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SOLVABILITY FOR CONTINUOUS CLASSICAL BOUNDARY OPTIMAL CONTROL OF COUPLE FOURTH ORDER LINEAR ELLIPTIC EQUATIONS

Eman Hussain Mukhalf Al-Rawdhanee

Department of Mathematics
College of Education
Mustansiriyah University
Baghdad, Iraq
e-mail: dr.eman_hussain@uomustansiriyah.edu.iq

Abstract

In this paper, we study continuous classical boundary optimal control problem for the couple fourth order of linear elliptic system with variable coefficients. The existence theorem of a unique couple vector state solution of the weak form obtaining from the couple fourth order of linear elliptic system with Neumann conditions (NCs) is demonstrated for fixed continuous classical couple boundary control vector (CCCPBCTV) utilizing Hermite finite element method. The existence theorem of a couple continuous classical boundary optimal

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control vector dominated with the considered problem is proved. The existence and uniqueness of the solution of the couple adjoint equations (CPAEs) is discussed, when the classical couple optimal boundary control is given. Finally, the Fréchet derivative (FrD) of the Hamiltonian is obtained to establish the theorem of the necessary condition for optimality.

1. Introduction

The use of optimal control problem (OCTPs) applications has become involved with various diverse subject areas like medicine [1], economic [2, 3], biology [4], electric power [5], aircraft [6] and other fields.

In the last century, many investigators interested to study the OCTPs which are either governing by ordinary differential equations (ODEs) as [7] or are involving by partial differential equations (PDEs) as [8]. In the recent years, the importance of OCTPs pushed many investigators interest to develop the continuous classical optimal boundary control problems (CCOBCTPs) which are involving either by a second order PDEs of elliptic type [9-11] or hyperbolic type [12] or parabolic type [13, 14]. These articles encourage us to study the optimal boundary control problem associated with a couple fourth order linear elliptic partial differential equations (LEPDEs). In this paper, the finite element method (FEM) with a piecewise cubic Hermite (PCH) basis function is applied to prove the existence and uniqueness of a couple state vector (CPSV) solution for a couple fourth order Neumann boundary value problem (NBVP) of LEPDEs with variable coefficients (VCs), when the CCCPBCTV is considered. Under essential assumptions, the existence theorem of a continuous classical couple optimal boundary control vector (CCCPOBCTV) associated with a couple fourth order LEPDEs is developed and proved. The existence and uniqueness of the solution of the CPAEs is discussed, when the CCCPOBCTV is given. The FrD of the Hamiltonian is introduced. In the end, the theorem of necessary condition (NEC) for optimality of the above considered problem is obtained.

2. Problem Statement

Let $\Lambda \subset \mathbb{R}^2$ be an open and bounded domain with Lipschitz boundary Γ. Consider the CCCPBCTV consisting of couple fourth order LEBVPs with VCs:

$$\sum_{i,j,k,l=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}\frac{\partial^2 y_1}{\partial x_k \partial x_l}\right) + a_1 y_1 - a_2 y_2 = G_1(x_1, x_2), \text{ on } \Lambda, \tag{1}$$

$$\sum_{i,j,k,l=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left(b_{ijkl}\frac{\partial^2 y_2}{\partial x_k \partial x_l}\right) + b_1 y_2 + a_2 y_1 = G_2(x_1, x_2), \text{ on } \Lambda, \tag{2}$$

$$\frac{\partial y_1}{\partial n} = 0, \text{ on } \Gamma, \tag{3}$$

$$\sum_{i,j,k,l=1}^{2} \frac{\partial a_{ijkl}\Delta y_1}{\partial n} = q_1, \text{ on } \Gamma, \tag{4}$$

$$\frac{\partial y_2}{\partial n} = 0, \text{ in } \Gamma, \tag{5}$$

$$\sum_{i,j,k,l=1}^{2} \frac{\partial b_{ijkl}\Delta y_2}{\partial n} = q_2, \text{ on } \Gamma, \tag{6}$$

where $a_{ijkl}, a_1, a_2, b_{ijkl}, b_1 \in L^\infty(\Lambda)$, and

$$\vec{y} = (y_1, y_2) = (y_1(x_1, x_2), y_2(x_1, x_2)) \in (H^4(\Lambda))^2$$

is the CPSV corresponding to continuous classical couple control vector (CCCPCTV) $(q_1, q_2) = (q_1(x_1, x_2), q_2(x_1, x_2)) \in (L^2(\Gamma))^2$ with a given vector function $(G_1, G_2) = (G_1(x_1, x_2), G_2(x_1, x_2)) \in (L^2(\Lambda))^2$ defined on $\Lambda \times \Lambda$, for all $x = (x_1, x_2) \in \Lambda$.

The set of admissible CCCPBCTV is

$$\overrightarrow{Q_a} \subset L^2(\Gamma) \times L^2(\Gamma)$$

$$= \{\vec{q} = (q_1, q_2) \in (L^2(\Gamma))^2 \,|\, (q_1, q_2) \in Q_1 \times Q_2 = \vec{Q} \subset \mathbb{R}^2 \text{ a.e. in } \Gamma\},$$

where $\vec{Q} \subset \mathbb{R}^2$ is convex.

