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ON 3D COMPRESSIBLE PRIMITIVE EQUATIONS APPROXIMATION OF ANISOTROPIC NAVIER-STOKES EQUATIONS: RIGOROUS JUSTIFICATION

Jules Ouya and Arouna Ouedraogo

Laboratoire de Mathématiques, Informatique et Applications (L@MIA)
Université Norbert ZONGO
BP 376 Koudougou
Burkina Faso
e-mail: ouyajules6@gmail.com
arounaoued2002@yahoo.fr

Abstract

In this paper, we obtain the 3D compressible primitive equations approximation without gravity by taking the small aspect ratio limit to the Navier-Stokes equations in the isothermal case with gravity. The aspect ratio (the ratio of the depth to horizontal width) is a geometrical constraint in general large scale geophysical motions that the vertical scale is significantly smaller than horizontal. We use the versatile relative entropy inequality to prove rigorously the limit from the compressible Navier-Stokes equations to the compressible primitive

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equations. In addition to the presence of gravity, we consider that the viscosity of the fluid depends on its density and that it is submitted to a quadratic friction force.

1. Introduction

Atmospheric flow in meteorology, water flow in oceanography, and limnology are all described by the Navier-Stokes equations. Due to the fact that the aspect ratio

$$\varepsilon = \frac{\text{characteristic depth}}{\text{characteristic width}}$$

is very small in most geophysical domains, asymptotic models have been used see, e.g., [10, 18, 27]. One such model is the primitive equations model; see, e.g., [13, 14], wherein the unknown flow variables are velocity, pressure, temperature, and salinity (in the case of an ocean). Besides, most geophysical fluids are stratified (i.e., density is a known function of the temperature (and salinity, if any)) and have a free surface. In this paper, we shall focus on the assumption that the pressure is hydrostatic, i.e., increases linearly with respect to the depth, as in the static case. This law agrees well with experiment and is frequently taken as a hypothesis in geophysical fluid dynamics. Therefore, many scientists suggest that the viscosity coefficients must be anisotropic.

We consider the following compressible anisotropic Navier-Stokes problem:

$$\begin{cases} \partial_t \rho + div(\rho U) = 0, \\ \partial_t(\rho U) + div(\rho U \otimes U) + \nabla p(\rho) - 2div_x(\mu_1 D_x(U)) - \partial_z(\mu_2 \partial_z U) = \rho f, \\ p(\rho) = c^2 \rho, \end{cases} \tag{1.1}$$

in the thin domain $(0, T) \times \Omega_\varepsilon$. Here $t > 0$, $x = (x_1, x_2)$ and z are time, horizontal and vertical variables, respectively,

$$\Omega_\varepsilon = \{(x, z);\ x \in \mathbb{T}^2,\ 0 < z < \varepsilon\}$$

with $\mathbb{T}^2$ a bi-dimensional torus. The unknown functions ρ, $U = (u, v)$ and p represent the density, velocities and the pressure of the medium, respectively. $divU = div_x u + \partial_z v$ (with $div_x = \partial_{x_1} + \partial_{x_2}$) and $\nabla = (\nabla_x, \partial_z)$ are the three-dimensional spatial divergence operator and gradient, respectively. $D_x(u)$ is the strain tensor with $D_x(u) = \dfrac{\nabla_x u + \nabla_x u^T}{2}$ along the horizontal directions. (μ_1, μ_2) are the turbulence viscosities in the horizontal and vertical directions, respectively, which depend on the variables t, x, z and the density ρ generally. The term f is the quadratic friction source term and the gravity strength is given as follows:

$$f = -ku|u| - g\kappa,$$

where k is a positive constant coefficient, g is the gravitational constant and $\kappa = (0, 0, 1)^T$ (where X^T stands for the transpose of tensor X). The pressure $p(\rho) = c^2\rho$ is a usual expression used in the isothermal case with c^2 a specific constant [12, 33].

We assume the density $\rho = \rho(t, x)$, that is, ρ is independent of z.

As atmosphere and ocean are the thin layers, where the fluid layer depth is small compared to radius of sphere, Pedlosky [36] pointed out that “the pressure difference between any two points on the same vertical line depends only on the weight of the fluid between these points”.

