Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
http://dx.doi.org/10.17654/0974324324008
Volume 31, Number 2, 2024, Pages 153-164 **P-ISSN: 0974-3243**

A NUMERICAL METHOD TO SOLVE THE VISCOSITY PROBLEM OF THE BURGERS EQUATION

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Abstract

Considering the viscosity problem of the Burgers equation, we give a numerical solution using the Cole-Hopf transformation.

Received: January 12, 2024; Accepted: February 27, 2024
2020 Mathematics Subject Classification: 35E05, 35Q35.
Keywords and phrases: Burgers equation, Cole-Hopf transformation, SBA method.

How to cite this article: Gérard ZONGO, Ousséni SO and Geneviève BARRO, A numerical method to solve the viscosity problem of the Burgers equation, Advances in Differential Equations and Control Processes 31(2) (2024), 153-164.
http://dx.doi.org/10.17654/0974324324008

Published Online: March 9, 2024

1. Introduction

Burgers equation is a partial differential equation which is important in the study of modeling turbulence, mass transport, the modeling of shock-wave formation and the gas dynamics. It was discovered in 1948 by physicist Johannes Martinus Burgers.

In the mathematical literature, we find a set of nonlinear partial differential equation problems. Among these equations, we have the equation of Schrödinger [15, 17], KdV equation [14], Burgers equation [8, 16], Burgers inhomogeneous equation, sine-Gordon [12], and others. Some methods have been invented to solve. We quote the inverse scattering method [3, 6], the decompositional method of Adomian [4, 5, 15], the SBA method [1, 2, 17], Backland transformation [14], the variational iteration method [9-11], and fractional differential equations [13]. In this article, we use the Cole-Hopf transformation to simplify the equation and then use the SBA method to solve it.

In the literature, we find the Burgers inhomogeneous equation that is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = F(x, t), \quad v > 0, \tag{1}$$

where the inhomogeneous term or the source term is the given function $F(x, t)$, t is the time variable and x is the space variable.

We have again Burgers equation with evanescent viscosity

$$\begin{cases} \dfrac{\partial u^{\varepsilon}}{\partial t} + u^{\varepsilon}\dfrac{\partial u^{\varepsilon}}{\partial x} - \varepsilon\dfrac{\partial^2 u^{\varepsilon}}{\partial x^2} = 0, & \text{in } \mathbb{R} \times \mathbb{R}_*^+, \\ u^{\varepsilon}(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $\varepsilon > 0$.

There is another equation which does not have source term. It is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where $\varepsilon > 0$ is the viscosity term. When $\varepsilon = 0,$ we get the inviscid equation (3) which will be the subject of our study.

2. Application of the SBA Method to the Viscosity Problem of the Burgers Equation

Consider Burgers viscosity equation which is a nonlinear parabolic PDE problem:

$$u_t + uu_x = \varepsilon u_{xx}, \tag{4}$$

where $\varepsilon > 0$ is the viscosity constant.

Now consider Burgers viscosity problem with an initial value:

$$\begin{cases} u_t + uu_x = \varepsilon u_{xx}, & x \in \mathbb{R},\ t > 0,\ \varepsilon > 0, \\ u(x, 0) = u_0(x) = -\dfrac{2\varepsilon\pi}{L}\operatorname{cotan}\left(\dfrac{\pi x}{L}\right), & x \in \mathbb{R}. \end{cases} \tag{5}$$

2.1. Operation with the Cole-Hopf transformation

The Cole-Hopf transformation is defined by

$$u = -2\varepsilon \frac{\varphi_x}{\varphi}. \tag{6}$$

By making a change in (6), we find that

$$\begin{cases} u_t = \dfrac{2\varepsilon(\varphi_t \cdot \varphi_x - \varphi \cdot \varphi_{xt})}{\varphi^2}, \\ u_x = \dfrac{2\varepsilon(\varphi_x^2 - \varphi \cdot \varphi_{xx})}{\varphi^2}, \\ uu_x = \dfrac{4\varepsilon^2 \varphi_x(\varphi \cdot \varphi_{xx} - \varphi_x^2)}{\varphi^3}, \\ u_{xx} = -2\varepsilon^2 \dfrac{2\varphi_x^3 - 3\varphi_x \cdot \varphi_{xx} \cdot \varphi + \varphi^2 \cdot \varphi_{xxx}}{\varphi^3}. \end{cases} \tag{7}$$

By substituting these expressions into (4), we have

$$\varphi = \varphi_t - \varepsilon\varphi_{xx}. \tag{8}$$

Therefore, if φ solves the heat equation $\varphi_t - \varepsilon\varphi_{xx}$, $x \in \mathbb{R}$, then $u(x, t)$ gives by transformation of (6) the solution of the Burgers viscosity equation (4).