The cost function (CF)

$$\operatorname{Min} J_0(\vec{q}) = \frac{1}{2}\| y_1 - y_{1d} \|^2 + \frac{1}{2}\| y_2 - y_{2d} \|^2 + \frac{\lambda_1}{2}\| q_1 \|_\Gamma^2 + \frac{\lambda_2}{2}\| q_2 \|_\Gamma^2,$$

$$(q_1, q_2) \in \overrightarrow{Q_a}, \tag{7}$$

where $\lambda_1, \lambda_2 > 0$, $(y_{1d}, y_{2d}) = (y_{1d}(x_1, x_2), y_{2d}(x_1, x_2))$ is the desired data and $(y_1, y_2) = (y_{1q_1}, y_{2q_2})$ is the solution of CPSV (1-6) corresponding to the CCCPCTV $\vec{q} = (q_1, q_2)$.

The continuous classical couple optimal boundary control problem is to minimize (7) subject to $\vec{q} = (q_1, q_2) \in \overrightarrow{Q_a}$, where the notations (u, u) and $(\vec{u}, \vec{u})_{(L^2(\Lambda))^2}$ denote the inner product in $L^2(\Lambda)$ and in $(L^2(\Lambda))^2$, respectively, $(u, u)_\Gamma$ and $(\vec{u}, \vec{u})_{(L^2(\Gamma))^2}$ denote the inner product in $L^2(\Gamma)$ and in $(L^2(\Gamma))^2$, respectively, $\| u \|$, and $\| \vec{u} \|_{(L^2(\Lambda))^2} = \sum_{i=1}^{2} \| u_i \|_{L^2(\Lambda)}$ denote the norm in $L^2(\Lambda)$ and in $(L^2(\Lambda))^2$, respectively, $\| u \|_{H^2(\Lambda)}$, and $\| \vec{u} \|_{(H^2(\Lambda))^2} = \sum_{i=1}^{2} \| u_i \|_{H^2(\Lambda)}$ denote the norm in $H^2(\Lambda)$ (Sobolev space) and in $(H^2(\Lambda))^2$, respectively. Notations $\rightharpoonup$ and $\rightarrow$ refer to the weak convergence and strong convergence of a sequence, respectively.

3. Solution of the CPSV Equations

First, we find the weak forms (WFs) of the CPSV equations (1-6). Let

$$\vec{S} = H^2(\Lambda) \times H^2(\Lambda)$$

$$= \{\vec{v} : \vec{v} = (v_1, v_2) = (v_1(x_1, x_2), v_2(x_1, x_2)) \in (H^2(\Lambda))^2,$$

$$\forall (x_1, x_2) \in \Lambda, \text{ with } \frac{\partial v_1}{\partial n} = \frac{\partial v_2}{\partial n} = 0 \text{ on } \Gamma\}.$$

When $\vec{y} \in (H^2(\Lambda))^2$ is obtained by multiplying both sides of equations (1) and (2) by $v_1 \in H^2(\Lambda)$ and $v_2 \in H^2(\Lambda)$, respectively, integrating both sides of the obtained equations over Λ, and then utilizing the generalized Green's theorem, we introduce the WFs

$$A_1(y_1, v_1) + (a_1 y_1, v_1) - (a_2 y_2, v_1) = (G_1(x), v_1) + (q_1, v_1)_\Gamma, \ \forall v_1 \in H^2(\Lambda) \tag{8}$$

and

$$A_2(y_2, v_2) + (b_1 y_2, v_2) + (a_2 y_1, v_2) = (G_2(x), v_2) + (q_2, v_2)_\Gamma, \ \forall v_2 \in H^2(\Lambda), \tag{9}$$

where

$$A_1(y_1, v_1) = \iint_\Lambda \sum_{i, j, k, l=1}^{2} \left(a_{ijkl} \frac{\partial^2 y_1}{\partial x_k \partial x_l} \frac{\partial^2 v_1}{\partial x_k \partial x_l} \right) dx,$$

$$A_2(y_2, v_2) = \iint_\Lambda \sum_{i, j, k, l=1}^{2} \left(b_{ijkl} \frac{\partial^2 y_2}{\partial x_k \partial x_l} \frac{\partial^2 v_2}{\partial x_k \partial x_l} \right) dx,$$

$$A_l(y_l, y_l) \geq \beta_l \| y_l \|^2_{H^2(\Lambda)}, \quad \beta_l > 0, \quad l = 1, 2$$

and

$$| A_l(y_l, v_l) | \leq C_l \| y_l \|_{H^2(\Lambda)} \| v_l \|_{H^2(\Lambda)}, \text{ where } C_l > 0, \ l = 1, 2.$$

