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THE LIMIT OF BLOW-UP DYNAMICS SOLUTIONS FOR A CLASS OF NONLINEAR CRITICAL SCHRÖDINGER EQUATIONS

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Abstract

This paper considers the asymptotic behavior of solutions of equations of evolutions, and concentrates on the analysis of the critical blow-up solutions for a class of evolutions for nonlinear Schrödinger equations in a bounded domain. More precisely, the numerical approximation of the blow-up rate below the one of the known explicit explosive solutions is studied, which has strictly positive energy for the following initial-boundary value problem:

$$(P)\begin{cases} u_t(t, x) - i\alpha\Delta u(t, x) - i\beta f(t, x) = 0, & x \in \mathbb{R}^d,\ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $i = \sqrt{-1}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $d \geq 1$, u is a complex-valued function of the variable $x \in \mathbb{R}^d$, Δ is the Laplace operator in $\mathbb{R}^d$ and the time $t \geq 0$.

The paper proposes a general setting to study and understand the behavior of the blow-up solutions in a finite time as a function of the parameters α, β, with initial condition $u(0, x) = u_0$, in the energy space $H^1 \in \mathbb{R}^d$, also in the case where $\mathbb{R}^d$ is large enough and its size d is taken as parameter. Some assumptions are found under which the solution of the above problem blows-up in a finite time, study the dynamics of blow-up solutions and estimate its blow-up time. Finally, some numerical experiments to illustrate the analysis have been provided.

1. Introduction

In this paper, we are interested in the numerical approximation for the following initial-boundary value problem for the critical nonlinear Schrödinger equation of the form:

$$u_t(t, x) - i\alpha\Delta u(t, x) - i\beta f(t, x) = 0, \quad x \in \mathbb{R}^d; \ \ t \in (0, T), \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{1.2}$$

which appears in a lot of models of nonlinear optics, energy transfer in molecular systems, quantum mechanics, seismology and plasma physics. They are widely used in several domains of applied physics, see [1, 29] to cite only a few cases. This equation is locally well-posed in H^1 the usual Sobolev space from [7]. The main objective of this paper is to present a relationship between the constants α, β and a sharp criterion for the existence of global solutions to the nonlinear Schrödinger equation. We will begin our mathematical discussion of (1.1) with a comparison to the *dD*.

The case, where $d = 1$, describes in certain regimes of the propagation of electromagnetic, it arises as an equation in water wave theory, see [25, 26], and the initial-value problem (1.1)-(1.2) could model an electron propagating in a 1*D*, where $q = 2$. The problem is globally well-posed for smooth enough initial data that decay sufficiently fast at infinity.

The case $d = 2$ arises in nonlinear optics: Blow-up of the solutions corresponds to a physical phenomenon and the solution u is then the envelope of an electromagnetic wave propagating. In the past, certain authors have used numerical methods to study the phenomenon of blow-up for nonlinear Schrödinger equations but they have considered the problem (1.1)-(1.2) in the case where the term $f(t, x) = |u(t, x)|^{q-1}u(t, x)$, see [2, 26]. In this case, one proves that the energy of the system is conserved and the method used to show blow-up solutions is based on the energy's method. It is classical from the conservation of the energy.

For $d = 2$ or $d = 3$, it is also well known that there exists a singular solution which blows-up in L^{∞} in finite time, see [21, 22, 30-32].

In this paper, we propose a method based on a modification of the method of Kaplan (see [9]) using eigenvalues and eigenfunctions to show that numerical solution blows-up in a finite time. We integrate the semi-discrete scheme and obtain some discrete schemes where the convergence and stability of the explicit schemes have been proved, see Dai [3].

This paper is organized as follows. In the next section, we give some results about the local solution. In the third section, we give some conditions under which the solution of (1.1)-(1.2) blows-up in a finite time and estimate its blow-up time. In the last section, we propose some schemes and algorithms to compute the numerical blow-up time. Some numerical values are given.

2. Local Solutions

In this section, we consider the following nonlinear Schrödinger equation:

$$u_t(t, x) - i\alpha\Delta u(t, x) = -\beta f(t, x), \quad x \in \mathbb{R}^d, \quad t \in (0, T), \tag{2.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{2.2}$$

where $u(t, x)$ is a complex-valued function in space time $I \times \mathbb{R}^d$. For the energy-critical local theory, it is convenient to introduce a number of scale invariant function spaces. We use $L_r^x(\mathbb{R})$ to denote the Banach space of functions $f : \mathbb{R} \to C$ whose norm

$$\| f \|_r = \left(\int_{R^d} | f(r) |^r dr \right)^{1/r}, \tag{2.3}$$

is finite, in the study of evolution equations, the terms "global" and "local" refer to the existence of the solution on some finite interval.

In this work, global solutions were constructed for small energy data and local solutions were constructed for large energy data, though, as is to be expected for a critical equation, the time of existence depends on the profile of the initial data and not simply on the energy. Furthermore, these solutions were unique in a certain space where they depended continuously on the initial data in the energy space $H^1(\mathbb{R}^d)$.

If $\alpha = 1,$ $\beta = 0$ or $f(u(x, t)) = 0,$ then the critical nonlinear Schrödinger equation (2.1) becomes:

$$iu_t(t, x) + \Delta u(t, x) = 0, \quad x \in \mathbb{R}^d, \quad t \in (0, T), \tag{2.4}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{2.5}$$

for $u_0 \in L^2(\Omega)$, the solution of (2.4) is

$$u(t, x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}} e^{-i\frac{|x-y|^2}{4t}} u_0(y)dy. \tag{2.6}$$

Equation (2.4) induces a spatial regularity gain for almost all t compared to the initial data that can be written as

$$\| \xi u(t) \|_{L^2([0, T]; G)} \le C \| u_0 \|_{L^2}. \tag{2.7}$$

Here G is a strict subspace of L^2 as a space of functions and ξ is a function C_c^∞. This property is called *regularizing effect*. For $\mathbb{R}^d$, the regularizing effect was shown with $G = H^{\frac{1}{2}}(\mathbb{R}^d)$.