To completely transform the problem (4), we still have to work with the initial condition of the function.

To do this, note that (6) can be written as

$$u = -2\varepsilon(\log\varphi)_x,$$

$$(\log\varphi(x, t))_x = -\frac{u(x, t)}{2\varepsilon},$$

$$\int(\log\varphi(x, t))_x dx = -\int\frac{u(x, t)}{2\varepsilon}dx,$$

$$\log\varphi(x, t) = -\int\frac{u(x, t)}{2\varepsilon}dx,$$

$$\varphi(x, t) = e^{-\int\frac{u(x,t)}{2\varepsilon}dx}. \tag{9}$$

It is clear from (6) that multiplying φ by a constant does not affect u, so we can write the last equation as

$$\varphi(x, t) = e^{-\int_0^x\frac{u(y,t)}{2\varepsilon}dy}. \tag{10}$$

The initial condition in (5) can therefore be transformed using (9) into

$$\varphi(x, 0) = \varphi_0(x) = e^{-\int\frac{u_0(y)}{2\varepsilon}dy}.$$

We have $u_0(y) = -\frac{2\varepsilon\pi}{L}\operatorname{cotan}\left(\frac{\pi y}{L}\right)$. Thus

$$-\frac{u_0(y)}{2\varepsilon} = \frac{\pi}{L}\frac{\cos\left(\frac{\pi y}{L}\right)}{\sin\left(\frac{\pi y}{L}\right)},$$

$$-\int \frac{u_0(y)}{2\varepsilon}dy = \frac{\pi}{L}\int \frac{\cos\left(\frac{\pi y}{L}\right)}{\sin\left(\frac{\pi y}{L}\right)}dy,$$

$$-\int \frac{u_0(y)}{2\varepsilon}dy = \frac{\pi}{L}\left[\frac{L}{\pi}\ln\left|\sin\left(\frac{\pi y}{L}\right)\right|\right],$$

$$-\int \frac{u_0(y)}{2\varepsilon}dy = \ln\left|\sin\left(\frac{\pi y}{L}\right)\right|,$$

$$e^{-\int \frac{u_0(y)}{2\varepsilon}dy} = e^{\ln\left|\sin\left(\frac{\pi y}{L}\right)\right|},$$

$$e^{-\int \frac{u_0(y)}{2\varepsilon}dy} = \sin\left(\frac{\pi y}{L}\right),$$

$$\varphi(y, 0) = \sin\left(\frac{\pi y}{L}\right).$$

In summary, we reduced the problem (4) to this one

$$\begin{cases}\varphi_t - \varepsilon\varphi_{xx} = 0, \ x \in \mathbb{R}, \ t > 0, \ \varepsilon > 0, \\ \varphi(x, 0) = \varphi_0(x) = \sin\left(\frac{\pi x}{L}\right), \ x \in \mathbb{R}.\end{cases}$$

3. Solving the New Equations Using the SBA Method

3.1. The case when $\varphi = 0$

Integrating the heat equation with respect to t, we obtain

$$\varphi(x, t) = \varphi(x, 0) + \varepsilon\int_0^t \frac{\partial^2\varphi(x, s)}{\partial x^2}ds. \tag{11}$$

Applying the SBA algorithm, we obtain

$$\begin{cases} \varphi_0^k(x, t) = \varphi^k(x, 0), \ \forall k \geq 1, \\ \varphi_n^k(x, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_{n-1}^k(x, s)}{\partial x^2} ds, \ \forall n \geq 1, \end{cases}$$

$$\begin{cases} \varphi_0^k(x, t) = \sin\left(\frac{\pi x}{L}\right), \ \forall k \geq 1, \\ \varphi_n^k(x, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_{n-1}^k(x, s)}{\partial x^2} ds, \ \forall n \geq 1. \end{cases}$$

For $k = 1$, we have $\varphi_0^1(x, t) = \sin\left(\frac{\pi x}{L}\right)$.