We suppose

$$\mu_1 = \mu(\rho) \quad \text{and} \quad \mu_2 = \nu(\rho)\varepsilon^2.$$

As stressed by Azérad and Guillén [2], it is necessary to consider the above anisotropic viscosities scaling, which is fundamental for the derivation of Primitive Equations (PE). Under this assumption, the system is rewritten as follow:

$$\begin{cases} \partial_t\rho + div_x(\rho u) + \partial_z(\rho v) = 0, \\ \partial_t(\rho u) + div_x(\rho u \otimes u) + \partial_z(\rho uv) + \nabla_x p(\rho) + k\rho u|u| \\ = 2div_x(\mu(\rho)D_x(u)) + \varepsilon^2\partial_z(\nu(\rho)\partial_z u), \\ \partial_t(\rho v) + div_x(\rho uv) + \partial_z(\rho v^2) + \partial_z p(\rho) \\ - 2div_x(\mu(\rho)D_x(v)) - \varepsilon^2\partial_z(\nu(\rho)\partial_z v) = -g\rho, \\ p(\rho) = c^2\rho. \end{cases} \tag{1.2}$$

We perform a vertical scaling to make the domain independent of ε, that is, $t = t'$, $x = x'$ and $z = \varepsilon z'$. The new fixed domain (without the «'» symbol) is $\Omega = \{(x, z); x \in \mathbb{T}^2, 0 < z < 1\}$.

As in [26], the corresponding kinematic scaling is

$$U_\varepsilon = (u_\varepsilon, v_\varepsilon), \quad u_\varepsilon(t, x, z) = u(t, x, \varepsilon z),$$

$$v_\varepsilon(t, x, z) = \frac{1}{\varepsilon}v(t, x, \varepsilon z), \rho_\varepsilon(t, x) = \rho(t, x),$$

for any $(x, z) \in \Omega := \mathbb{T}^2 \times (0, 1)$. Then, the system (1.2) becomes the following compressible scaled Navier-Stokes equations (CNS):

$$\begin{cases} \partial_t\rho_\varepsilon + div_x(\rho_\varepsilon u_\varepsilon) + \partial_z(\rho_\varepsilon v_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + div_x(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \partial_z(\rho_\varepsilon u_\varepsilon v_\varepsilon) + \nabla_x p(\rho_\varepsilon) + k\rho_\varepsilon u_\varepsilon|u_\varepsilon| \\ = 2div_x(\mu(\rho_\varepsilon)D_x(u_\varepsilon)) + \partial_z(\nu(\rho_\varepsilon)\partial_z u_\varepsilon), \\ \varepsilon^2(\partial_t(\rho_\varepsilon v_\varepsilon) + div_x(\rho_\varepsilon u_\varepsilon v_\varepsilon) + \partial_z(\rho_\varepsilon v_\varepsilon^2) - 2div_x(\mu(\rho_\varepsilon)D_x(v_\varepsilon)) \\ - \partial_z(\nu(\rho_\varepsilon)\partial_z v_\varepsilon)) + \partial_z p(\rho_\varepsilon) = -\varepsilon g\rho_\varepsilon, \\ p(\rho_\varepsilon) = c^2\rho_\varepsilon. \end{cases} \tag{1.3}$$

We make the following boundary conditions:

u_ε and ρ_ε are periodic in the directions x_1, x_2, respectively,

$$v_\varepsilon|_{z=0} = v_\varepsilon|_{z=1} = 0,$$

$$\partial_z u_\varepsilon|_{z=0} = \partial_z u_\varepsilon|_{z=1} = 0, \quad u_\varepsilon|_{z=0} = u_\varepsilon|_{z=1} = 0 \tag{1.4}$$

and the initial conditions:

$$\rho_\varepsilon(0, x)U_\varepsilon(0, x, z) = m_0(x, z), \quad \rho_\varepsilon(0, x) = \rho_0(x). \tag{1.5}$$

In this work, our goal is to prove that as $\varepsilon \to 0$, the system (1.3) converges in a certain sense to the following compressible primitive equations (CPEs):

$$\begin{cases} \partial_t \rho + div_x(\rho u) + \partial_z(\rho v) = 0, \\ \partial_t(\rho u) + div_x(\rho u \otimes u) + \partial_z(\rho u v) + \nabla_x p(\rho) + k\rho u| u | \\ = 2div_x(\mu(\rho) D_x(u)) + \partial_z(\nu(\rho)\partial_z u), \\ \partial_z p(\rho) = 0, \\ p(\rho) = c^2\rho. \end{cases} \tag{1.6}$$