If $\alpha = 1$, $\beta \neq 0$ and for $f(u(t, x)) = |u(t, x)|^q u_x(t, x)$, equation (2.1) becomes a fractional modified nonlinear Schrödinger equation, namely,

$$iu_t(x, t) + u_{xx}(x, t) - \beta |u(x, t)|^q u_x(x, t) = 0, \quad x \in \mathbb{R}, \quad t \in (0, T) \tag{2.8}$$

which describes the propagation of rogue waves in deep water with a general nonlinearity by Mio et al. [17]. The study of DNLS (the derivative nonlinear Schrödinger equation) is more difficult than the corresponding cubic nonlinear Schrödinger equation (NLS), namely,

$$iu_t(x, t) + u_{xx}(x, t) - \beta |u(x, t)|^2 u(x, t) = 0, \quad x \in \mathbb{R}^2, \quad t \in (0, T). \tag{2.9}$$

The DNLS equation in (2.8) with general power nonlinearity $(q > 0)$ and $\beta = 1$ admits a family of solitary waves solutions [8] given explicitly by

$$u(x, t) = \vartheta_{\lambda, a}(x - at) e^{i\left\{\lambda t + \frac{a}{2}(x - at) - \frac{1}{q}\int_{-\infty}^{x-at} \vartheta_{\lambda, a}^q(y)dy\right\}}, \tag{2.10}$$

where admissible values of (λ, a) satisfy the conditions $\lambda > \frac{a^2}{4}$, $a \in R$, and

$$\vartheta^q_{\lambda, a}(y) = \frac{(2+q)(4\lambda - a^2)}{4\sqrt{\lambda}\left(\cosh\left(\frac{q}{2}\sqrt{4\lambda - a^2 y}\right) - \frac{a}{2\sqrt{\lambda}}\right)}.$$

For the case, where $q = 2\rho$, equation (2.8) is in the form

$$iu_t(x, t) + u_{xx}(x, t) - |u(x, t)|^{2\rho} u_x = 0, \quad x \in \mathbb{R}^d, \quad t \in (0, T) \quad (2.11)$$

and the solitary waves solution is

$$u(x, t) = \vartheta_{\lambda, a}(x - at) e^{i\left\{\lambda t + \frac{a}{2}(x - at) - \frac{1}{q}\int_{-\infty}^{x-at} \vartheta^{2\rho}_{\lambda, a}(y)dy\right\}}, \quad (2.12)$$

where

$$\vartheta^{2\rho}_{\lambda, a}(y) = \frac{(\rho+1)(4\lambda - a^2)}{2\sqrt{\lambda}\left(\cosh\left(\rho\sqrt{4\lambda - a^2 y}\right) - \frac{a}{2\sqrt{\lambda}}\right)}$$

is the positive solution to this hyperbolic partial differential equation:

$$-\partial^2_y \vartheta_{\lambda, a} + \left(\lambda - \frac{a^2}{4}\right)\vartheta_{\lambda, a} + \frac{\lambda}{2}|\vartheta_{\lambda, a}|^{2\rho}\vartheta_{\lambda, a} - \frac{2\rho+1}{(2\rho+2)^2}|\vartheta_{\lambda, a}|^{4\rho}\vartheta_{\lambda, a} = 0$$

(2.13)

with the solution satisfying

$$u(x, t) = e^{i\lambda t}\phi_{\lambda, a}(x - at), \quad (2.14)$$

where $\phi_{\lambda, a}(y) = \vartheta_{\lambda, a}(y)e^{i\theta_{\lambda, a}(y)}$ for

$$\theta_{\lambda, a}(y) \equiv \frac{a}{2}y - \frac{1}{2\rho+2}\int_{-\infty}^{y} \vartheta^{2\rho}_{\lambda, a}(\nu)d\nu.$$

The complex function $\phi_{\lambda, a}(y)$ satisfies

$$\partial_y^2 \phi_{\lambda, a} + \lambda \phi_{\lambda, a} + ia\phi_{\lambda, a} - i|\phi_{\lambda, a}|^{2\rho} \partial_y \phi_{\lambda, a} = 0, \quad y \in \mathbb{R}. \tag{2.15}$$

For $\alpha = 0$, $\beta = -1$ and $f(u(t, x)) = u(t, x)u_x(t, x)$, equation (2.1) becomes a fractional modified nonlinear Schrödinger equation:

$$u_t(x, t) + u(x, t)u_x(x, t) = 0, \quad x \in \mathbb{R}, \; t \in (0, T), \tag{2.16}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \tag{2.17}$$

If $u_0(x) = g(x)$, then we obtain Burgers' equation whose solution is given implicitly for $t > 0$ by

$$u(x, t) = g(x - t(u(x, t))), \quad x \in \mathbb{R}. \tag{2.18}$$

If g is a bounded, continuously differentiable function, the (continuously differentiable) solution remains bounded as long as it exists. However, if $g'(x)$ is somewhere negative, then from an easy calculation using implicit differentiation, we see that u_t and u both become unbounded in finite time. Therefore, we shall take the phrase "finite time blow-up" to mean that either the solution or some derivative of the solution becomes unbounded in some norm in finite time.

For $\alpha = 1$, $\beta \neq 0$ and $f(u(x, t)) = -\beta|u(x, t)|^q u(x, t)$, the critical nonlinear Schrödinger equation (2.1) takes the form:

$$iu_t(x, t) + \Delta u(x, t) - \beta|u(x, t)|^q u(x, t) = 0, \quad x \in \mathbb{R}^d, \; t \in (0, T), \tag{2.19}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d. \tag{2.20}$$

In the case $d = 1$, the problem (2.19) is globally well-posed for smooth enough initial data that decay sufficiently fast at infinity. It is straightforward for $q = 2$, to check it in the L^2 norm. In one-dimensional space, equation (2.19) exhibits solitary wave solutions, that can furthermore be solved exactly by the inverse scattering method, see [28]. Some results in this direction have already been obtained by Strauss [23], in particular on low energy scattering.

In the cases $d \geq 2,$ the main result is the existence and uniqueness of global solutions of the Cauchy problem for equation (2.19) under suitable assumptions on f and for initial data in the Sobolev space $H^1 \equiv H^1(\mathbb{R}^d).$ From this, it follows that the time interval of existence of a local solution with initial data in H^1 depends only on the H^1-norm of the initial data. Therefore, such a solution can be continued into a global one if one succeeds in finding an a priori estimate for the H^1-norm. The assumptions on f include continuous differentiability, the condition $f(0) = 0$ and suitable power bounds both at zero and at infinity. They cover the case of a single power $f(u(x, t)) = |u(x, t)|^{q-1}u(x, t),$ where $1 < q < (d+2)/(d-2)$ if $\beta > 0$ and $1 < q < (d+4)/d$ if $\beta < 0.$ For more refined estimates of the Strichartz inequalities where we no longer have punctual information, but a time average of the norm $L^m(\mathbb{R}^d),$ if we denote by $e^{it\Delta}$ the linear semigroup associated with the Schrödinger operator, we have

$$\| e^{it\Delta}u_0 \|_{L^p(\mathbb{R}, L^m(\mathbb{R}^d))} \leq C\| u_0 \|_{L^2(\mathbb{R}^d)}, \tag{2.21}$$

where (p, m) is a d-couple admissible, i.e., $p,$ m satisfy $\frac{2}{p} + \frac{d}{m} = \frac{d}{2}.$ Furthermore, $2 \leq p \leq \infty$ and $(p, m, d) \neq (2, \infty, 2).$ We can find the scaling d-admissibility condition with the invariance of the linear Schrödinger equation under the change of $t \to \lambda^2 t,$ $x \to \lambda x.$