For $n = 1$, we have

$$\varphi_1^1(x, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_0^1(x, s)}{\partial x^2} ds,$$

$$\frac{\partial \varphi_0^1(x, t)}{\partial x} = \frac{\pi}{L} \cos\left(\frac{\pi x}{L}\right),$$

$$\varphi_1^1(x, t) = \varepsilon \int_0^t -\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) ds$$

$$= -\varepsilon \left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) \cdot t.$$

For $n = 2$, we have

$$\varphi_2^1(x, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_1^1(x, s)}{\partial x^2} ds,$$

$$\varphi_2^1(x, t) = (-1)^2 \varepsilon^2 \left(\frac{\pi}{L}\right)^4 \int_0^t \sin\left(\frac{\pi x}{L}\right) \cdot s ds$$

$$= (-1)^2 \varepsilon^2 \left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi x}{L}\right) \cdot \frac{t^2}{2}.$$

For $n = 3,$ we have

$$\varphi_3^1(x, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_2^1(x, s)}{\partial x^2} ds,$$

$$\varphi_3^1(x, t) = \varepsilon \int_0^t (-1)^3 \varepsilon^2 \left(\frac{\pi}{L}\right)^6 \sin\left(\frac{\pi x}{L}\right) \cdot \frac{s^2}{2} ds$$

$$= (-1)^3 \varepsilon^3 \left(\frac{\pi}{L}\right)^6 \sin\left(\frac{\pi x}{L}\right) \cdot \frac{t^3}{3!}.$$

So step by step, we arrive at

$$\varphi_n^1(x, t) = (-1)^n \varepsilon^n \left(\frac{\pi}{L}\right)^{2n} \frac{t^n}{n!} \cdot \sin\left(\frac{\pi x}{L}\right)$$

$$= (-1)^n \varepsilon^n \left(\left(\frac{\pi}{L}\right)^2\right)^n \frac{t^n}{n!} \cdot \sin\left(\frac{\pi x}{L}\right)$$

$$= \frac{\left(-\varepsilon\left(\frac{\pi}{L}\right)^2 t\right)^n}{n!} \cdot \sin\left(\frac{\pi x}{L}\right).$$

Therefore, the approximate solution is

$$\varphi^1(x, t) = \sum_{n \geq 0} \varphi_n^1(x, t)$$

$$= \sum_{n \geq 0} \frac{\left(-\varepsilon\left(\frac{\pi}{L}\right)^2 t\right)^n}{n!}$$

$$= \sin\left(\frac{\pi x}{L}\right) \sum_{n \geq 0} \frac{\left(-\varepsilon\left(\frac{\pi}{L}\right)^2 t\right)^n}{n!}$$

$$= \sin\left(\frac{\pi x}{L}\right) e^{-\varepsilon\left(\frac{\pi}{L}\right)^2 t},$$

$$\varphi(x, t) = e^{-\int \frac{u(x,t)}{2\varepsilon} dx},$$

$$\ln \varphi(x, t) = -\int \frac{u(x, t)}{2\varepsilon} dx,$$

$$\ln \sin\left(\frac{\pi x}{L}\right) e^{-\varepsilon\left(\frac{\pi}{L}\right)^2 t} = -\int \frac{u(x, t)}{2\varepsilon} dx,$$

$$\int \frac{\pi}{L} \operatorname{cotan}\left(\frac{\pi x}{L}\right) e^{-\varepsilon\left(\frac{\pi}{L}\right)^2 t} dx = -\int \frac{u(x, t)}{2\varepsilon} dx,$$

$$\frac{\pi}{L} \operatorname{cotan}\left(\frac{\pi x}{L}\right) e^{-\varepsilon\left(\frac{\pi}{L}\right)^2 t} = -\frac{u(x, t)}{2\varepsilon},$$

$$-\frac{2\varepsilon\pi}{L} \operatorname{cotan}\left(\frac{\pi x}{L}\right) e^{-\varepsilon\left(\frac{\pi}{L}\right)^2 t} = u(x, t),$$

$$u(x, t) = -\frac{2\varepsilon\pi}{L} \operatorname{cotan}\left(\frac{\pi x}{L}\right) e^{-\varepsilon\left(\frac{\pi}{L}\right)^2 t}.$$

3.2. The case when $\varphi \neq 0$

$$\begin{cases} \varphi_t - \varepsilon\varphi_{xx} = \varphi, \ x \in \mathbb{R}, \ t > 0, \ \varepsilon > 0, \\ \varphi(x, 0) = \varphi_0(x) = \sin\left(\frac{\pi x}{L}\right), \ x \in \mathbb{R}, \end{cases}$$

$$\int_0^t \varphi_s(x, s) ds = \varepsilon \int_0^t \varphi_{xx}(x, s) ds + \int_0^t \varphi(x, s) ds,$$

$$\varphi(x, t) = \varphi(x, 0) + \varepsilon \int_0^t \frac{\partial^2 \varphi(x, s)}{\partial x^2} ds + \int_0^t \varphi(x, s) ds.$$