By adding (8) and (9), we find $\forall \vec{y} \in (H^2(\Lambda))^2$,

$$\mathcal{B}(\vec{y}, \vec{v}) = l(\vec{v}), \quad \forall (v_1, v_2) \in \vec{S}, \tag{10}$$

where the symmetric bilinear form (BLF) $\mathcal{B}(\vec{y}, \vec{v})$ and the continuous linear form $l(\vec{v})$ are explained in the following, when $\vec{q} \in (L^2(\Gamma))^2$ is fixed:

$$\mathcal{B}(\vec{y}, \vec{v}) = A_1(y_1, v_1) + (a_1 y_1, v_1) - (a_2 y_2, v_1) + A_2(y_2, v_2)$$
$$+ (b_1 y_2, v_2) + (a_2 y_1, v_2), \tag{11}$$

$$l(\vec{v}) = (G_1(x), v_1) + (q_1, v_1)_\Gamma + (G_2(x), v_2) + (q_2, v_2)_\Gamma, \forall (v_1, v_2) \in \vec{S}. \tag{12}$$

Assumptions (A)

(1) The BLF $\mathcal{B}(\cdot, \cdot)$ satisfies the following properties:

(a) $\mathcal{B}(\vec{y}, \vec{v})$ is coercive, i.e., $\forall \vec{y} \in \vec{S}$, $\exists c_0 > 0$ such that $\mathcal{B}(\vec{y}, \vec{v}) \geq c_0 \| \vec{y} \|^2_{(H^2(\Lambda))^2}$.

(b) $\mathcal{B}(\vec{y}, \vec{v})$ is continuous, i.e.,

$\exists c_1 > 0$ such that $|\mathcal{B}(\vec{y}, \vec{v})| \leq c_1 \| \vec{y} \|_{(H^2(\Lambda))^2} \| \vec{v} \|_{(H^2(\Lambda))^2}, \forall \vec{y}, \vec{v} \in \vec{S}$.

(2) $l(\vec{v})$ is a bounded functional on $\vec{S}$, where $\vec{q}$ is bounded, i.e.,

$$\exists c_2 > 0 \text{ such that } |l(\vec{v})| \leq c_2 \| \vec{v} \|_{(H^2(\Lambda))^2}, \forall \vec{v} \in \vec{S}.$$

To obtain the solution of the general classical problem (10), the FEM is utilized by choosing a finite approximation subspace $\vec{S}_n \subset \vec{S}$ and the problem (10) reduces to the discrete Galerkin WF: find $\vec{y}_n \in \vec{S}_n$ such that

$$\mathcal{B}(\vec{y}_n, \vec{v}) = l(\vec{v}), \quad \forall \vec{v} \in \vec{S}_n. \tag{13}$$

Theorem 3.1. *For any fixed CCCPCTV* $\vec{q} = (q_1, q_2) \in (L^2(\Gamma))^2$, *there is a unique approximation solution* $\vec{y}_n = (y_{1n}, y_{2n}) \in \vec{S}_n$ *for problem* (13).

Proof. For each n, let $\vec{S}_n$ be the set of continuous and PCH type polynomials functions in Λ. We define two Hermite basis functions namely $\vec{\varphi}_j$ and $\vec{\overline{\varphi}}_j$, i.e., $\{\vec{\varphi}_1, \vec{\varphi}_2, ..., \vec{\varphi}_n, \vec{\overline{\varphi}}_1, \vec{\overline{\varphi}}_2, ..., \vec{\overline{\varphi}}_n\}$ is a finite Hermite basis.

We now express $\vec{y}_n = \vec{y}_n(x_1, x_2)$ as a finite linear combination

$$\vec{y}_n = \sum_{j=1}^{n} (c_j \vec{\varphi}_j(x_1, x_2) + \overline{c}_j \vec{\overline{\varphi}}_j(x_1, x_2))$$

$$\equiv \left(\sum_{j=1}^{n} c_j \varphi_{1j} + \overline{c}_j \overline{\varphi}_{1j}, \sum_{j=1}^{n} c_j \varphi_{2j} + \overline{c}_j \overline{\varphi}_{2j} \right), \tag{14}$$

where c_j, $\overline{c}_j$ are unknown constant vectors, $\forall j = 1, 2, ..., n$.

By substituting the solution $\vec{y}_n$ in equation (13) and $\vec{v} = \vec{\varphi}_i + \vec{\overline{\varphi}}_i$, equation (13), we have

$$Kc = b, \tag{15}$$

where $K = (k_{ij})_{n \times n}$, $k_{ij} = \mathcal{B}(\vec{\varphi}_j + \vec{\overline{\varphi}}_j, \vec{\varphi}_i + \vec{\overline{\varphi}}_i)$, $b = (b_i)_{n \times 1}$, $b_i = l(\vec{\varphi}_i + \vec{\overline{\varphi}}_i)$ and $c = (c_1, ..., c_n, \overline{c}_1, ..., \overline{c}_n)^T$.

To obtain the uniqueness of the solution, set $\sum_{j=1}^{n} c_j k_{ij} = 0$, $\forall i = 1, 2, ..., n$.

Now, by utilizing assumption (A(1-a)), the system (15) has a unique solution which gives the existence of a uniqueness solution of (13).