Definition 2.1. We say a *critical exponent* q for $u_0 \in H^m(\mathbb{R}^d)$ exponent such as this equation

$$iu_t(x, t) + \Delta u(x, t) - \beta|u(x, t)|^q u(x, t) = 0, \quad x \in \mathbb{R}^d, \ \ t \in (0, T)$$

remains invariant by $u(t, x) \to \lambda^\theta u(\lambda^2 t, \lambda x),$ also its norm in $H^m(\mathbb{R}^d),$

$\| (-\Delta)^{\frac{m}{2}} \|_{L^2}$. This involves $m < \frac{d}{2}$ and $\theta = \frac{4}{d-2m}$. The nonlinearity $\beta |u|^q u$ is invariant by the change mentioned, but to have a regularity in H^1 with $q \geq 1$.

Theorem 2.1. *Assume that* $d \geq 2, 1 \leq q < \frac{4}{d-2}$ *and* $u_0 \in H^1(\mathbb{R}^d)$. *Then there exists a unique solution* $u \in C(R, H^1(\mathbb{R}^d)) \cap L^p_{loc}(R, W^{1,m}(\mathbb{R}^d))$, *where* (p, m) *are d-admissible solutions of equation*

$$iu_t(x, t) + \Delta u(x, t) - |u(x, t)|^q u(x, t) = 0, \quad x \in \mathbb{R}^d, \quad t \in (0, T).$$

Proof. We prove a local existence and uniqueness theorem by a fixed point technique of this equation

$$\phi(u)(t) := e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} (|u(\tau)|^q u(\tau)) d\tau.$$

The existence problem for $d = 2$ and 3 has also been studied in [8]. There is a large amount of flexibility in the choice of the spaces. The mathematical theory of the initial-value problem for (2.8) relies to a considerable extent on two invariance properties satisfied by the solution, see [5-33].

The cases $d = 3$, $d = 4$ were treated in [5] and [15], respectively (see also [1, 3, 9, 21, 33] treating the radial case). The behavior of the solution, say u, or its derivatives near the blow-up time T has been observed. It is often conjectured that the growth of u near the singularity can be described by

$$\max_x |u(x, t)| \propto (T - t)^{-\sigma}.$$

Since the convergence properties of any numerical scheme depend on the good behavior of u and its derivatives, it is clear that the calculation of T and σ poses some difficulty.

3. Time of Schrödinger Equation with Blow-up Rate

In this section, we compute the blow-up rates in the case where $f(t, x) = |u(t, x)|^q u(t, x)$ for an initial-value $u_0(r)$ and for several norms of the solution of the radial problem in a finite interval $0 \leq r \leq R$, with Dirichlet boundary condition, then the problem (1.1) becomes in the form:

$$u_t(x, t) - i\alpha\Delta u(x, t) - i\beta |u(x, t)|^q u(x, t) = 0, \quad x \in \mathbb{R}^d; \; t \in (0, T),$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d.$$

The numerical approximation of radially symmetric solution $u(r, t)$ for $t > 0$ on finite interval in $(d+1)$ dimensional space-time and equation (1.1) becomes

$$u_t = i\alpha\left(u_{rr} + \frac{d-1}{r} u_r\right) + i\beta |u(x, t)|^q u, \quad 0 < r \leq 1, \quad t \in (0, T), \tag{3.1}$$

$$u_r(0, t) = 0, \quad t \in (0, T), \tag{3.2}$$

$$u(r, 0) = u_0(r), \quad r > 0. \tag{3.3}$$

For $q = 2$, with $r = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$ of the initial valued problem (1.1)-(1.2), the mathematical theory of the initial-value problem for (3.1) relies to a considerable extent on two invariance properties satisfied by the solution, see [5-33]. In the presence of radial symmetry solution of the initial-valued problem, these invariants are

$$\int_0^{\infty} |u(r, t)|^2 r^{d-1} dr = C, \text{ for } t \geq 0, \tag{3.4}$$

$$\int_0^{\infty} \left(|u_r(r, t)|^2 - \frac{1}{2}|u(r, t)|^4\right) r^{d-1} dr = S, \text{ for } t \geq 0, \tag{3.5}$$

for $0 < t < T$. In what follows, we shall consider the L^p norms, $1 \leq p < \infty$, of the radial function defined on $[0, 1]$ by

$$\| f \|_{L^p} = \left(\int_0^1 | f(r) |^p r^{d-1} dr \right)^{1/p}. \tag{3.6}$$

If $q \geq 2,$ we give a regular initial condition, the solution of equation (1.1) exists up to some time $T > 0,$ see [24] and satisfies the invariance properties

$$\int_{\mathbb{R}} | u(r, t) |^2 dr = C, \text{ for } t \geq 0, \tag{3.7}$$

$$\int_{\mathbb{R}} \left(| u_r(r, t) |^2 - \frac{2}{2+q} | u(r, t) |^{q+2} \right) dr = S, \text{ for } t \geq 0. \tag{3.8}$$

If $d = 1,$ then the constancy of C and S ensures the boundedness of the solution. The solution u must cease to exist at some finite value of $t = T.$ More precisely,

$$\lim_{t \to T} \| u(t) \|_{L^{+\infty}} = +\infty.$$

The case $d = 2$ is the critical dimension case for the cubic nonlinearity, and the blow-up slows down somewhat, making the numerical integration of the equation harder. We give a regular initial condition such that the solution of equation (3.1) exists up to some $T > 0.$ Many authors have conjectured that the singular solutions of equation (3.1) exhibit a self-similar structure as t approaches T, see Rypdal and Rasmussen [21], for the behavior of the solution u near the singularity. We have

$$\lim_{t \to T} \| u(r, t) \| = \lim_{t \to T} B(t) \Psi(B(t) r), \tag{3.9}$$

where $B(t)$ is singular at $t = T.$ We need to determine which particular self similar structure acts as an attractor for a given class of initial conditions. However, disagreement has existed regarding the nature of the singularity $B(t).$ In fact,

$$B(t) \sim (T - t)^{-\rho} \text{ as } t \uparrow T. \tag{3.10}$$

For $d = 2$, we computed the solution of the problem (3.1) at $r = 0$ and the nature of the singularity $B(t)$ is in the form $B(t) = (T - t)^{\frac{1}{2}}$ by Kelley, see [11], and the blow-up slows down somewhat, making the numerical integration of the equation harder. Thus, $B(t) = (T - t)^{\frac{-2}{3}}$ law for the blow-up of the amplitude was conjectured by Zakharov et al., see [29-31]. This behavior was perturbed by a more slowly varying factor, using computational evidence and asymptotic techniques. LeMesurier et al. [15] suggested the form

$$B(t) \sim \left(\frac{\ln\left(\frac{1}{T - t}\right)}{T - t}\right)^{\frac{1}{2}} \text{ as } t \uparrow T. \tag{3.11}$$

