Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
http://dx.doi.org/10.17654/0974324324009
Volume 31, Number 2, 2024, Pages 165-185 **P-ISSN: 0974-3243**

ON THE EXISTENCE OF CLASSICAL SOLUTION TO ONE-DIMENSIONAL FOURTH ORDER SEMILINEAR EQUATIONS

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Received: December 22, 2023; Accepted: February 6, 2024
2020 Mathematics Subject Classification: 35A01, 35A09, 35G31.
Keywords and phrases: semilinear equations, mixed problem, classical solution.
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How to cite this article: Samed J. Aliyev, Maftun N. Heydarova and Arzu G. Aliyeva, On the existence of classical solution to one-dimensional fourth order semilinear equations, Advances in Differential Equations and Control Processes 31(2) (2024), 165-185.
http://dx.doi.org/10.17654/0974324324009

Published Online: March 16, 2024

Abstract

In this paper, we prove the existence in small of classical solution of one-dimensional mixed problem for one class of fourth order semilinear Sobolev type equations by combining the generalized contracted mapping principle with Schauder's fixed point principle.

1. Introduction

This work is dedicated to the study of the existence of classical solution for the following one-dimensional mixed problem:

$$u_{txx}(t, x) - \alpha \cdot u_{xxxx}(t, x) = F(t, x, u(t, x), u_x(t, x), u_{xx}(t, x), u_{xxx}(t, x))$$

$$(0 \le t \le T, 0 \le x \le \pi), \quad (1)$$

$$u(0, x) = \varphi(x) (0 \le x \le \pi), \quad (2)$$

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0 \ (0 \le t \le T), \quad (3)$$

where $\alpha > 0$ is a fixed number, $0 < T < +\infty$; F and φ are given functions, and $u(t, x)$ is a sought function.

We call a function $u(t, x)$ a *classical solution* of the problem (1)-(3) if this function and all its derivatives involved in equation (1) are continuous in $[0, T] \times [0, \pi]$ and the conditions (1)-(3) are satisfied in the usual sense.

There have been many works devoted to the study of initial boundary value problem for nonlinear Sobolev equations (see [1, 6, 8, 10, 11, 13] and references therein), where the problem of existence and uniqueness in appropriate Sobolev spaces, the problem of blow up of solutions and the problems of asymptotic behavior of solutions are studied.

In [2], by means of Schauder's strong principle on a fixed point for any dimension *n*, the existence in large theorem (i.e., for any finite value of *T*) of generalized solution of the problem (1)-(3) has been proved. But in [3] using the method of a priori estimates for any dimension *n*, the existence in large theorem of almost everywhere solution of problem (1)-(3) is proved.

We also note works [4, 5, 12], of which some approaches are used in this work.

2. Auxiliaries

In this section, we introduce a number of concepts, notations, and facts to be used later.

(1) As the system $\{\sin nx\}_{n=1}^{\infty}$, $n = 1, 2, \ldots$ forms a basis in the space $L_2(0, \pi)$, it is obvious that every classical solution $u(t, x)$ of the problem (1)-(3) has the following form:

$$u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin nx, \tag{4}$$

where

$$u_n(t) = \frac{2}{\pi} \int_0^{\pi} u(t, x) \sin nx dx \ (n = 1, 2, \ldots; t \in [0, T]). \tag{5}$$

By using the Fourier's method, we can easily see that $u_n(t) \, (n = 1, 2, \ldots)$ satisfy the following system of countable many nonlinear integral equations:

$$u_n(t) = \varphi_n \cdot e^{-\alpha n^2 t} - \frac{2}{\pi n^2} \cdot \int_0^t \int_0^{\pi} E(u(\tau, x)) \sin nx$$

$$\cdot e^{-\alpha n^2 (t-\tau)} dx d\tau \ (n = 1, 2, \ldots; t \in [0, T]), \tag{6}$$

where

$$\varphi_n \equiv \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin nx dx \ (n = 1, 2, \ldots), \tag{7}$$

$$E(u(t, x)) \equiv F(t, x, u(t, x), u_x(t, x), u_{xx}(t, x), u_{xxx}(t, x)). \tag{8}$$

Proceeding from the definition of classical solution of the problem (1)-(3), it is easy to prove the following:

Lemma. *If* $u(t,\ x) = \sum_{n=1}^{\infty} u_n(t)\sin nx$ *is any classical solution of the problem* (1)-(3), *then functions* $u_n(t)\,(n = 1,\ 2,\ \ldots)$ *satisfy the system* (6).