Applying the SBA algorithm, we obtain

$$\begin{cases} \varphi_0^k(x,\, t) = \varphi^k(x,\, 0),\ \forall k \geq 1, \\ \varphi_n^k(x,\, t) = \varepsilon \int_0^t \dfrac{\partial^2 \varphi_{n-1}^k(x,\, s)}{\partial x^2} ds + \int_0^t \varphi_{n-1}^k(x,\, s) ds,\ \forall n \geq 1, \end{cases}$$

$$\begin{cases} \varphi_0^k(x,\, t) = \sin\left(\dfrac{\pi x}{L}\right),\ \forall k \geq 1, \\ \varphi_n^k(x,\, t) = \varepsilon \int_0^t \dfrac{\partial^2 \varphi_{n-1}^k(x,\, s)}{\partial x^2} ds + \int_0^t \varphi_{n-1}^k(x,\, s) ds,\ \forall n \geq 1. \end{cases}$$

For $k = 1,$ we have $\varphi_0^1(x,\, t) = \sin\left(\frac{\pi x}{L}\right).$

For $n = 1,$ we have

$$\varphi_1^1(x,\, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_0^1(x,\, s)}{\partial x^2} ds + \int_0^t \varphi^0(x,\, s) ds$$

$$= -\varepsilon \left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) \cdot t + \sin\left(\frac{\pi x}{L}\right) \cdot t.$$

For $n = 2,$ we have

$$\varphi_2^1(x,\, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_1^1(x,\, s)}{\partial x^2} ds + \int_0^t \varphi_1^0(x,\, s) ds$$

$$= \left[1 + (-1)^1 \varepsilon \left(\frac{\pi}{L}\right)^2\right] \left[\sin\left(\frac{\pi x}{L}\right) + (-1)^1 \varepsilon \left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right)\right] \cdot \frac{t^2}{2!}.$$

For $n = 3,$ we have

$$\varphi_3^1(x,\, t) = \varepsilon \int_0^t \frac{\partial^2 \varphi_2^1(x,\, s)}{\partial x^2} ds + \int_0^t \varphi_2^1(x,\, s) ds$$

$$= \left[1 + (-1)^1 \varepsilon \left(\frac{\pi}{L}\right)^2\right]^2 \left[\sin\left(\frac{\pi x}{L}\right) + (-1)^1 \varepsilon \left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right)\right] \cdot \frac{t^3}{3!}.$$

For $n = 4$, we have

$$\varphi_4^1(x, t) = \varepsilon\int_0^t \frac{\partial^2\varphi_3^1(x, s)}{\partial x^2} ds + \int_0^t \varphi_3^1(x, s) ds$$

$$= \left[1 + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]^3 \left[\sin\left(\frac{\pi x}{L}\right) + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right)\right] \cdot \frac{t^4}{4!}.$$

So step by step, we arrive at

$$\varphi_n^1(x, t) = \left[1 + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]^{n-1} \left[\sin\left(\frac{\pi x}{L}\right) + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right)\right] \cdot \frac{t^n}{n!}.$$

The solution $\varphi^1(x, t)$ is therefore written as

$$\varphi^1(x, t) = \sum_{n\geq 1}\varphi_n^1(x, t) + \varphi_0^1(x, t)$$

$$= \sin\left(\frac{\pi x}{L}\right) + \frac{\left[\sin\left(\frac{\pi x}{L}\right) + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right)\right]}{\left[1 + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]}\left(e^{\left[1+(-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]t} - 1\right),$$

$$\varphi_x = \frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) + \frac{\frac{\pi}{L}\left[\cos\left(\frac{\pi x}{L}\right) + (-1)^2\varepsilon\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right)\right]}{\left[1 + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]}\left(e^{\left[1+(-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]t} - 1\right),$$

$$u = -2\varepsilon\frac{\varphi_x}{\varphi}$$

$$= \frac{\dfrac{-2\varepsilon\pi}{L}\cos\left(\dfrac{\pi x}{L}\right) - \dfrac{2\varepsilon\pi}{L}\dfrac{\left[\cos\left(\frac{\pi x}{L}\right) + (-1)^2\varepsilon\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right)\right]}{\left[1 + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]}\left(e^{\left[1+(-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]t} - 1\right)}{\sin\left(\dfrac{\pi x}{L}\right) + \dfrac{\left[\sin\left(\frac{\pi x}{L}\right) + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right)\right]}{\left[1 + (-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]}\left(e^{\left[1+(-1)^1\varepsilon\left(\frac{\pi}{L}\right)^2\right]t} - 1\right)}.$$

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