Using the behavior technique on the basis of asymptotic estimates, the rate becomes in the form

$$B(t) \sim \left(\left(\frac{\ln\left(\frac{1}{T - t}\right)}{T - t}\right)^{\delta}\right)^{\frac{1}{2}} \text{ as } t \uparrow T, \tag{3.12}$$

where $0 \le \delta \le 1$, or

$$B(t) \sim \left(\frac{\ln \ln\left(\frac{1}{T - t}\right)}{T - t}\right)^{\frac{1}{2}} \text{ as } t \uparrow T. \tag{3.13}$$

It is concluded that solutions emanating from several types of initial data blow-up in finite time as $t \uparrow T$. We illustrate the development of the singularity for an initial condition of the type

$$u(r, 0) = u_0(r) = be^{-\left(\frac{r}{l}\right)^2}. \tag{3.14}$$

For $q = 2$, $l = 1$, $d = 2$ or $d = 3$, we computed the solution of the problem (3.1) at $r = 0$. This equation is integrated numerically and the details of the blow-up are inferred from the long-time asymptotic of the numerical solution and the scale factors such that the scaling factors are chosen so that suitable functionals of solution are preserved. We assumed that the phase of the solution at $r = 0$ is of the form

$$|u(0, t)| \propto (T - t)^{-\rho} \text{ as } t \uparrow T. \tag{3.15}$$

For the case $d = 3$, McLaughlin et al. in [16] have derived a semi-linear elliptic equation for Ψ. They carried out calculations with initial conditions of the type (3.14) which support the conjecture (3.9) with $B(t) = (T - t)^{\frac{-1}{2}}$ as t approaches the blow-up time $T < \infty$, and suggest another self-similar blow-up regime where $B(t) = (T - t)^{1/2}$ weak collapses and $B(t) = (T - t)^{3/5}$ strong collapses, see [18, 30, 31].

4. Numerical Approximation for Nonlinear Schrödinger Equations with Finite Element Methods

In this section, we report the results of numerical experiments performed with our adaptive code in the critical dimensional case. We computed blow-up rates for the amplitude of the solution of (3.1) as well as for several of its norms and norms of its radial derivative.

Let $\Delta_h : 0 = r_1 < r_2 < \cdots < r_{n+1} = R$ be a repartition of the interval $[0, R]$. We introduce the finite-dimensional S_h consisting of all the functions defined on $[0, R]$ which are continuous and piecewise linear with respect to Δ_h. We also introduce the discrete time levels $0 = t_0 < t_1 < \cdots < t_m < \cdots$. To normalize matters, we scale the radial variable so that it takes values in $[0, 1]$, and thus we have the fully discrete Galerkin approximation of the solutions. Therefore, we consider the radial problem in a finite interval with Dirichlet boundary condition. To that effect, after scaling $r \to r/R$, for $T > 0$ at $T \to \infty$, we have

$$u_t = i\alpha\left(u_{rr} + \frac{d-1}{r}u_r\right) + i\beta|u(x, t)|^q u, \quad 0 < r \leq 1, \quad t \in (0, T), \quad (4.1)$$

$$u_r(0, t) = 0, \quad t \in (0, T), \quad (4.2)$$

$$u(1, t) = 0, \quad t \in (0, T), \quad (4.3)$$

$$u(r, 0) = v(r), \quad 0 \leq r \leq 1, \quad (4.4)$$

where $\alpha = \frac{1}{R}$, $v(x) = u_0(xR)$, $0 \leq x \leq 1$ and $d = 2$ or $d = 3$. The solution of (1.1)-(1.2) by fully discrete Galerkin finite element method uses continuous, piecewise linear polynomials in r and the implicit midpoint time-stepping rule in t.

Another reason for introducing finite elements is to get a new result for a new code in two dimension which worked equally well with the resulting finite difference scheme. There is an evidence on the dynamics of blow-up of radially symmetric solution of NLS equations in two and three dimensions, see [19-21]. It has been concluded that solutions emanating from several types of initial data blow-up in finite time as $t \uparrow T$.

Let $h = \max_{1<i\leq n}(r_{i+1} - r_i)$ and use a time step

$$\Delta t = \max_{0\leq m}(t_{m+1} - t_m),$$

which is constant.

Let S_h be the space of complex-valued continuous functions defined on $[0, 1]$ that vanish at $r = 1$ and are the linear polynomials on each interval in (r_{i-1}, r_i), where $U_h^{(m)} \in S_h$ an approximation to $u(t_m)$ is defined as the solution of the problem, see [6]. We shall consider two different ways of constructing an approximation to u at the next time level. In the first method, due to Delfour et al. [4] (DFP for short), $U_h^{m+1} \in S_h$, is defined as the solution of the problem (4.1)-(4.4):

$$i\int_0^R \frac{U_h^{m+1} - U_h^m}{t_{m+1} - t_m}\phi r^{d-1}dr - \int_0^R \frac{d}{dr}U_h^{\left(m+\frac{1}{2}\right)}\frac{d\phi}{dr}r^{d-1}dr + \tag{4.5}$$

$$\int_0^R \frac{|U^{(m+1)}|^2 + |U^{(m)}|^2}{2}U^{m+\frac{1}{2}}\phi r^{d-1}dr = 0, \tag{4.6}$$

for all $\phi \in S_h$, where

$$U^{m+\frac{1}{2}} = \frac{U^{(m+1)} + U^{(m)}}{2},$$

then there exists a unique solution U^n of the fully discrete scheme satisfying

$$\| u(t^n) - U^n \|_{L^2(\Omega)} \leq C(\Delta t^2 + h^2), \tag{4.7}$$

where the constant C depends on the solution u. The stability and convergence of fully discrete finite element methods have been analyzed in detail in [2] and [16] for the NLS equation. To derive an estimate for the error $u(t^n) - U^n$ in L^2, we assume that the solution of (4.1) is sufficiently smooth on $[0, 1] \times [0, T]$. Then, arguing along the lines of [2], we may prove that there exists a unique solution U^n of the fully discrete scheme of (4.1) satisfying

$$\max_{0 \leq n \leq J} \| u(t^n) - U^n \|_{L^2(\Omega)} \leq C(\Delta t^2 + h^2) \tag{4.8}$$

provided Δt is sufficiently small and that $\Delta t = o(h^{d/4})$ as $h \to 0$, where C is a constant independent of the discretization parameters. We may approximate second-order accuracy of the solution of (4.1).