Besides, with the aim to study the existence of classical solution of the problem (1)-(3) in this work, assuming that

$$E(u(t,\ x)),\ \frac{\partial}{\partial x}\{E(u(t,\ x))\} \in C([0,\ T]\times[0,\ \pi]),$$

$$\frac{\partial^2}{\partial x^2}\{E(u(t,\ x))\} \in C([0,\ T];\ L_2(0,\ \pi)), \tag{9}$$

$$E(u(t,\ x))\big|_{x=0} = E(u(t,\ x))\big|_{x=\pi} = 0, \quad \forall t \in [0,\ T], \tag{10}$$

and integrating by parts in x twice on the right side of (6), we transform system (6) to the following form:

$$u_n(t) = \varphi_n \cdot e^{-\alpha n^2 t} + \frac{2}{\pi n^4} \cdot \int_0^t \int_0^\pi \frac{\partial^2}{\partial x^2}\{E(u(\tau,\ x))\}\sin nx$$

$$\cdot\, e^{-\alpha n^2 (t-\tau)} dx d\tau \ (n = 1,\ 2,\ \ldots;\ t \in [0,\ T]). \tag{11}$$

(2) We denote by $B^{\alpha_0,\ldots,\alpha_l}_{\beta_0,\ldots,\beta_l,T}$ a totality of all the functions $u(t,\ x)$ of the form (4) considered in $[0,\ T]\times[0,\ \pi]$ for which all the functions $u_n(t) \in C^{(l)}([0,\ T])$ and

$$J_T(u) \equiv \sum_{i=0}^{l} \left\{ \sum_{n=1}^{\infty} \left(n^{\alpha_i} \cdot \max_{0\le t\le T} |\, u_n^{(i)}(t)\,| \right)^{\beta_i} \right\}^{\frac{1}{\beta_i}} < +\infty,$$

where $l \ge 0$ is an integer, $\alpha_i \ge 0\ (i = \overline{0,\ l})$, $1 \le \beta_i \le 2\ (i = \overline{0,\ l})$. We define the norm in this set as $\|\, u \,\| = J_T(u)$. It is evident that all these spaces are Banach spaces [9, p. 50].

Throughout this paper, we use the following notations for functions: $u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin nx \in B^{\alpha_0, \ldots, \alpha_l}_{\beta_0, \ldots, \beta_l, T}$:

$$\| u \|_{B^{\alpha_0, \ldots, \alpha_l}_{\beta_0, \ldots, \beta_l, t}} \equiv \sum_{i=0}^{l} \left\{ \sum_{n=1}^{\infty} \left(n^{\alpha_i} \cdot \max_{0 \le \tau \le t} | u_n^{(i)}(\tau) | \right)^{\beta_i} \right\}^{\frac{1}{\beta_i}} \quad (0 \le t \le T).$$

(3) It is obvious that if $u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin nx \in B^k_{2,T}$ ($k \ge 1$ is a positive integer), then $\forall t \in [0, T]$:

$$\| u \|_{B^{k-1}_{1,t}} \equiv \sum_{n=1}^{\infty} n^{k-1} \cdot \max_{0 \le \tau \le t} | u_n(\tau) | \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}}$$

$$\cdot \left\{ \sum_{n=1}^{\infty} \left(n^k \cdot \max_{0 \le \tau \le t} | u_n(\tau) | \right)^2 \right\}^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}} \cdot \| u \|_{B^k_{2,t}}. \tag{12}$$

Let $u(t, x) \equiv \sum_{n=1}^{\infty} u_n(t) \sin nx \in B^5_{2,T}$. Then, using estimate (12), we have for $k = 5$, $\forall t \in [0, T]$ and $x \in [0, \pi]$,

$$\left| \frac{\partial^i u(t, x)}{\partial x^i} \right| \le \sum_{n=1}^{\infty} n^i \cdot | u_n(t) | \le \sum_{n=1}^{\infty} n^i \cdot \max_{0 \le \tau \le t} | u_n(\tau) |$$

$$\le \sum_{n=1}^{\infty} n^4 \cdot \max_{0 \le \tau \le t} | u_n(\tau) |$$

$$= \| u \|_{B^4_{1,t}} \le \frac{\pi}{\sqrt{6}} \cdot \| u \|_{B^5_{2,t}} \quad (i = \overline{0, 4}). \tag{13}$$

From estimates (13) and structure of space $B^5_{2,T}$, it follows that

$$u(t, x),\ u_x(t, x),\ u_{xx}(t, x),\ u_{xxx}(t, x),\ u_{xxxx}(t, x) \in C([0, T] \times [0, \pi]). \tag{14}$$

Besides, it is obvious that $\forall t \in [0, T]$,

$$\int_0^{\pi} u_{xxxxx}^2(t, x)dx = \frac{\pi}{2} \cdot \sum_{n=1}^{\infty} (n^5 \cdot u_n(t))^2$$

$$\leq \frac{\pi}{2} \cdot \sum_{n=1}^{\infty} \left(n^5 \cdot \max_{0 \leq \tau \leq t} |u_n(\tau)| \right)^2 = \frac{\pi}{2} \cdot \| u \|_{B_{2,t}^5}^2 .$$