Geophysical fluid dynamics is a crucial field for understanding the behavior of the atmosphere and the ocean. However, when it comes to analyzing and simulating the complex flows in these systems, using the complete hydrodynamical and thermodynamical equations is mathematically and numerically challenging. To overcome this, scientists have introduced the Primitive Equation (PE) model in geophysical fluid dynamics. The PE model was initially derived by Richardson in the 1920s for weather prediction. However, due to stability issues in calculations, it did not achieve much success. It was Bryan in [7] who improved the PE model in 1969 by incorporating the hydrostatic approximation. Although the PE model showed promising results in early simulations and applications, mathematical research on the PE model started much later. In the 1990s, Lions et al. [27, 28] were the first to study the PE model and made significant contributions to this field. Since then, the PE model has progressed through the precise analysis of simpler models. There is a vast amount of literature dedicated to the PE model, with numerous studies and references exploring its various aspects. Some notable research topics include the works by Bresch et al. [4-6], Cao et al. [8-10], Guo et al. [22, 23], Ju [25], Lions et al. [29, 30], Temam and Ziane [38] and Wang and Yang [40], among others. These studies have contributed to the understanding and development of the PE model in geophysical fluid dynamics.

The research in geophysical fluid dynamics has traditionally focused on the incompressible case of the Primitive Equation (PE) model, mainly due to historical reasons. However, it is well known that the atmosphere and ocean exhibit compressible properties. Therefore, it is natural and interesting to consider the compressible version of the PE model, known as the CPE model. In recent years, several researchers have made significant contributions to the study of the CPE model. Gatapov and Kazhikhov [20], as well as Ersoy and Ngom [13], investigated the CPE model with constant viscosity coefficients and proved the global existence of weak solutions in the two-dimensional case. Liu and Titi [31, 33] extended the analysis to the three-dimensional case and established the local existence of strong solutions for the CPE model. They considered the zero Mach number limit of the CPE model. This limit corresponds to the situation where the fluid flow becomes nearly incompressible. Ersoy et al. [12] introduced the concept of dimensionless numbers and employed asymptotic analysis to study the CPE model with viscosity coefficients that depend on the density. They obtained interesting results in this setting. The stability of weak solutions in the CPE model has also been investigated. Ersoy et al. [12] and Tang and Gao [37] demonstrated the stability of weak solutions, which means that under certain uniform bounds, a subsequence of weak solutions will converge to another weak solution. In recent developments, Liu and Titi [32] and independently Wang et al. [39], utilized the B-D entropy to prove the global existence of weak solutions for the CPE model. This entropy-based approach has provided valuable insights into the behavior of the solutions. Overall, the research on the compressible version of the PE model, the CPE model, has made significant progress in recent years, with studies focusing on global existence, stability, and entropy-based analysis of weak solutions.

According to the studies by Azérad and Guillén [2] and Li and Titi [26], the hydrostatic approximation is a significant aspect of the PE model, as emphasized by these authors. Establishing the rigorous justification for the transition from the anisotropic Navier-Stokes equations to the hydrostatic approximation through the small aspect limit is evidently of great practical importance. Numerous studies have been conducted on the convergence of incompressible flows. For instance, Azérad and Guillén [2] demonstrated the convergence of weak solutions from the anisotropic Navier-Stokes equations to weak solutions of the PE model. Li and Titi [26] employed the method of weak-strong uniqueness to prove the aspect ratio limit of the incompressible anisotropic Navier-Stokes equations, showing the convergence from weak solutions to strong solutions of the incompressible PE model.

Our main objective is to provide a rigorous justification for the limit passage in the context of weak solutions of the compressible Navier-Stokes equations (CNS). Recent studies by Bella et al. [3] and Maltese and Novotný [34] have proven the limit passage from the 3D compressible Navier-Stokes equations to the 1D and 2D compressible Navier-Stokes equations in thin domains. Drawing inspiration from their work, we have developed and adapted the idea of the relative entropy inequality for the compressible Navier-Stokes equations. However, there are significant mathematical differences between the Navier-Stokes equations and the CPE model. The hydrostatic approximation in the CPE model eliminates information about the vertical velocity in the momentum equation, and the vertical velocity is determined by the horizontal velocity through the continuity equation. As a result, analyzing the CPE model becomes considerably challenging. Consequently, the classical methods used in the Navier-Stokes system cannot be straightforwardly applied to the CPE model.