For $d = 3$, and the initial value $u_0(r) = 6\sqrt{2}e^{-r^2}$, take $R = 5$ for the radius of the sphere Ω. Assume that the behavior of u near T takes the form $|u(0, t)| \propto (T - t)^{-\rho}$. The problem (4.8) has been considered by McLaughlin et al. [16] who computed $T = 0.034302$ and $\rho = \frac{1}{2}$. As far as the implementation of the numerical method is concerned, we have used uniform grids in space and time and the fixed-point iteration procedure. For an example, starting with $M = 500$ subintervals and an initial mesh length $h = 1/500 = 0.002$, assume that the finest mesh region has $M = 200$ subintervals with $h = 1/200 = 0.005$.

Numerical experiments of the blow-up rate for $u^0(r) = 6\sqrt{2}e^{-r^2}$, $d = 3$

Δt	h	T_h	Δt	h	T_h
10^{-3}	0.002	0.0343041880	10^{-3}	0.001	0.034304430
10^{-4}	0.002	0.0343062450	10^{-4}	0.001	0.034304450
10^{-5}	0.002	0.0343042998	10^{-3}	0.001	0.0343042998
10^{-10}	0.005	0.0343041880	10^{-10}	0.001	0.0343007519
10^{-10}	0.00125	0.0343005490	10^{-10}	0.005	0.0343007539
10^{-12}	0.005	0.0343007519	10^{-12}	0.001	0.0343041880
10^{-12}	0.00125	0.0343041885	10^{-12}	0.005	0.0343041880

For $d = 2$, we computed approximations as in (3.12) of the previous section,

$$B(t) \sim \left(\left(\frac{\ln\left(\frac{1}{T - t}\right)}{T - t}\right)^{\delta}\right)^{\frac{1}{2}},$$

where $0 \leq \delta \leq 1$.

Numerical experiments of the blow-up rate $B(t)$ **for** $u^0(r) = 6\sqrt{2}e^{-r^2}$, $r = 0$ **and** $d = 2$

i	δ = 0.45	δ = 0.50	δ = 0.6	δ = 1	δ = 0	lnln
20	0.48976	0.49580	0.49737	0.49888	0.50553	0.50060
21	0.49004	0.49583	0.49788	0.49878	0.50514	0.50049
22	0.49030	0.49587	0.49780	0.49871	0.50479	0.50039
23	0.49057	0.49594	0.49729	0.49867	0.50448	0.50033
24	0.49081	0.49603	0.49730	0.49852	0.50422	0.50026
25	0.49103	0.49605	0.49730	0.49856	0.50398	0.50018
26	0.49126	0.49604	0.49731	0.49854	0.50323	0.49999
27	0.49138	0.49598	0.49712	0.48827	0.50292	0.49982

For $d = 1$, we computed approximations with $u_0(r) = 2\sqrt{2}e^{-r^2}$ and $q > 3$. The behavior of the solution u near T at $r = 0$ takes the value of $T \simeq 0.087$ under some condition [15-31].

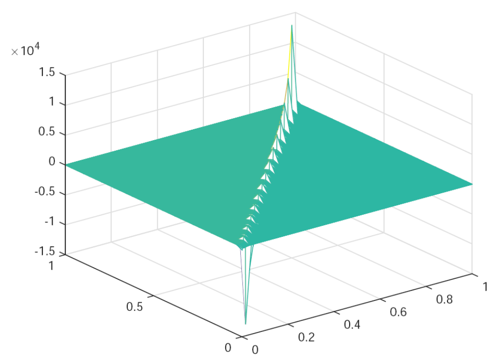

Figure 1. Evolution of the solution $u(0, t) = (T - t)^{-1/2}$, $r = 0$, $\Delta t = 0.001$, $d = 3$.

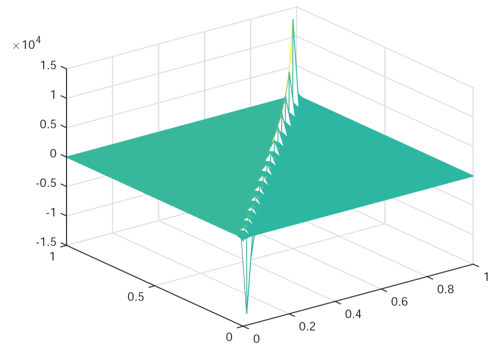

Figure 2. Evolution of the initial solution, $u(0, t) = (T - t)^{-1/2}$, $r = 0$, $\Delta t = 0.0001$, $d = 3$.

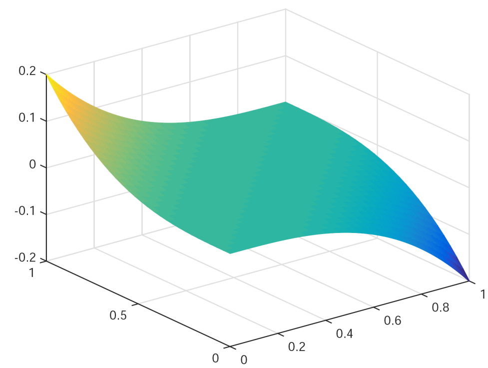

Figure 3. Evolution of the singular solution $u(0, t) = (T - t)^{3/5}$, $r = 0$, $\Delta t = 0.001$, $d = 3$.

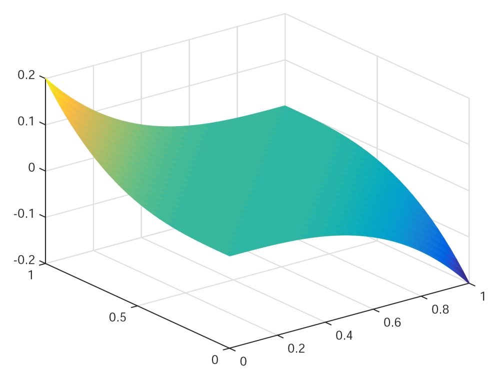

Figure 4. Evolution of the singular solution, $u(0, t) = (T - t)^{3/5}$, $r = 0$, $\Delta t = 0.0001$, $d = 3$.

