \tag{28}$$

Calculating the following terms, we obtain:

for $n = 0$,

$$u_1(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_0(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u_0(x, t)}{\partial x}\right)$$

$$= [(-\lambda A\omega^2 - \gamma B\omega)\sin(\omega x) + (-\lambda B\omega^2 + \gamma A\omega)\cos(\omega x)]\frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

$$= [A_1 \sin(\omega x) + B_1 \cos(\omega x)]\frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

with

$$\begin{cases} A_1 = -\lambda A\omega^2 - \gamma B\omega \\ B_1 = -\lambda B\omega^2 + \gamma A\omega \end{cases}$$

for $n = 1$,

$$u_2(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_1(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u_1(x, t)}{\partial x}\right)$$

$$= [(-\lambda A_1\omega^2 - \gamma B_1\omega)\sin(\omega x) + (-\lambda B_1\omega^2 + \gamma A_1\omega)\cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$= [A_2 \sin(\omega x) + B_2 \cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

with

$$A_2 = -\lambda A_1\omega^2 - \gamma B_1\omega$$

$$= \lambda^2 A\omega^4 + 2\lambda\gamma B\omega^3 - \gamma^2 A\omega^2,$$

$$B_2 = -\lambda B_1\omega^2 + \gamma A_1\omega$$

$$= \lambda^2 B\omega^4 - 2\lambda\gamma A\omega^3 - \gamma^2 B\omega^2$$

for $n = 2$,

$$u_3(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_2(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u_2(x, t)}{\partial x}\right)$$

$$= [(-\lambda A_2\omega^2 - \gamma B_2\omega)\sin(\omega x) + (-\lambda B_2\omega^2 + \gamma A_2\omega)\cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$= [A_3 \sin(\omega x) + B_3 \cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

with

$$A_3 = -\lambda\omega^2 A_2 - \gamma\omega B_2$$

$$= -\lambda^3 A\omega^6 - 3\lambda^2\gamma B\omega^5 + 3\lambda\gamma^2 A\omega^4 + \gamma^3 B\omega^3,$$

$$B_3 = -\lambda\omega^2 B_2 + \gamma\omega A_2$$

$$= -\lambda^3 B\omega^6 + 3\lambda^2\gamma A\omega^5 + 3\lambda\gamma^2 B\omega^4 - \gamma^3 A\omega^3$$

for $n = 3$,

$$u_4(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_3(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u_3(x, t)}{\partial x}\right)$$

$$= [(-\lambda A_3\omega^2 - \gamma B_3\omega)\sin(\omega x) + (-\lambda B_3\omega^2 + \gamma A_3\omega)\cos(\omega x)]\frac{t^{4\alpha}}{\Gamma(4\alpha + 1)}$$

$$= [A_4 \sin(\omega x) + B_4 \cos(\omega x)]\frac{t^{4\alpha}}{\Gamma(4\alpha + 1)}$$

with

$$A_4 = -\lambda A_3\omega^2 - \gamma B_3\omega$$

$$= \lambda^4\omega^8 A + 4\lambda^3\gamma\omega^7 B - 6\lambda^2\gamma^2\omega^6 A - 4\lambda\gamma^3\omega^5 B + \gamma^4\omega^4 A,$$

$$B_4 = -\lambda\omega^2 B_3 + \gamma\omega A_3$$

$$= \lambda^4\omega^8 B - 4\lambda^3\gamma\omega^7 A - 6\lambda^2\gamma^2\omega^6 B + 4\lambda\gamma^3\omega^5 A + \gamma^4\omega^4 B.$$

The approximate solution to the problem (24) is

$$u(x, t) \simeq u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t)$$

$$\simeq A\sin(\omega x) + B\cos(\omega x) + [A_1 \sin(\omega x) + B_1 \cos(\omega x)]\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ [A_2 \sin(\omega x) + B_2 \cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+ [A_3 \sin(\omega x) + B_3 \cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$+ [A_4 \sin(\omega x) + B_4 \cos(\omega x)]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}.$$

7. Homogeneous Linear Fractional Diffusion-convection-reaction Equation

$$\begin{cases} \dfrac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = \lambda \dfrac{\partial^2 u(x, t)}{\partial x^2} + \gamma \dfrac{\partial u(x, t)}{\partial x} + \beta u(x, t) \\ u(x, 0) = A\sin(\omega x) + B\cos(\omega x) \end{cases} \tag{29}$$

with $\lambda, \gamma, \beta \in \mathbb{R}_+$; $t \geq 0$; $\omega \in \mathbb{R}_+^*$, $A, B \in \mathbb{R}$; $x \in \mathbb{R}$ and $0 < \alpha \leq 1$.