Remark 3.1. $\forall \vec{v} \in (H_0^2(\Lambda))^2$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n \in \vec{S}_n$, $\forall n$, and $\vec{v}_n \to \vec{v}$ in $\vec{S}$, problem (13) has a unique solution $\vec{y}_n$, hence corresponding to the sequence $\{\vec{S}_n\}_{n=1}^{\infty}$, we have a sequence of (13), for each $n = 1, 2, ...$, i.e., $\vec{y}_n \in \vec{S}_n$ such that

$$\mathcal{B}(\vec{y}_n, \vec{v}_n) = l(\vec{v}_n), \quad \forall \vec{v}_n \in \vec{S}_n, \forall n \tag{16}$$

which has a sequence of $\{\vec{y}_n\}_{n=1}^{\infty}$.

Theorem 3.2 (Existence solution of the CPSV equations)**.** *The sequence of solution* $\{\vec{y}_n\}_{n=1}^{\infty}$ (*of the sequence of WF* (16)) *converges to* $\vec{y}$ (*solution of* (12)).

Proof. Since $\vec{y}_n$ is a solution of (16), using assumptions (A(1-a)) and (A(2)), we verify that $\| \vec{y}_n \|_{(H_0^2(\Lambda))^2} \leq c_2$, where $c_2 > 0, \forall n$.

From Alaoglu theorem [9], there exists a subsequence of $\{\vec{y}_n\}$ (say $\{\vec{y}_n\}$) such that $\vec{y}_n \rightharpoonup \vec{y}$ in $\vec{S}$. We want to show that the sequence $\{\vec{y}_n\}_{n=1}^{\infty}$ of the solutions of (16) converges to the solution $\vec{y}$ of (12).

First, we prove that the left hand side of (16) $\to$ the left hand side of (12).

Since $\vec{y}_n \rightharpoonup \vec{y}$ in $\vec{S}$ from above and $\vec{v}_n \to \vec{v}$ in $\vec{S}$, we obtain

$$\begin{aligned} |\mathcal{B}(\vec{y}_n, \vec{v}_n) - \mathcal{B}(\vec{y}, \vec{v})| &= |\mathcal{B}(\vec{y}_n, \vec{v}_n - \vec{v}) + \mathcal{B}(\vec{y}_n - \vec{y}, \vec{v})| \\ &\leq c_1 \| \vec{y}_n \|_{(H^2(\Lambda))^2} \| \vec{v}_n - \vec{v} \|_{(H^2(\Lambda))^2} \\ &\quad + c_1 \| \vec{y}_n - \vec{y} \|_{(H^2(\Lambda))^2} \| \vec{v} \|_{(H^2(\Lambda))^2} \to 0 \end{aligned}$$

$\Rightarrow \mathcal{B}(\vec{y}_n, \vec{v}_n) \to \mathcal{B}(\vec{y}, \vec{v})$.

Next, we show that the right hand side of (16) $\to$ the right hand side of (12).

Since $\vec{v}_n \to \vec{v}$ in $\vec{S}$, we get $\vec{v}_n \rightharpoonup \vec{v}$ in $\vec{S}$.

Now, for fixed $\vec{v} \in \vec{S}$, we have

$$l(\vec{v}_n) \to l(\vec{v}). \tag{17}$$

This gives $\mathcal{B}(\vec{y}, \vec{v}) = l(\vec{v}), \forall \vec{v} \in \vec{S}$.

Therefore, $\vec{y}$ is a solution of (12).

To demonstrate $\vec{y}_n \to \vec{y}$ in $\vec{S}$.

From assumption (A(1-a)) and (17), it follows that

$$c_0 \| \vec{y} - \vec{y}_n \|^2_{(H_0^2(\Lambda))^2}$$

$$\leq \mathcal{B}(\vec{y} - \vec{y}_n, \vec{y} - \vec{y}_n) = \mathcal{B}(\vec{y} - \vec{y}_n, \vec{y}) - \mathcal{B}(\vec{y} - \vec{y}_n, \vec{y}_n)$$

$$= \mathcal{B}(\vec{y} - \vec{y}_n, \vec{y}) - \mathcal{B}(\vec{y}, \vec{y}_n) + \mathcal{B}(\vec{y}_n, \vec{y}_n)$$

$$= \mathcal{B}(\vec{y} - \vec{y}_n, \vec{y}) + l(\vec{v}) - l(\vec{v}_n) \to 0.$$

Therefore, $\{\vec{y}_n\}$ converges to $\vec{y}$ strongly with respect to $\| \cdot \|_{(H^2(\Lambda))^2}$.

Now, let $\vec{y}_1, \vec{y}_2$ be two solutions of (12). Then

$$\mathcal{B}(\vec{y}_1, \vec{v}) = F(\vec{v}), \quad \forall \vec{v} \in \vec{S},$$

$$\mathcal{B}(\vec{y}_2, \vec{v}) = F(\vec{v}), \quad \forall \vec{v} \in \vec{S}.$$

The above two equations give

$$\mathcal{B}(\vec{y}_1 - \vec{y}_2, \vec{v}) = 0, \quad \forall \vec{v} \in \vec{S}. \tag{18}$$

Now, by inserting $\vec{v} = \vec{y}_1 - \vec{y}_2$ in (18) and using assumption (A(1-a)), we find that $\vec{y}_1 = \vec{y}_2$, i.e., the solution is unique.