From here, due to the structure of space $B_{2,T}^5$, it follows that

$$u_{xxxxx}(t, x) \in C([0, T]; L_2(0, \pi)). \tag{15}$$

(4) For a positive integer k, let

$$\varphi(x) \in C^{(k-1)}([0, \pi]), \ \varphi^{(k)}(x) \in L_2(0, \pi),$$

$$\varphi^{(2s)}(0) = \varphi^{(2s)}(\pi) = 0 \left(s = \overline{0, \left[\frac{k-1}{2} \right]} \right).$$

Then, integrating by parts, using Bessel's inequality (for odd values of k) and Parseval's equality (for even values of k), it is easy to obtain that

$$\sum_{n=1}^{\infty} (n^k \cdot \varphi_n)^2 \leq \frac{2}{\pi} \cdot \| \varphi^{(k)}(x) \|_{L_2(0, \pi)}^2, \tag{16}$$

where the numbers φ_n $(n = 1, 2, ...)$ are defined by (7). Note that it is evident that the estimate (16) is also true for $k = 0$ if $\varphi(x) \in L_2(0, \pi)$.

3. Main Result

In this section, by combining the generalized contracted mapping principle and Schauder's fixed point principle, the following existence in small (that is, true for sufficiently small values of T) theorem for the classical solution of the problem (1)-(3) is proved:

Theorem. *Let*

(1)

$$\varphi(x) \in C^{(4)}([0, \pi]),\ \varphi^{(5)}(x) \in L_2(0, \pi) \ \textit{and}$$

$$\varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = \varphi^{(4)}(0) = \varphi^{(4)}(\pi) = 0.$$

(2)

$$F(t, \xi_0, \xi_1, \dots, \xi_4),\ F_{\xi_i}(t, \xi_0, \xi_1, \dots, \xi_4)\,(i = \overline{0, 4}),$$

$$F_{\xi_i \xi_j}(t, \xi_0, \xi_1, \dots, \xi_4)\,(i,\ j = \overline{0, 4}) \in C([0, T] \times [0, \pi] \times (-\infty, \infty)^4).$$

(3)

$$F(t, 0, 0, \xi_2, 0, \xi_4) = F(t, \pi, 0, \xi_2, 0, \xi_4) = 0,$$

$$\forall t \in [0, T],\ \xi_2, \xi_4 \in (-\infty, \infty).$$

Then there exists in small a classical solution of the problem (1)-(3).

Proof. For each fixed $u \in B^4_{1,T}$, we define in $B^5_{2,T}$, the operator P_u relative to V:

$$P_u(V(t, x)) = \tilde{V}(t, x) \equiv \sum_{n=1}^{\infty} \tilde{V}_n(t) \sin nx, \tag{17}$$

where

$$\tilde{V}_n(t) = \varphi_n \cdot e^{-\alpha n^2 t} + \frac{2}{\pi n^4} \cdot \int_0^t \int_0^\pi \Phi_u(V(\tau, x)) \sin nx$$

$$\cdot\, e^{-\alpha n^2 (t-\tau)} dx d\tau \ (n = 1, 2, \dots;\ t \in [0, T]), \tag{18}$$

the numbers φ_n $(n = 1, 2, \dots)$ are defined by (7),

$$\Phi_u(V(t, x)) \equiv G(u(t, x)) + g(u(t, x)) \cdot V_{xxxxx}(t, x), \tag{19}$$

$$g(u(t, x)) \equiv F_{\xi_4}(t, x, u(t, x), u_x(t, x), u_{xx}(t, x), u_{xxx}(t, x)), \tag{20}$$

$$G(u(t,\ x)) \equiv \frac{\partial^2}{\partial x^2}\{E(u(t,\ x))\} - g(u(t,\ x)) \cdot u_{xxxxx}(t,\ x), \tag{21}$$

the operator E is defined by (8), and ξ_4 is a variable of the function $F(t,\ \xi_0,\ \xi_1,\ ...,\ \xi_4)$.

It is evident that

$$\forall u \in B^5_{2,T}, \quad \Phi_u(u(t,\ x)) = \frac{\partial^2}{\partial x^2}\{E(u(t,\ x))\}. \tag{22}$$

From (18), we obtain for any fixed $u \in B^4_{1,T}$, $\forall V \in B^5_{2,T}$ and $t \in [0,\ T]$,

$$\|\tilde{V}\|^2_{B^5_{2,t}} \equiv \sum_{n=1}^{\infty}\left(n^5 \cdot \max_{0\le\tau\le t}|\tilde{V}_n(\tau)|\right)^2$$

$$\le a_0 + \frac{2}{\alpha\pi} \cdot \int_0^t\int_0^\pi \{\Phi_u(V(\tau,\ x))\}^2 dxd\tau, \tag{23}$$

where

$$a_0 \equiv 2\sum_{n=1}^{\infty}(n^5 \cdot \varphi_n)^2, \tag{24}$$

when a_0 follows from (16) for $k = 5$.