Proposition 3. *The ADM method provides an approximate* 3*rd-order solution to the diffusion-convection-reaction problem*:

$$u(x, t) \simeq A\sin(\omega x) + B\cos(\omega x) + [L\sin(\omega x) + K\cos(\omega x)]\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ [L_1 \sin(\omega x) + K_1 \cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+ [L_2 \sin(\omega x) + K_2 \cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

Proof. Let us put $Lu(x, t) = \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}$. We have

$$Lu(x, t) = \lambda \frac{\partial^2 u(x, t)}{\partial x^2} + \gamma \frac{\partial u(x, t)}{\partial x} + \beta u(x, t). \tag{30}$$

Applying $L_t^{-1}(\cdot) = I_0^{\alpha}(\cdot)$ to (30), we obtain

$$u(x, t) = A \sin(\omega x) + B \cos(\omega x) + \lambda I_0^{\alpha}\left(\frac{\partial^2 u(x, t)}{\partial x^2}\right)$$

$$+ \gamma I_0^{\alpha}\left(\frac{\partial u(x, t)}{\partial x}\right) + \beta I_0^{\alpha}(u(x, t)). \tag{31}$$

The solution of the problem is sought in the form of a convergent series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t); \;\; n \geq 0. \tag{32}$$

Replacing (32) in (31), we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x, t) = A \sin(\omega x) + B \cos(\omega x) \\ u_{n+1}(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_n(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u_n(x, t)}{\partial x}\right) + \beta I_0^{\alpha}(u_n(x, t)); \; n \geq 0. \end{cases} \tag{33}$$

Calculating the following terms, we obtain:

for $n = 0$,

$$u_1(x, t) = \lambda I_0^{\alpha}\left(\frac{\partial^2 u_0(x, t)}{\partial x^2}\right) + \gamma I_0^{\alpha}\left(\frac{\partial u_0(x, t)}{\partial x}\right) + \beta I_0^{\alpha}(u_0(x, t))$$

$$= [(-\lambda A \omega^2 - \gamma B \omega + \beta A) \sin(\omega x)$$

$$+ (-\lambda B \omega^2 + \gamma A \omega + \beta B) \cos(\omega x)] \frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

$$= [L \sin(\omega x) + K \cos(\omega x)] \frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

with

$$\begin{cases} L = -\lambda A\omega^2 - \gamma B\omega + \beta A \\ K = -\lambda B\omega^2 + \gamma A\omega + \beta B \end{cases}$$

for $n = 1$,

$$u_2(x, t) = \lambda I_0^\alpha\left(\frac{\partial^2 u_1(x, t)}{\partial x^2}\right) + \gamma I_0^\alpha\left(\frac{\partial u_1(x, t)}{\partial x}\right) + \beta I_0^\alpha(u_1(x, t))$$

$$= [(-\lambda L\omega^2 - \gamma K\omega + \beta L)\sin(\omega x)$$

$$+ (-\lambda K\omega^2 + \gamma L\omega + \beta K)\cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$= [L_1 \sin(\omega x) + K_1 \cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

with

$$L_1 = -\lambda L\omega^2 - \gamma K\omega + \beta L$$

$$= \lambda^2 A\omega^4 + 2\lambda\gamma B\omega^3 - 2\lambda\beta A\omega^2 - 2\beta\gamma B\omega - \gamma^2 A\omega^2 + \beta^2 A,$$

$$K_1 = -\lambda K\omega^2 + \gamma L\omega + \beta K$$

$$= \lambda^2 B\omega^4 - 2\lambda\gamma A\omega^3 - 2\lambda\beta B\omega^2 + 2\beta\gamma A\omega - \gamma^2 B\omega^2 + \beta^2 B$$

for $n = 2$,

$$u_3(x, t) = \lambda I_0^\alpha\left(\frac{\partial^2 u_2(x, t)}{\partial x^2}\right) + \gamma I_0^\alpha\left(\frac{\partial u_2(x, t)}{\partial x}\right) + \beta I_0^\alpha(u_2(x, t))$$

$$= [(-\lambda L_1\omega^2 - \gamma K_1\omega + \beta L_1)\sin(\omega x)$$

$$+ (-\lambda K_1\omega^2 + \gamma L_1\omega + \beta K_1)\cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$= [L_2 \sin(\omega x) + K_2 \cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

with

$$L_2 = -\lambda L_1\omega^2 - \gamma K_1\omega + \beta L_1$$

$$= -\lambda^3 A\omega^6 - 3\lambda^2\gamma B\omega^5 + 3\lambda^2\beta A\omega^4 + 6\lambda\gamma\beta B\omega^3 + 3\lambda\gamma^2 A\omega^4$$

$$- 3\lambda\beta^2 A\omega^2 - 3\beta\gamma^2 A\omega^2 - 3\gamma\beta^2 B\omega + \gamma^3 B\omega^3 + \beta^3 A,$$

$$K_2 = -\lambda K_1\omega^2 + \gamma L_1\omega + \beta K_1$$

$$= -\lambda^3 B\omega^6 + 3\lambda^2\gamma A\omega^5 + 3\lambda^2\beta B\omega^4 - 6\lambda\gamma\beta A\omega^3 + 3\lambda\gamma^2 B\omega^4$$

$$- 3\lambda\beta^2 B\omega^2 - 3\beta\gamma^2 B\omega^2 + 3\gamma\beta^2 A\omega - \gamma^3 A\omega^3 + \beta^3 B.$$

The general solution to the problem is

$$u(x, t) \simeq u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t)$$

$$\simeq A\sin(\omega x) + B\cos(\omega x) + [L\sin(\omega x) + K\cos(\omega x)]\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ [L_1\sin(\omega x) + K_1\cos(\omega x)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+ [L_2\sin(\omega x) + K_2\cos(\omega x)]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

8. Conclusion

In this work, we have given a brief review of fractional calculations. We used the ADM method to successfully solve the diffusion problem, while we found approximate solutions for the diffusion-convection and diffusion-convection-reaction problems. The analytical solution yielded one exact and two approximate solutions.

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