4. Existence of a Couple Boundary Optimal Classical Control

In this part, the following lemmas are important in the proof of the existence of a couple boundary optimal classical control.

Lemma 4.1. *The operator* $\vec{q} \mapsto \vec{y}_{\vec{q}}$ *from* $\overrightarrow{Q_a}$ *to* $(L^2(\Lambda))^2$ *is Lipschitz continuous*, *i.e.*,

$$\| \overrightarrow{\delta y} \|_{(L^2(\Lambda))^2} \leq k \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}, \text{ for } k > 0.$$

Proof. Let $\vec{\bar{q}} = (\bar{q}_1, \bar{q}_2) \in \overrightarrow{Q_a}$ be a given couple boundary control vector of the WF (10) and $\vec{\bar{\bar{y}}} = (\bar{\bar{y}}_1, \bar{\bar{y}}_2)$ be the corresponding vector of state solution. Subtracting the obtained WF from (10), and substituting $\overrightarrow{\delta y} = \vec{y} - \vec{\bar{\bar{y}}}$, $\overrightarrow{\delta q} = \vec{q} - \vec{\bar{q}}$ in the obtained equation, and inserting $v_1 = \delta y_1$ and $v_2 = \delta y_2$, we get

$$\mathcal{B}(\overrightarrow{\delta y}, \overrightarrow{\delta y}) = (\delta q_1, \delta y_1) + (\delta q_2, \delta y_2). \tag{19}$$

Taking the absolute value of (19) with using assumption (A(1-a)) and the Cauchy-Schwarz inequality (C-SI), we deduce that

$$c_0 \| \overrightarrow{\delta y} \|^2_{(H^2(\Lambda))^2} \leq \| \delta q_1 \|_{L^2(\Gamma)} \| \delta y_1 \| + \| \delta q_2 \|_{L^2(\Gamma)} \| \delta y_2 \|. \tag{20}$$

Since

$$\| \delta y_i \| \leq \| \overrightarrow{\delta y} \|_{(L^2(\Lambda))^2} \leq \| \overrightarrow{\delta y} \|_{(H^2(\Lambda))^2}$$

and

$$\| \delta q_i \|_{L^2(\Gamma)} \leq \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}, \quad \forall i = 1, 2,$$

(20) becomes

$$\| \overrightarrow{\delta y} \|_{(L^2(\Lambda))^2} \leq k \| \overrightarrow{\delta q} \|_{L^2(\Gamma)}, \text{ with } k = \frac{2}{c_0}. \tag{21}$$

Lemma 4.2 [15]**.** *The CF* (7) *is weakly lower semicontinuous* (*WLSC*).

Lemma 4.3 [15]**.** *The norm* $\| \cdot \|^2$ *and* $(\| \cdot \|^2_{L^2(\Gamma)})$ *is strictly convex.*

Theorem 4.1. *If* $J_0(\vec{q})$ *is coercive, then there exists a couple classical optimal boundary control for the problem.*

Proof. Since Q_i, for each $i = 1, 2$, is convex, hence $\overrightarrow{Q_a}$ is convex.

Since $J_0(\vec{q}) \geq 0$, and $J_0(\vec{q})$ is coercive, there exists a minimizing sequence $\{\vec{q}_n\} = \{(q_{1n}, q_{2n})\} \in \overrightarrow{Q_a}$, $\forall n$ such that

$$\lim_{n\to\infty} J_0(\vec{q}_n) = \inf_{\vec{w}\in\overrightarrow{Q_a}} J_0(\vec{w}).$$

Therefore, there exists a constant $C > 0$ such that $\| \vec{q}_n \|_{(L^2(\Gamma))^2} \leq C$, $\forall n$, i.e.,

$$\| q_{1n} \|_{L^2(\Gamma)} \leq C_1 \quad \text{and} \quad \| q_{2n} \|_{L^2(\Gamma)} \leq C_2, \ (C_1, C_2 > 0), \forall n. \qquad (22)$$

From Alaoglu theorem, there exists a subsequence of $\{\vec{q}_n\}$ (say again $\{\vec{q}_n\}$) such that $\vec{q}_n \rightharpoonup \vec{\bar{q}}$ in $(L^2(\Gamma))^2$.

Since from (22), the state equation has a unique $\vec{y}_n = \vec{y}_{\vec{q}_n}$ $((\forall n)$ by Theorem 3.1).