Thus, for any fixed $u \in B^4_{1,T}$ by virtue of (23), $\forall V \in B^5_{2,T}$, we have

$$\|P_u(V)\|^2_{B^5_{2,T}} \equiv \|\tilde{V}\|^2_{B^5_{2,T}} \equiv \left\|\sum_{n=1}^{\infty}\tilde{V}_n(t)\sin nx\right\|^2_{B^5_{2,T}}$$

$$= \sum_{n=1}^{\infty}\left(n^5 \cdot \max_{0\le\tau\le T}|\tilde{V}_n(\tau)|\right)^2$$

$$\le a_0 + \frac{2}{\alpha\pi} \cdot \int_0^T\int_0^\pi \{\Phi_u(V(\tau,\ x))\}^2 dxd\tau. \tag{25}$$

Next, due to the structure of space $B_{1,T}^4$, we have for every $u \in B_{1,T}^4$ and $\tau \in [0, T]$,

$$\left\| \frac{\partial^i u(\tau, x)}{\partial x^i} \right\|_{C([0,\pi])} \leq \| u \|_{B_{1,\tau}^i} \leq \| u \|_{B_{1,\tau}^4} \leq \| u \|_{B_{1,T}^4} \quad (i = \overline{0, 4}). \tag{26}$$

Besides, for any $V = V(t, x) = \sum_{n=1}^{\infty} V_n(t) \sin nx \in B_{2,T}^5$ and $\forall t \in [0, T]$,

$$\int_0^\pi V_{xxxxx}^2(t, x)\, dx = \frac{\pi}{2} \cdot \sum_{n=1}^{\infty} (n^5 \cdot V_n(t))^2$$

$$\leq \frac{\pi}{2} \cdot \sum_{n=1}^{\infty} \left(n^5 \cdot \max_{0 \leq \tau \leq t} | V_n(\tau) | \right)^2 = \frac{\pi}{2} \cdot \| V \|_{B_{2,t}^5}^2. \tag{27}$$

Now, using condition (2) of this theorem and estimates (26), it is easy to obtain that, for any $u \in B_{1,T}^4$,

$$\| g(u(t, x)) \|_{C(Q_T)} \leq C(u), \quad \| G(u(t, x)) \|_{C(Q_T)} \leq C(u), \tag{28}$$

where $g(u(t, x))$ and $G(u(t, x))$ are defined by (20) and (21), $Q_T \equiv [0, T] \times [0, \pi]$, $C(u) > 0$ is a constant.

Thus, from (25) using estimates (28), (27) and relation (19), we obtain for any fixed $u \in B_{1,T}^4$, $\forall V \in B_{2,T}^5$,

$$\| P_u(V) \|_{B_{2,T}^5}^2 \leq a_0 + \frac{2}{\alpha\pi} \cdot \int_0^T \int_0^\pi \{\Phi_u(V(\tau, x))\}^2 dx d\tau$$

$$\leq a_0 + \frac{4T}{\alpha} \cdot C^2(u) + \frac{2T}{\alpha} \cdot C^2(u) \cdot \| V \|_{B_{2,t}^5}^2. \tag{29}$$

It follows from (29) that for any fixed $u \in B_{1,T}^4$, the operator P_u acts in $B_{2,T}^5$, boundedly.

Now, using relations (17)-(21) and estimates (28), (27) $\left(\text{for } V = V_1 - V_2\right)$, similar to (23), we obtain that for any fixed $u \in B_{1,T}^4$, $\forall V_1, V_2 \in B_{2,T}^5$:

$$\| P_u(V_1) - P_u(V_2) \|_{B_{2,t}^5}^2$$

$$\leq \frac{2}{\alpha\pi} \cdot \int_0^t \int_0^\pi \{\Phi_u(V_1(\tau, x)) - \Phi_u(V_2(\tau, x))\}^2 dx d\tau$$

$$\leq \frac{2}{\alpha\pi} \cdot C^2(u) \cdot \int_0^t \left\{ \int_0^\pi [V_{1,xxxxx}(\tau, x) - V_{2,xxxxx}(\tau, x)]^2 dx \right\} d\tau$$

$$\leq \frac{2}{\alpha\pi} \cdot C^2(u) \cdot \frac{\pi}{2} \int_0^t \| V_1 - V_2 \|_{B_{2,\tau}^5}^2 d\tau$$

$$\leq \frac{1}{\alpha} \cdot C^2(u) \cdot \| V_1 - V_2 \|_{B_{2,T}^5}^2 \cdot t, \tag{30}$$

$$\| P_u^k(V_1) - P_u^k(V_2) \|_{B_{2,t}^5}^2$$

$$= \| P_u(P_u^{k-1}(V_1)) - P_u(P_u^{k-1}(V_2)) \|_{B_{2,t}^5}^2$$

$$\leq \left\{ \frac{1}{\alpha} \cdot C^2(u) \right\}^k \cdot \| V_1 - V_2 \|_{B_{2,T}^5}^2 \cdot \frac{t^k}{k!}$$

$$\leq \left\{ \frac{1}{\alpha} \cdot C^2(u) \right\}^k \cdot \| V_1 - V_2 \|_{B_{2,T}^5}^2 \cdot \frac{T^k}{k!},$$

where k is a positive integer.