Additionally, in [1], Andrášik et al. have proven the existence of weak-strong solutions for the problem (1.6) with general external forces f. However, in our work, we focus on the hydrostatic case and specifically consider the case where $f = -ru|u| - g\kappa$. Our study is among the first to use the relative entropy inequality to establish the hydrostatic approximation in the compressible case. A similar approach was taken in [19], where the pressure is assumed to be of the form ρ^{γ} with $\gamma > 4$, viscosities are constants, and there are no external forces. The introduction of the versatile relative entropy inequality can be found in [19]. It is important to mention that the cornerstone of our analysis is based on the relative energy inequality, which was originally introduced by Dafermos [11]. Subsequently, Germain [21] applied it to the compressible Navier-Stokes equations. Feireisl and his co-authors [16, 17] further generalized the relative energy inequality to solve various problems related to compressible fluid models.

The rest of paper is organized as follows. In Section 2, we recall some useful inequalities. We introduce the definition of weak solutions, strong solution, relative energy and state the main theorem in Section 3. Section 4 is devoted to proof of the convergence.

2. Preliminaries

In this section, we introduce some basic inequalities needed in the later proof. The first inequality is the so-called generalized Poincaré inequality.

Lemma 2.1 (See [15])**.** *Let* $2 \leq p \leq 6$ *and* $\rho \geq 0$ *such that*

$$0 < \int_{\Omega} \rho dx = M \leq \infty \quad and \quad \int_{\Omega} \rho^{\gamma} dx \leq E_0$$

for some $(\gamma > 1)$. *Then*

$$\| f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{L^p(\Omega)} + \| \rho^{\frac{1}{2}} f \|_{L^2(\Omega)},$$

where C depends on M and E_0.

The following is the famous Gagliardo-Nirenberg inequality (for the proof, see [35]).

Lemma 2.2. *For a function* $u : \Omega \to \mathbb{R}$ *defined on a bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$, $\forall 1 \leq q, r \leq \infty$, *and a natural number m*, *suppose that a real number* θ *and a natural number j are such that*

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\theta + \frac{1-\theta}{q} \quad and \quad \frac{j}{m} \leq \theta \leq 1.$$

Then there exists a constant C independent of u such that

$$\| D^j u \|_p \le C \| D^m u \|^{\theta}_{L^r(\Omega)} \| u \|^{1-\theta}_{L^q(\Omega)}.$$

3. Main Results

Before showing our main result, we give the definition of a weak solution for CNS and a strong solution for CPE.

3.1. Dissipative weak solutions of CNS

Providing the definition of weak solutions to (1.3), we give the energy inequality.

Definition 3.1. We say that $(\rho_\varepsilon, u_\varepsilon, v_\varepsilon)$ is a *finite weak energy solution* to the system of (1.3), with boundary (1.4) and initial conditions (1.5) if

$$\begin{cases} \rho_\varepsilon \in L^\infty(0, T; L^1(\Omega)), & \sqrt{\rho_\varepsilon} \in L^\infty(0, T; H^1(\Omega)), \\ \sqrt{\rho_\varepsilon} u_\varepsilon \in L^2(0, T; L^2(\Omega)^2), & \sqrt{\rho_\varepsilon} v_\varepsilon \in L^\infty(0, T; L^2(\Omega)), \\ \sqrt{\rho_\varepsilon} D_x u_\varepsilon \in L^2(0, T; (L^2(\Omega))^{2\times 2}), & \sqrt{\rho_\varepsilon} \nabla v_\varepsilon \in L^2(0, T; L^2(\Omega)^3), \\ \nabla_x \sqrt{\rho_\varepsilon} \in L^2(0, T; L^2(\Omega)^2), & \rho_\varepsilon^{\frac{1}{3}} u \in L^3([0, T]; L^3(\Omega)); \end{cases} \tag{3.1}$$