By utilizing assumptions (A(1-a)) and (A(2)), the C-SI, the trace theorem and using (22), we have

$$c_0 \| \vec{y}_n \|^2_{(H_0^2(\Lambda))^2}$$

$$\leq \mathcal{B}(\vec{y}_n, \vec{y}_n) = l(\vec{y}_n)$$

$$\leq \| G_1 \| \| y_{1n} \| + \| q_{1n} \|_{L^2(\Gamma)} \| y_{1n} \| + \| G_2 \| \| y_{2n} \| + \| q_{2n} \|_{L^2(\Gamma)} \| y_{2n} \|$$

$$\leq \ell_1 \| y_{1n} \| + C_1 \| y_{1n} \| + \ell_2 \| y_{2n} \| + C_2 \| y_{2n} \|$$

$$\leq (r_1 + r_2) \| \vec{y}_n \|_{(H^2(\Lambda))^2} = \mathcal{C} \| \vec{y}_n \|_{(H^2(\Lambda))^2},$$

where $\mathcal{C} = r_1 + r_2$, then $\| \vec{y}_n \|_{(H_0^2(\Lambda))^2} \leq K$, $\forall n$, where $K = \dfrac{\mathcal{C}}{c_0}$, $K \geq 0$.

Then there exists a subsequence of $\{\vec{y}_n\}$ (say again $\{\vec{y}_n\}$) such that $\vec{y}_n \rightharpoonup \vec{\bar{y}}$ in $\vec{S}$ (by Alaoglu theorem).

Since for each n, $\vec{y}_n = (y_{1n}, y_{2n})$ satisfies the WF (13), we have

$$\mathcal{B}(\vec{y}_n, \vec{v}) = (G_1, v_1) + (q_{1n}, v_1) + (G_2, v_2) + (q_{2n}, v_2), \forall (v_1, v_2) \in \vec{S}, \forall n. \tag{23}$$

To show that (23) converges to

$$\mathcal{B}(\vec{\bar{y}}, \vec{v}) = (G_1, v_1) + (\bar{q}_1, v_1) + (G_2, v_2) + (\bar{q}_2, v_2), \quad \forall (v_1, v_2) \in \vec{S}. \tag{24}$$

Note that

$$y_{in} \rightharpoonup \bar{y}_i \text{ in } H^2(\Lambda) \overset{\forall i=1,2}{\Rightarrow} \begin{cases} y_{in} \rightharpoonup \bar{y}_i \\ \Delta y_{in} \rightharpoonup \Delta \bar{y}_i \end{cases} \text{ in } L^2(\Lambda).$$

By using the C-SI,

$$\begin{aligned}
&| A_1(y_{1n}, v_1) + (a_1 y_{1n}, v_1) - (a_2 y_{2n}, v_1) + A_2(y_{2n}, v_2) \\
&\quad + (b_1 y_{2n}, v_2) + (a_2 y_{1n}, v_2) - A_1(\bar{y}_1, v_1) - (a_1 \bar{y}_1, v_1) \\
&\quad + (a_2 \bar{y}_2, v_1) - A_2(\bar{y}_2, v_2) - (b_1 \bar{y}_2, v_2) - (a_2 \bar{y}_1, v_2) | \\
&\le k_1 \| (y_{1n} - \bar{y}_1) \| \| v_1 \| + k_2 \| y_{1n} - \bar{y}_1 \| \| v_1 \| + k_3 \| y_{2n} - \bar{y}_2 \| \| v_1 \| \\
&\quad + k_4 \| (y_{2n} - \bar{y}_2) \| \| v_2 \| + k_5 \| y_{2n} - \bar{y}_2 \| \| v_2 \| + k_6 \| y_{1n} - \bar{y}_1 \| \| v_2 \| \to 0.
\end{aligned}$$

Since $q_{1n} \rightharpoonup \bar{q}_1$ in $L^2(\Gamma)$ and $q_{2n} \rightharpoonup \bar{q}_2$ in $L^2(\Gamma)$, the R.H.S. of (23) converges to the R.H.S. of (24).

Since $J_0(\vec{q})$ is WLSC from Lemma 4.2, and $\vec{q}_n \rightharpoonup \vec{\bar{q}}$ in $(L^2(\Gamma))^2$, we observe that

$$J_0(\vec{q}) \le \lim_{n\to\infty} \inf_{\vec{q}_n \in \vec{Q}_a} J_0(\vec{q}_n) = \lim_{n\to\infty} J_0(\vec{q}_n) = \inf_{\vec{w} \in \vec{Q}_a} J_0(\vec{w}),$$

$$J_0(\vec{q}) = \inf_{\vec{w} \in \vec{Q}_a} J_0(\vec{w}).$$

Therefore, $\vec{q}$ is a couple classical boundary optimal control.

To prove $\vec{q}$ is unique, from strict convexity of $J_0(\vec{q})$, we conclude the uniqueness of $\vec{q}$.

5. The NECs for Optimality

In order to formulate the NECs for a couple classical optimal boundary control, we drive the FrD of the Hamiltonian to establish the theorem of the NECs for optimality.