Thus, for any fixed $u \in B_{1,T}^4$, $\forall V_1, V_2 \in B_{2,T}^5$, we have

$$\| P_u^k(V_1) - P_u^k(V_2) \|_{B_{2,T}^5} \leq q_k(u) \cdot \| V_1 - V_2 \|_{B_{2,T}^5},$$

where

$$q_k(u) \equiv \frac{1}{\sqrt{k!}} \cdot \left\{\frac{1}{\alpha} \cdot C^2(u) \cdot T\right\}^{\frac{k}{2}}.$$

It is obvious that for sufficiently large k, $q_k(u) < 1$. For such a k, the operator P_u^k is a contraction in space $B_{2,T}^5$. Hence, by virtue of the generalized contracted mapping principle, the unique fixed point V of the operator P_u^k in $B_{2,T}^5$ is the unique fixed point of the operator P_u in $B_{2,T}^5$:

$$V = P_u(V), \quad V \in B_{2,T}^5.$$

Matching to each $u \in B_{1,T}^4$, the unique fixed point V of the operator P_u in $B_{2,T}^5$, we generate the operator H:

$$H(u) = V = P_u(V).$$

To show the continuity of the operator H, consider

$$B_{1,T}^4 \ni u_k(t, x) \xrightarrow{B_{1,T}^4} u_0(t, x) \in B_{1,T}^4 \text{ as } k \to \infty.$$

Then, due to (26) for $u = u_k - u_0$, it is evident that

$$\frac{\partial^i u_k(t, x)}{\partial x^i} \xrightarrow{C(Q_T)} \frac{\partial^i u_0(t, x)}{\partial x^i} \text{ as } k \to \infty \ (i = \overline{0, 4})$$

and there exists a number $R_0 > 0$ such that $\forall k$ $(k = 1, 2, \ldots)$ and $t \in [0, T]$, $x \in [0, \pi]$,

$$-R_0 \le u_k(t, x), u_{k,x}(t, x), u_{k,xx}(t, x), u_{k,xxx}(t, x), u_{k,xxxx}(t, x) \le R_0.$$

Consequently,

$$\| G(u_k(t, x)) - G(u_0(t, x)) \|_{C(Q_T)} \equiv \varepsilon_k \to 0 \text{ as } k \to \infty, \tag{31}$$

$$\| g(u_k(t, x)) - g(u_0(t, x)) \|_{C(Q_T)} \equiv \delta_k \to 0 \text{ as } k \to \infty, \tag{32}$$

$$\| g(u_k(t, x)) \|_{C(Q_T)} \le C_0 \ (k = 0, 1, \ldots), \tag{33}$$

where $C_0 > 0$ is a constant.

We use the notations:

$$H(u_k) = V_k \ (V_k = P_{u_k}(V_k)), \quad k = 0, 1, \ldots.$$

Then, using relations (19)-(21), (31), (32), estimate (33) and estimate (27) for $V = V_k - V_0$ and $V = V_0$, similar to (30), we obtain that $\forall k$ $(k = 1, 2, \ldots)$ and $t \in [0, T]$,

$$\| H(u_k) - H(u_0) \|^2_{B^5_{2,t}}$$

$$\equiv \| V_k - V_0 \|^2_{B^5_{2,t}} \equiv \| P_{u_k}(V_k) - P_{u_0}(V_0) \|^2_{B^5_{2,t}}$$

$$\le \frac{2}{\alpha\pi} \cdot \int_0^t \int_0^\pi \{\Phi_{u_k}(V_k(\tau, x)) - \Phi_{u_0}(V_0(\tau, x))\}^2 dx d\tau$$

$$\le \frac{3T}{\alpha} \left(2\varepsilon_k^2 + \delta_k^2 \cdot \| V_0 \|^2_{B^5_{2,T}} \right) + \frac{3}{\alpha} \cdot C_0^2 \cdot \int_0^t \| V_k - V_0 \|^2_{B^5_{2,\tau}} d\tau. \tag{34}$$

From (34), on applying Bellman's inequality [7, pp. 188-189], we obtain that

$$\| H(u_k) - H(u_0) \|^2_{B^5_{2,T}} \equiv \| V_k - V_0 \|^2_{B^5_{2,T}} \le \frac{3T}{\alpha} \left(2\varepsilon_k^2 + \delta_k^2 \cdot \| V_0 \|^2_{B^5_{2,T}} \right)$$

$$\cdot \exp\left\{ \frac{3}{\alpha} \cdot C_0^2 \cdot T \right\}, \quad k = 1, 2, \ldots.$$

From here, due to (31) and (32), it follows that

$$H(u_k) \xrightarrow{B^5_{2,T}} H(u_0) \text{ as } k \to \infty.$$

Thus, the operator H acts continuously from $B_{1,T}^4$ to $B_{2,T}^5$ and, moreover, it acts continuously on $B_{1,T}^4$.