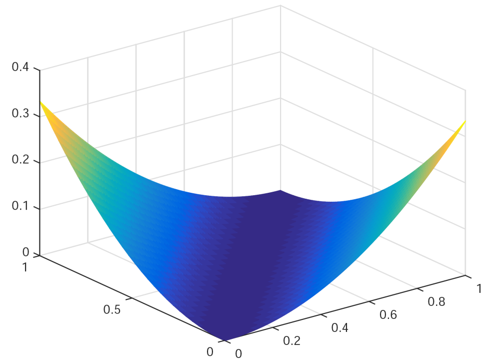

Figure 5. Evolution of the singular solution $u(0, t) = (T - t)^{2/3}$, $r = 0$, $\Delta t = 0.001$, $d = 2$.

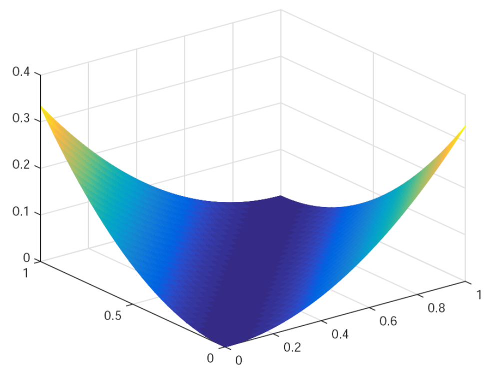

Figure 6. Evolution of the singular solution, $u(0, t) = (T - t)^{2/3}$, $r = 0$, $\Delta t = 0.0001$, $d = 2$.

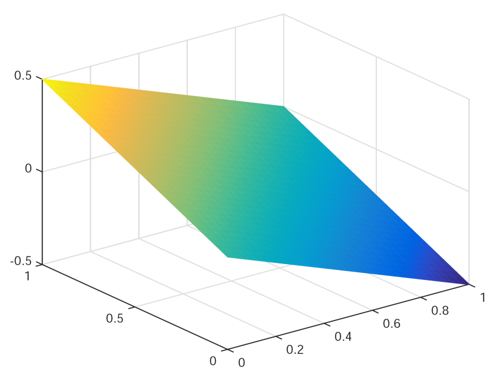

Figure 7. Evolution of the singular solution $u(0, t) = (T - t)^{1/2}$, $r = 0$, $\Delta t = 0.001$, $d = 2$.

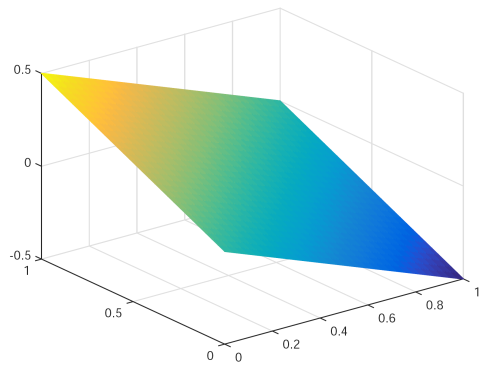

Figure 8. Evolution of the singular solution, $u(0, t) = (T - t)^{1/2}$, $r = 0$, $\Delta t = 0.0001$, $d = 2$.

5. Numerical Approximation for Nonlinear Schrödinger Equations with Finite Difference Methods

In this section, we are interesting in the numerical study for the following initial-boundary value problem (1.1) for $f(x, t) = \beta|u(x, t)|^q,$. The semi-linear Schrödinger equation is of the form:

$$u_t(x, t) - i\alpha\Delta u(x, t) - i\beta|u(x, t)|^q = 0, \quad x \in \mathbb{R}^d;\ t \in (0, T), \tag{5.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d. \tag{5.2}$$

Suppose that $\beta = 1$, $d = 1$, and assume the Dirichlet boundary condition. To that effect, for $T > 0$ at $T \to \infty$, we have also

$$u_t(x, t) = i\alpha u_{xx}(x, t) - i\beta|u(x, t)|^q, \quad x \in (0, 1),\ t \in (0, T), \tag{5.3}$$

$$u(0, t) = 0,\ u(1, t) = 0, \quad t \in (0, T), \tag{5.4}$$

$$u(x, 0) = u_0(x), \quad x \in (0, 1). \tag{5.5}$$

Here $q > 1$, $\alpha \in \mathbb{R}$, $a \neq 0$, $\beta > 0$, $u_0 \in C([0, 1])$, $u_0(0) = 0$, $u_0(1) = 0$, $(0, T)$ is the maximal time interval of existence for the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u

exists globally. When T is finite, the solution u develops a singularity in a finite time, namely,

$$\lim_{t \to T} \| u(x, t) \|_\infty = \infty,$$

where $\| u(x, t) \|_\infty = \sup_{x \in (0,1)} | u(x, t) |$. In this case, we say that the solution u of (5.1)-(5.2) blows-up in a finite time and the time T is called the *blow-up time* of the solution u. Let I be a positive integer and let $h = 1/I$. Define the grid $x_j = jh$, $0 \le j \le I$ and approximate the solution u of (5.1)-(5.2) by the solution $U_h(t) = (U_0(t), U_1(t), ..., U_I(t))^T$ of the following semidiscrete equations:

$$\frac{d}{dt} U_j(t) = i\alpha \delta^2 U_j(t) - i\beta |U_j(t)|^q U_j(t), \quad 0 \le j \le I - 1, \ t \in (0, T_h), \tag{5.6}$$

$$U_0(t) = 0, U_I(t) = 0, \quad t \in (0, T_h), \tag{5.7}$$

$$U_j(0) = \varphi_j, \quad 0 \le j \le I, \tag{5.8}$$

where

$$\delta^2 U_j(t) = \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2}.$$

Here $(0, T)$ is the maximal time interval on which $\| U_h(t) \|_\infty < \infty$ with

$$\| U_h(t) \|_\infty = \max_{0 \le j \le I} |U_j(t)|.$$

When the time T is finite, we say that the solution $U_h(t)$ of (5.6)-(5.8) *blows-up* in a finite time and the time T is called the *semidiscrete blow-up time* of the solution $U_h(t)$. Under some conditions, we show that the solution $U_h(t)$ of (5.6)-(5.8) blows-up in a finite time and estimates its semidiscrete blow-up time. We also show that the solution $U_h(t)$ of (5.6)-(5.8) blows-up in a finite time. We need the following lemma:

Lemma 5.1. $\sum_{j=1}^{I-1} \sin(j\pi h) = \text{cotan}\left(\frac{\pi h}{2}\right).$

Proof. A routine calculation yields

$$\sum_{j=1}^{I-1} \sin(j\pi h) = \text{Im}\left(\sum_{j=1}^{I-1} e^{ij\pi h}\right) = \text{Im}\left(\sum_{j=1}^{I-1} (e^{i\pi h})^j\right)$$

$$= \text{Im}\left(e^{i\pi h}\frac{1 - e^{i\pi h(I-1)}}{1 - e^{i\pi h}}\right) = \text{Im}\left(\frac{e^{i\pi h} - e^{i\pi}}{1 - e^{i\pi h}}\right)$$

because $hI = 1$. Since $e^{i\pi} = -1$, we arrive at

$$\sum_{j=1}^{I-1} \sin(j\pi h) = \text{Im}\left(\frac{e^{i\pi h} + 1}{1 - e^{i\pi h}}\right) = \text{Im}\left(-\frac{e^{i\frac{\pi h}{2}} + e^{-i\frac{\pi h}{2}}}{e^{i\frac{\pi h}{2}} - e^{-i\frac{\pi h}{2}}}\right)$$

$$= \text{Im}\left(i\text{cotan}\left(\frac{\pi h}{2}\right)\right) = \text{cotan}\left(\frac{\pi h}{2}\right)$$

and the proof is complete. □

Lemma 5.2. *Let* $U_h(t)$ *and* $V_h(t)$ *be two vectors such that*

$$U_0(t) = 0, \quad U_I(t) = 0, \quad V_0(t) = 0, \quad V_I(t) = 0.$$

Then

$$\sum_{j=1}^{I-1} hU_j\delta^2 V_j = \sum_{j=1}^{I-1} hV_j\delta^2 U_j. \tag{5.9}$$

Proof. A straightforward computation reveals that

$$\sum_{j=1}^{I-1} hU_j\delta^2 V_j = \sum_{j=1}^{I-1} hV_j\delta^2 U_j + \frac{V_I U_{I-1} - U_I V_{I-1} + V_0 U_1 - U_0 V_1}{h} \tag{5.10}$$

and the result follows using the assumptions of the lemma. □

Now let us state our first result on blow-up.

Theorem 5.1. *Assume that* $1 - \frac{\alpha\lambda_h A^{1-q}}{\beta(q-1)} > 0,$ *where* $\lambda_h = \frac{2 - 2\cos(\pi h)}{h^2}$ *and*

$$A = \sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(j\pi h) \operatorname{Re}(\varphi_j).$$

Then the solution U_h *of* (5.1)-(5.2) *blows-up in a finite time T estimated by*

$$T \le \frac{1}{\alpha\lambda_h} \arccos\left(1 - \frac{\alpha\lambda_h A^{1-q}}{\beta(q-1)}\right). \tag{5.11}$$

Proof. Since $(0, T)$ is the maximal time interval on which $\| U_h(t) \|_\infty < \infty,$ our aim is to show that T_h is finite and satisfies the above inequality. Introduce the functions $v(t)$ and $w(t)$ defined by

$$v(t) = \sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(j\pi h) U_j(t) \quad \text{and} \quad w(t) = \sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(j\pi h) \overline{U}_j(t).$$

Taking the derivative of v in t and using (5.6), we get

$$v'(t) = i\alpha \sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(j\pi h) \delta^2 U_j(t)$$

$$- i\beta \sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(j\pi h) |U_j(t)|^q U_j(t).$$

We observe that $\delta^2 \sin(j\pi h) = -\lambda_h \sin(j\pi h).$ Due to Lemma 5.2, we arrive at

$$v'(t) = -i\alpha\lambda_h v(t) - i\beta \sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(j\pi h) |U_j(t)|^q U_j(t),$$

which implies that

$$\frac{d}{dt}(e^{i\alpha\lambda_h t}v(t)) = -i\beta e^{i\alpha\lambda_h t}\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|^q U_j(t).$$

We also observe that taking the derivative of w in t and using (5.6), we discover that

$$w'(t) = -i\alpha\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)\delta^2\overline{U}_j(t)$$

$$+ i\beta\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|^q U_j(t).$$

Reasoning as above, we find that

$$\frac{d}{dt}(e^{-i\alpha\lambda_h t}w(t)) = i\beta e^{-i\alpha\lambda_h t}\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|^q U_j(t).$$

We deduce that

$$Z'(t) = \beta\sin(\alpha\lambda_h t)\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|^q U_j(t),$$

where

$$Z(t) = \frac{e^{i\alpha\lambda_h t}v(t) + e^{-i\alpha\lambda_h t}w(t)}{2}.$$

From Lemma 5.1, we see that $\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)$ equals one. Thus applying Jensen's inequality, we find that

$$\sum_{j=1}^{I-1}\tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|^q U_j(t)$$

is bounded from below by

$$\left(\sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|\right)^q U_j(t).$$

Applying the triangle inequality, we discover that $|Z(t)|$ is bounded from above by $\sum_{j=1}^{I-1} \tan\left(\frac{\pi}{2}h\right)\sin(j\pi h)|U_j(t)|$. Since $\sin(\alpha\lambda_h t)$ is nonnegative when t is between 0 and $\frac{\pi}{\alpha\lambda_h}$, we deduce that

$$Z'(t) \geq \beta\sin(\lambda_h t)|Z(t)|^q \text{ for } t \in \left(0, \frac{\pi}{a\lambda_h}\right).$$

This inequality implies that the function $Z(t)$ is increasing. Since $Z(0)$ is positive, we find that

$$Z'(t) \geq b\sin(a\lambda_h t)(Z(t))^q \text{ for } t \in \left(0, \frac{\pi}{a\lambda_h}\right),$$

which implies that

$$\frac{dZ}{Z^q} \geq \beta\sin(\alpha\lambda_h t)dt \text{ for } t \in \left(0, \frac{\pi}{a\lambda_h}\right).$$

Let $T_h^* = \min\left(\frac{\pi}{a\lambda_h}, T\right)$. Integrating this inequality over $(0, T^*)$, we conclude that

$$\frac{(Z(0))^{1-q}}{q-1} \geq \frac{\beta}{\alpha\lambda_h}(1 - \cos(\alpha\lambda_h T^*)).$$

Therefore, we have

$$\cos(\alpha\lambda_h T^*) \geq 1 - \frac{\alpha\lambda_h}{\beta}\frac{(Z(0))^{1-q}}{q-1}.$$