Theorem 5.1. *Consider the CF which is defined by* (7), *and the couple adjoint* $(z_1, z_2) = (z_{1q_1}, z_{2q_2})$ *equations of the couple state equations* (1-6) *are obtained by*

$$\sum_{i,j,k,l=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}\frac{\partial^2 z_1}{\partial x_k \partial x_l}\right) + a_1 z_1 + a_2 z_2 = (y_1 - y_{1d}),\ on\ \Lambda, \tag{25}$$

$$\sum_{i,j,k,l=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}\frac{\partial^2 z_2}{\partial x_k \partial x_l}\right) + b_1 z_2 - a_2 z_1 = (y_2 - y_{2d}),\ on\ \Lambda, \tag{26}$$

$$\frac{\partial z_1}{\partial n} = 0,\ on\ \Gamma, \tag{27}$$

$$\frac{\partial \Delta z_1}{\partial n} = 0,\ on\ \Gamma, \tag{28}$$

$$\frac{\partial z_2}{\partial n} = 0,\ on\ \Gamma, \tag{29}$$

$$\frac{\partial \Delta z_2}{\partial n} = 0,\ on\ \Gamma. \tag{30}$$

Then the FrD of J_0 *is given by*

$$(J_0'(\vec{q}), \overrightarrow{\delta q}) = (z_1 + \lambda_1 q_1, \delta q_1)_\Gamma + (z_2 + \lambda_2 q_2, \delta q_2)_\Gamma.$$

Proof. Rewriting the CPAEs (25-30) by their WFs, we get

$$A_1(z_1, v_1) + (a_1 z_1, v_1) + (a_2 z_2, v_1) = (y_1 - y_{1d}, v_1),\ \forall v_1 \in H^2(\Lambda) \tag{31}$$

and

$$A_2(z_2, v_2) + (b_1 z_2, v_2) - (a_2 z_1, v_2) = (y_2 - y_{2d}, v_2), \forall v_2 \in H^2(\Lambda). \quad (32)$$

By adding (31) and (32), for fixed couple classical control vector $\vec{q} = (q_1, q_2) \in (L^2(\Gamma))^2$, we obtain that the WF of the CPAEs has a unique solution $(z_1, z_2) = (z_{1q_1}, z_{2q_2}) \in \vec{S}$ (this can be proved in the same way as the proof of Theorem 3.1), and

$$(\Delta z_1, \Delta v_1) + (z_1, v_1) + (z_2, v_1) + (\Delta z_2, \Delta v_2) + (z_2, v_2) - (z_1, v_2)$$

$$= (y_1 - y_{1d}, v_1) + (y_2 - y_{2d}, v_2), \forall (v_1, v_2) \in \vec{S}. \quad (33)$$

By substituting y_1 once and $y_1 + \delta y_1$ once again in (8), subtracting the two resulted equations one from the other, and inserting $v_1 = z_1$, we obtain

$$A_1(\delta y_1, z_1) + (a_1 \delta y_1, z_1) - (a_2 \delta y_2, z_1) = (\delta q_1, z_1)_\Gamma, \forall z_1 \in H^2(\Lambda). \quad (34)$$

Also, substituting $v_1 = \delta y_1$ in (31), then subtracting the resulted equation with (34), we get

$$(a_2 \delta y_2, z_1) + (a_2 z_2, \delta y_1) = -(\delta q_1, z_1)_\Gamma + (y_1 - y_{1d}, \delta y_1). \quad (35)$$

By substituting y_2 once and $y_2 + \delta y_2$ once again in (9), subtracting the resulted equations one from the other, with substituting $v_2 = z_2$, we obtain

$$A_2(\delta y_2, z_2) + (b_1 \delta y_2, z_2) + (a_2 \delta y_1, z_2) = (\delta q_1, z_2)_\Gamma, \forall z_2 \in H^2(\Lambda). \quad (36)$$

Also, $v_2 = \delta y_2$ in (32), then subtracting the obtained equation with (36), we get

$$-(a_2 \delta y_1, z_2) - (a_2 z_1, \delta y_2) = -(\delta q_2, z_2)_\Gamma + (y_2 - y_{2d}, \delta y_2). \quad (37)$$

Adding (35) and (37), we get

$$(\delta q_1, z_1)_\Gamma + (\delta q_2, z_2)_\Gamma = (y_1 - y_{1d}, \delta y_1) + (y_2 - y_{2d}, \delta y_2). \quad (38)$$

Then from the CF, we observe that

$$J_0(\vec{q} + \overrightarrow{\delta q}) - J_0(\vec{q})$$

$$= (y_1 - y_{1d}, \delta y_1) + \lambda_1 (q_1, \delta q_1)_\Gamma + (y_2 - y_{2d}, \delta y_2) + \lambda_2 (q_2, \delta q_2)_\Gamma$$

$$+ \frac{1}{2} \| \overrightarrow{\delta y} \|^2_{(L^2(\Lambda))^2} + \frac{\lambda_1}{2} \| \delta q_1 \|^2_{L^2(\Gamma)} + \frac{\lambda_2}{2} \| \delta q_2 \|^2_{L^2(\Gamma)}.$$