Now, we show the compactness of the operator H in $B_{1,T}^4$. Let $K = K_R$ be any closed ball of the space $B_{1,T}^4$ with the center at zero having the radius R. Then it evident that for any $u \in K_R$, due to (26), $\forall t \in [0, T]$ and $x \in [0, \pi]$,

$$-R \le u(t, x), u_x(t, x), u_{xx}(t, x), u_{xxx}(t, x), u_{xxxx}(t, x) \le R.$$

Besides, it is evident that $\forall u \in K_R$,

$$\| g(u(t, x)) \|_{C(Q_T)} \le C_R, \quad \| G(u(t, x)) \|_{C(Q_T)} \le C_R, \tag{35}$$

where $C_R > 0$ is a constant.

Using estimates (35), (27) and relations (17)-(21), similar to (23), we obtain that $\forall u \in K_R$ and $\forall t \in [0, T]$,

$$\| H(u) \|_{B_{2,t}^5}^2 \equiv \| V \|_{B_{2,t}^5}^2 \equiv \| P_u(V) \|_{B_{2,t}^5}^2$$

$$\le a_0 + \frac{2}{\alpha\pi} \cdot \int_0^t \int_0^\pi \{\Phi_u(V(\tau, x))\}^2 dx d\tau$$

$$\le a_0 + \frac{4T}{\alpha} \cdot C_R^2 + \frac{2}{\alpha} \cdot C_R^2 \cdot \int_0^t \| V \|_{B_{2,\tau}^5}^2 d\tau. \tag{36}$$

From (36), on applying Bellman's inequality, we obtain that $\forall u \in K_R$,

$$\| H(u) \|_{B_{2,T}^5}^2 \equiv \| V \|_{B_{2,T}^5}^2 \le \left(a_0 + \frac{4T}{\alpha} \cdot C_R^2 \right) \cdot \exp\left\{ \frac{2}{\alpha} \cdot C_R^2 \cdot T \right\} \equiv a_R^2. \tag{37}$$

Further, since $\forall u \in B_{1,T}^4$, $H(u) = V \equiv \sum_{n=1}^{\infty} V_n(t) \sin nx$, where $V_n(t)$ $(n = 1, 2, ...)$ is equal to the right side of (18), it follows that

$$V_n'(t) = -\alpha n^2 \cdot \varphi_n \cdot e^{-\alpha n^2 t} - \frac{2\alpha}{\pi n^2}$$

$$\cdot \int_0^t \int_0^\pi \Phi_u(V(\tau, x)) \sin nx \cdot e^{-\alpha n^2 (t-\tau)} dx d\tau$$

$$+ \frac{2}{\pi n^4} \cdot \int_0^\pi \Phi_u(V(t, x)) \sin nx dx \quad (n = 1, 2, \ldots; t \in [0, T]).$$

From here, it is evident that $\forall n$ $(n = 1, 2, \ldots)$ and $t \in [0, T]$,

$$|V_n'(t)| \leq \alpha n^2 \cdot |\varphi_n| + \frac{\sqrt{2\alpha}}{\pi n^3} \cdot \left\{ \int_0^t \left[\int_0^\pi \Phi_u(V(\tau, x)) \sin nx dx \right]^2 d\tau \right\}^{\frac{1}{2}}$$

$$+ \frac{2}{\sqrt{\pi} \cdot n^4} \cdot \left\{ \int_0^t [\Phi_u(V(t, x))]^2 dx \right\}^{\frac{1}{2}}. \tag{38}$$

On the other hand, using relations (19)-(21) and estimates (35), (27), (37), we have $\forall u \in K_R$ and $t \in [0, T]$,

$$\int_0^\pi [\Phi_u(V(t, x))]^2 dx$$

$$\leq C_R^2 \cdot \int_0^\pi [1 + |V_{xxxxx}(t, x)|]^2 dx \leq 2C_R^2 \cdot \int_0^\pi [1 + V_{xxxxx}^2(t, x)] dx$$

$$\leq 2 \cdot C_R^2 \cdot \left(\pi + \frac{\pi}{2} \| V \|_{B_{2,t}^5}^2 \right) \leq \pi \cdot C_R^2 \cdot (2 + \| V \|_{B_{2,T}^5}^2)$$

$$\leq \pi \cdot C_R^2 \cdot (2 + a_R^2). \tag{39}$$

Then, using estimate (39), from (38), we obtain that $\forall n$ $(n = 1, 2, \ldots)$ and $t \in [0, T]$,