Since the quantity on the right hand side of the above inequality is positive, we see that the time T^* is estimated as follows:

$$T^* \leq \frac{1}{\alpha\lambda_h} \arccos\left(1 - \frac{\alpha\lambda_h}{\beta} \frac{(Z(0))^{1-q}}{q-1}\right).$$

Since

$$1 - \frac{\alpha\lambda_h}{\beta} \frac{(Z(0))^{1-q}}{q-1}$$

is positive, we deduce that $T^* \leq \frac{\pi}{2\alpha\lambda_h}$. Consequently, $T^* = T$ is finite. Use the fact that $Z(0) = A$ to complete the rest of the proof. □

Numerical results for $u_0 = 20\sin(\pi hi),\ q = 2,\ \alpha = \frac{1}{25},\ \beta = 1,\ d = 1$

Table 1. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with explicit scheme

I	T^n	n	CPU time	s
16	0.08894760719774	216	0.015	-
32	0.08789556888927	924	0.031	-
64	0.08894741304892	4152	0.156	1.994
128	0.08894778127283	12037	5.296	1.998
256	0.08894762835822	103254	112.062	1.999
512	0.08894760778907	174408	6115.546	2.000

Table 2. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with implicit scheme

I	T^n	n	CPU time	s
16	0.08894760719774	216	0.016	-
32	0.08789556888923	824	0.032	-
64	0.08894741330489	3152	0.126	1.994
128	0.08894778127283	14146	3.286	1.998
256	0.08894768358222	103255	113.062	1.999
512	0.08894761788917	141406	62015.512	2.000

Numerical results for $u_0 = 6\sqrt{2}e^{-r^2}, \; q = 2, \; \alpha = 1, \; \beta = 1, \; d = 1$

Table 3. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with explicit scheme

I	T^n	n	CPU time	s
16	0.493839215838980	3152	0.015	-
32	0.493839108911286	115351	0.031	-
64	0.493832286048925	3152	0.156	1.994
128	0.490087978127283	12037	5.296	1.998
256	0.492861272835822	156602	112.062	1.999
512	0.490028683478907	1850408	6115.546	2.000

Numerical results for $u_0 = 2\sqrt{2}e^{-r^2}, \; q = 4, \; \alpha = 1/25, \; \beta = 1, \; d = 1$

Table 4. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with implicit scheme

I	T^n	n	CPU time	s
16	0.031938360418998	3223	0.015	-
32	0.036137438898042	3094	0.031	-
64	0.031000081591358	33356	0.156	1.994
128	0.020786010349181	124852	5.296	1.998
256	0.020786010349181	45874	112.062	1.999
512	0.034127283478907	184408	6115.546	2.000

Numerical results for $u_0 = 2e^{-r^2}, \ q = 4, \ \alpha = 1/25, \ \beta = 1, \ d = 1$

Table 5. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with implicit scheme

I	T^n	n	*CPU time*	s
16	0.081832555854175	7470	0.016	-
32	0.082325558898042	13094	0.031	-
64	0.085834218578467	23017	0.146	1.994
128	0.080981859466907	86050	5.296	1.998
256	0.089001131888300	529161	113.062	1.999
512	0.080127283478907	1884408	6225.546	2.000

Remark 5.1. It is not hard to see that for $d = 1$, the initial-value problem u_0 in (1.2) is globally well-posed for smooth enough initial data that decay sufficiently fast at infinity. For $d \geq 2$, there exists a singular solution blowing-up in L^∞ in finite time, see [21, 22].

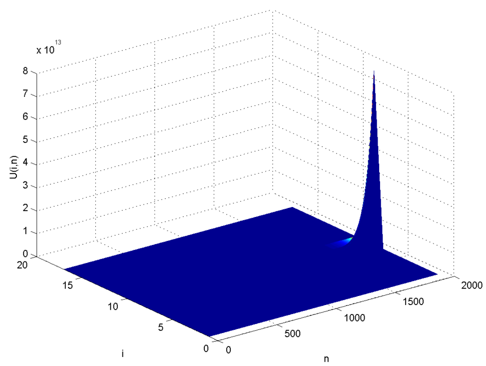

Figure 9. Evolution of the discrete solution, $\alpha = 1/25$, $q = 4$, $\beta = 1$, $d = 1$, $I = 16$ (explicit scheme).

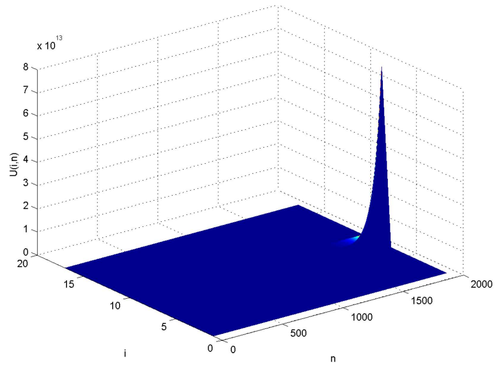

Figure 10. Evolution of the discrete solution, $\alpha = 1/25$, $q = 4$, $\beta = 1$, $d = 1$, $I = 16$ (implicit scheme).

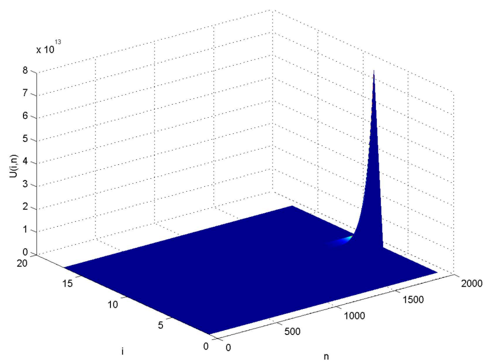

Figure 11. Evolution of the discrete solution, $\alpha = 1/25$, $q = 4$, $\beta = 1$, $d = 1$, $I = 32$ (explicit scheme).

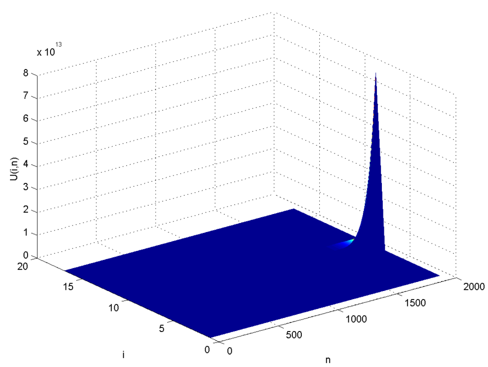

Figure 12. Evolution of the discrete solution, $\alpha = 1/25$, $q = 4$, $\beta = 1$, $d = 1$, $I = 32$ (implicit scheme).

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