From the CF and using (38), we have

$$J_0(\vec{q} + \overrightarrow{\delta q}) - J_0(\vec{q})$$

$$= (\delta q_1, z_1)_\Gamma + \lambda_1 (q_1, \delta q_1)_\Gamma + (\delta q_2, z_2)_\Gamma + \lambda_2 (q_2, \delta q_2)_\Gamma$$

$$+ \frac{1}{2} \| \overrightarrow{\delta y} \|^2_{(L^2(\Lambda))^2} + \frac{\lambda_1}{2} \| \delta q_1 \|^2_{L^2(\Gamma)} + \frac{\lambda_2}{2} \| \delta q_2 \|^2_{L^2(\Gamma)}$$

$$= (z_1 + \lambda_1 q_1, \delta q_1)_\Gamma + (z_2 + \lambda_2 q_2, \delta q_2)_\Gamma$$

$$+ \frac{1}{2} \| \overrightarrow{\delta y} \|^2_{(L^2(\Lambda))^2} + \frac{\lambda_1}{2} \| \delta q_1 \|^2_{L^2(\Gamma)} + \frac{\lambda_2}{2} \| \delta q_2 \|^2_{L^2(\Gamma)}. \tag{39}$$

From Lemma 4.1,

$$\frac{1}{2} \| \overrightarrow{\delta y} \|^2_{(L^2(\Lambda))^2} \leq 2 \| \overrightarrow{\delta q} \|^2_{(L^2(\Gamma))^2} = \varepsilon_1(\overrightarrow{\delta q}) \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}, \tag{39a}$$

where $\varepsilon_1(\overrightarrow{\delta q}) \to 0$, as $\| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2} \to 0$, where $\varepsilon_1(\overrightarrow{\delta q}) = 2 \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}$.

Since $\| \delta q_i \|^2_{L^2(\Gamma)} \leq \| \overrightarrow{\delta q} \|^2_{(L^2(\Gamma))^2}$, we have

$$\frac{\lambda_1}{2} \| \delta q_1 \|^2_{L^2(\Gamma)} + \frac{\lambda_2}{2} \| \delta q_2 \|^2_{L^2(\Gamma)}$$

$$\leq C \| \overrightarrow{\delta q} \|^2_{(L^2(\Gamma))^2} = C \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2} \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}$$

$$= \varepsilon_2(\overrightarrow{\delta q}) \| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}, \tag{39b}$$

where $C = \max\left\{\frac{\lambda_1}{2}, \frac{\lambda_2}{2}\right\}$ with $\varepsilon_2(\overrightarrow{\delta q}) \to 0,$ as $\| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2} \to 0,$ where

$$\varepsilon_2(\overrightarrow{\delta q}) = C\| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}.$$

Hence the FrD of J_0 is

$$J_0(\vec{q} + \overrightarrow{\delta q}) - J_0(\vec{q})$$

$$= (z_1 + \lambda_1 q_1, \delta q_1)_\Gamma + (z_2 + \lambda_2 q_2, \delta q_2) + \varepsilon(\overrightarrow{\delta q})\| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2}, \tag{40}$$

where $\varepsilon(\overrightarrow{\delta q}) = \varepsilon_1(\overrightarrow{\delta q}) + \varepsilon_2(\overrightarrow{\delta q}) \to 0,$ as $\| \overrightarrow{\delta q} \|_{(L^2(\Gamma))^2} \to 0.$

Finally, from (39) and (40), we get

$$(J_0'(\vec{q}), \overrightarrow{\delta q}) = (z_1 + \lambda_1 q_1, \delta q_1)_\Gamma + (z_2 + \lambda_2 q_2, \delta q_2)_\Gamma.$$

Theorem 5.2. *The continuous classical couple optimal control of the considered problem is* $J_0'(\vec{q}) = (z_1 + \lambda_1 q_1, \delta q_1)_\Gamma + (z_2 + \lambda_2 q_2, \delta q_2)_\Gamma = 0$ *with* $\vec{y} = \vec{y}_{\vec{q}}$ *and* $\vec{z} = \vec{z}_{\vec{q}}$.

Proof. If $\vec{q}$ is an optimal control vector of the considered problem, then

$$J_0(\vec{q}) = \min_{\vec{\bar{q}} \in \vec{Q}} J_0(\vec{\bar{q}}),$$

$\forall \vec{\bar{q}} \in (L^2(\Gamma))^2,$ i.e.,

$$J_0'(\vec{q}) = 0 \Rightarrow (z_1 + \lambda_1 q_1, \delta q_1)_\Gamma + (z_2 + \lambda_2 q_2, \delta q_2)_\Gamma = 0$$

$$\Rightarrow \lambda_1 q_1 = -z_1 \text{ and } \lambda_2 q_2 = -z_2.$$

$\overrightarrow{\delta q} = \vec{\bar{q}} - \vec{q} \Rightarrow$ the NEC of the optimality is

$$(J_0'(\vec{q}), \overrightarrow{\delta q}) \geq 0, \quad \forall \vec{\bar{q}} \in L^2(\Gamma) \times L^2(\Gamma).$$

6. Conclusion

In this work, the FEM with a PCH basis function is suitable to study the existence of a unique couple state vector solution for a couple fourth NBVP of LEPDEs with VCs, when the CCCPBCTV is considered. Under essential conditions, the existence theorem of a couple continuous classical optimal boundary control vector associated with a couple fourth order linear elliptic boundary value problem is introduced. In the end, the FrD of the Hamiltonian is obtained to establish the theorem of the NEC for optimality of the above considered problem.

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