$$|V_n'(t)| \le \alpha n^2 \cdot |\varphi_n| + \frac{\sqrt{2\alpha}}{\pi n^3} \cdot \left\{\int_0^T \left[\int_0^\pi \Phi_u(V(\tau,\ x)) \sin nx dx\right]^2 d\tau\right\}^{\frac{1}{2}}$$

$$+ \frac{2}{\sqrt{\pi} \cdot n^4} \cdot \sqrt{\pi C_R^2 \cdot (2 + a_R^2)}.$$

Consequently,

$$\sum_{n=1}^{\infty} \left(n^3 \cdot \max_{0 \le t \le T} |V_n(t)|\right)^2$$

$$\le 3\alpha^2 \cdot \sum_{n=1}^{\infty} (n^5 \cdot \varphi_n)^2 + 12 C_R^2 \cdot (2 + a_R^2) \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{6\alpha}{\pi^2}$$

$$\cdot \frac{\pi}{2} \int_0^T \sum_{n=1}^{\infty} \left\{\int_0^\pi \Phi_u(V(t,\ x)) \cdot \sqrt{\frac{2}{\pi}} \sin nx dx\right\}^2 dt$$

$$= 3\alpha^2 \cdot \sum_{n=1}^{\infty} (n^5 \cdot \varphi_n)^2 + 12(2 + a_R^2) \cdot C_R^2 \cdot \frac{\pi^2}{6} + \frac{3\alpha}{\pi}$$

$$\cdot \int_0^T \int_0^\pi \{\Phi_u(V(t,\ x))\}^2 dx dt.$$

Hence, using estimate (39), we have

$$\| (H(u))_t \|_{B_{2,T}^3}^2$$

$$\equiv \| V_t \|_{B_{2,T}^3}^2 \equiv \sum_{n=1}^{\infty} \left(n^3 \cdot \max_{0 \le t \le T} |V_n'(t)|\right)^2$$

$$\le 3\alpha^2 \cdot \sum_{n=1}^{\infty} (n^5 \cdot \varphi_n)^2 + 2\pi^2(2 + a_R^2) \cdot C_R^2 + 3\alpha \cdot (2 + a_R^2) \cdot C_R^2 \equiv b_R^2. \quad (40)$$

Thus, from the estimates (37) and (40), it follows that $\forall u \in K_R$,

$$\| H(u) \|^2_{B^{5,3}_{2,2,T}} = \| H(u) \|_{B^5_{2,T}} + \| (H(u))_t \|_{B^3_{2,T}} \le a_R + b_R \equiv c_R. \tag{41}$$

Consequently, the set $H(K_R)$ is bounded in $B^{5,3}_{2,2,T}$. It follows the validity of the following two facts:

(i) for each fixed n $(n = 1, 2, ...)$, the set of nth components of all elements from $H(K_R)$ is bounded in $C^{(1)}([0, T])$ and, therefore, compact in $C([0, T])$;

(ii) by virtue of estimates

$$\sum_{n=N}^{\infty} n^4 \cdot \max_{0 \le t \le T} | V_n(t) |$$

$$\le \left(\sum_{n=N}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \cdot \left\{ \sum_{n=1}^{\infty} \left(n^5 \cdot \max_{0 \le t \le T} | V_n(t) | \right)^2 \right\}^{\frac{1}{2}} \le a_R \cdot \left(\sum_{n=N}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}},$$

for any $\varepsilon > 0$, there exists a number n_ε, the same for all

$$H(u) = V \equiv \sum_{n=1}^{\infty} V_n(t) \sin nx \in H(K_R),$$

such that

$$\sum_{n=n_\varepsilon}^{\infty} n^4 \cdot \max_{0 \le t \le T} | V_n(t) | < \varepsilon, \quad \forall V \in H(K_R),$$

where N is a positive integer and a_R is defined by (37).

Consequently, by Theorem 1.1 [9, p. 51], the set $H(K_R)$, considered as a subset of the space $B^4_{1,T}$, is compact in $B^4_{1,T}$. Thus, the operator H acts compactly in $B^4_{1,T}$. Since the operator H acts (as proved above) in $B^4_{1,T}$ and is continuous, it acts in $B^4_{1,T}$ completely continuously. Next, due to estimates (12) for $k = 5$ and (37), we have $\forall u \in K_R$,

$$\| H(u) \|_{B^4_{1,T}} \leq \frac{\pi}{\sqrt{6}} \cdot \| H(u) \|_{B^5_{2,T}} \leq \frac{\pi}{\sqrt{6}} \cdot a_R$$

$$= \frac{\pi}{\sqrt{6}} \cdot \left(a_0 + \frac{4T}{\alpha} \cdot C_R^2 \right)^{\frac{1}{2}} \cdot \exp\left\{ \frac{1}{\alpha} \cdot C_R^2 \cdot T \right\}, \tag{42}$$

where the numbers a_0 and C_R are defined by relations (24) and (35).

From (42), it follows that, if the fixed number

$$R > \frac{\pi}{\sqrt{6}} \cdot \sqrt{a_0} = \frac{\pi}{\sqrt{3}} \cdot \sqrt{\sum_{n=1}^{\infty} (n^5 \cdot \varphi_n)^2}, \tag{43}$$

then for sufficiently small values of T, we have

$$\forall u \in K_R \ \| H(u) \|_{B^4_{1,T}} \leq R, \text{ i.e., } H(K_R) \subset K_R.$$

Thus, for any fixed R satisfying condition (43), for sufficiently small values of T, the operator H transforms the ball K_R into itself completely continuously. Consequently, by the Schauder's principle about fixed point, for sufficiently small values of T, the operator H has at least one fixed point u in $K_R \subset B^4_{1,T}$:

$$u = H(u). \tag{44}$$

As $u = H(u) = V = P_u(V)$, $u = V$, and consequently,

$$u = H(u) = P_u(u),$$

while due to (44) and (41), we have $u(t, x) \in B^{5,3}_{2,2,T}$.

Next, by virtue of (22),

$$\Phi_u(u) = \frac{\partial^2}{\partial x^2}\{E(u(t,\ x))\},$$

and consequently, for the found fixed point

$$u = u(t,\ x) = \sum_{n=1}^{\infty} u_n(t)\sin nx \in B^{5,\,3}_{2,\,2,\,T}, \tag{45}$$

the functions $u_n(t)$ $(n = 1, 2, ...)$ satisfy the system (11).

Now, we show that the obtained function (45) is a classical solution of the problem (1)-(3).

From $u(t,\ x) \in B^4_{2,T}$, due to estimates (26) and the structure of space $B^4_{1,T}$, it follows that

$$u(t,\ x),\ u_x(t,\ x),\ u_{xx}(t,\ x),\ u_{xxx}(t,\ x),\ u_{xxxx}(t,\ x) \in C([0,\ T]\times[0,\ \pi]). \tag{46}$$

Next, from $u_t(t,\ x) \in B^3_{2,T}$, due to estimate (12) for $k = 3$, it follows that $u_t(t,\ x) \in B^2_{1,T}$. From here, by virtue of the structure of space $B^2_{1,T}$, we have

$$u_t(t,\ x),\ u_{tx}(t,\ x),\ u_{txx}(t,\ x) \in C([0,\ T]\times[0,\ \pi]). \tag{47}$$

Thus, as follows from (46) and (47), the function $u(t,\ x)$ is continuous in the closed domain $[0,\ T]\times[0,\ \pi]$ together with all its derivatives entering into equation (1).

Further, the fulfillment of conditions (3) for the function

$$u(t,\ x) = \sum_{s=1}^{\infty} u_n(t)\sin nx$$

is obvious, because, as follows from the relation $u(t,\ x) \in B_{1,T}^{4}$,

$$\sum_{n=1}^{\infty} n^{4} \cdot \max_{0 \le t \le T} |u_n(t)| < +\infty$$

and, moreover

$$\sum_{n=1}^{\infty} n^{2} \cdot \max_{0 \le t \le T} |u_n(t)| < +\infty.$$

Then, it is obvious that for a function $u(t,\ x) \in B_{1,T}^{5}$, by virtue of conditions (2) and (3) of this theorem and properties (14), (15), (3), conditions (9) and (10) are satisfied. Therefore, functions $u_n(t)\,(n = 1,\ 2,\ \ldots)$ satisfying system (11) also satisfy system (6).

Further, from system (6), it follows that

$$u(0,\ x) = \sum_{n=1}^{\infty} u_n(0) \sin nx = \sum_{n=1}^{\infty} \varphi_n \cdot \sin nx = \varphi(x)\ (0 \le x \le \pi),$$

because, by virtue of condition (1) of this theorem and estimate (16) for $k = 5$,

$$\sum_{n=1}^{\infty} (n^{5} \cdot \varphi_n)^{2} < +\infty$$

and, moreover

$$\sum_{n=1}^{\infty} |\varphi_n| < +\infty.$$

Thus, the function $u(t,\ x)$ satisfies conditions (2) and (3) in the usual sense.

Now, using the fact that the function $u(t,\ x)$ has properties (46) and (47), it is easy to obtain from system (6) that for each fixed $t \in [0,\ T]$ and

$x \in [0, \pi]$,

$$u_{txx}(t, x) - \alpha \cdot u_{xxxx}(t, x) = \sum_{n=1}^{\infty} \left\{ \frac{2}{\pi} \int_0^{\pi} E(u(t, x)) \sin nx dx \right\} \cdot \sin nx$$

$$\equiv \sum_{n=1}^{\infty} E_n(u; t) \sin nx = E(u(t, x)).$$

Consequently, the function $u(t, x)$ satisfies equation (1) everywhere in $[0, T] \times [0, \pi]$.

Thus, the function $u(t, x) \in B_{2,2,T}^{5,3}$ is a classical solution of the problem (1)-(3).

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