• the continuity equation

$$\left[\int_\Omega \rho_\varepsilon \varphi dxdz \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon u_\varepsilon \cdot \nabla_x \varphi + \rho_\varepsilon v_\varepsilon \partial_z \varphi) dxdzdt \tag{3.2}$$

holds for all $\varphi \in C_c^\infty([0, T) \times \Omega)$;

• the momentum equations

$$\left[\int_\Omega \rho_\varepsilon u_\varepsilon \phi_H \, dxdz \right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \rho_\varepsilon u_\varepsilon \partial_t \phi_H \, dxdzdt - \int_0^\tau \int_\Omega \rho_\varepsilon u_\varepsilon v_\varepsilon \partial_z \phi_H \, dxdzdt$$

$$+ \int_0^\tau \int_\Omega (2\mu(\rho_\varepsilon) D_x(u_\varepsilon) - \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) : \nabla_x \phi_H \, dxdzdt$$

$$+\int_0^\tau\int_\Omega \nu(\rho_\varepsilon)\partial_z u_\varepsilon \partial_z \phi_H\, dxdzdt - \int_0^\tau\int_\Omega p(\rho_\varepsilon) div_x \phi_H\, dxdzdt$$

$$+\int_0^\tau\int_\Omega k\rho_\varepsilon u_\varepsilon |\, u_\varepsilon\, |\phi_H\, dxdzdt = 0, \tag{3.3}$$

and

$$\varepsilon^2\Big[\int_\Omega \rho_\varepsilon v_\varepsilon \phi_3 dxdz\Big]_{t=0}^{t=\tau} - \varepsilon^2\int_0^\tau\int_\Omega \rho_\varepsilon v_\varepsilon \partial_t \phi_3\, dxdzdt$$

$$-\varepsilon^2\int_0^\tau\int_\Omega \rho_\varepsilon u_\varepsilon v_\varepsilon \cdot \nabla_x \phi_3\, dxdzdt$$

$$+\varepsilon^2\int_0^\tau\int_\Omega 2\mu(\rho_\varepsilon) D_x(v_\varepsilon) : \nabla_x \phi_3\, dxdzdt$$

$$+\varepsilon^2\int_0^\tau\int_\Omega (\nu(\rho_\varepsilon)\partial_z v_\varepsilon - \rho_\varepsilon v_\varepsilon^2)\partial_z \phi_3\, dxdzdt$$

$$-\int_0^\tau\int_\Omega p(\rho_\varepsilon)\partial_z \phi_3\, dxdzdt$$

$$= -\varepsilon\int_0^\tau\int_\Omega g\rho_\varepsilon \phi_3\, dxdzdt \tag{3.4}$$

hold for all $\phi_H,\ \phi_3 \in C_c^\infty([0, T)\times\Omega)$ and for a.e $\tau \in (0, T)$.

Combining (3.3) and (3.4), we obtain

$$\Big[\int_\Omega \rho_\varepsilon(u_\varepsilon \phi_H + \varepsilon^2 v_\varepsilon \phi_3) dxdz\Big]_{t=0}^{t=\tau}$$

$$-\int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon \partial_t \phi_H + \varepsilon^2 v_\varepsilon \partial_t \phi_3) dxdzdt$$

$$-\int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon v_\varepsilon \partial_z \phi_H + \varepsilon^2 v_\varepsilon^2 \partial_z \phi_3) dxdzdt$$

$$+\int_0^\tau\int_\Omega 2\mu(\rho_\varepsilon)(D_x(u_\varepsilon) : \nabla_x \phi_H + \varepsilon^2 D_x(v_\varepsilon) : \nabla_x \phi_3) dxdzdt$$

$$-\int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon \otimes u_\varepsilon : \nabla_x\phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x\phi_3\, dxdzdt)$$

$$+\int_0^\tau\int_\Omega \nu(\rho_\varepsilon)(\partial_z u_\varepsilon \partial_z \phi_H + \varepsilon^2 \partial_z v_\varepsilon \partial_z \phi_3)dxdzdt$$

$$-\int_0^\tau\int_\Omega p(\rho_\varepsilon)(div_x\phi_H + \partial_z\phi_3)dxdzdt$$

$$= -\int_0^\tau\int_\Omega k\rho_\varepsilon u_\varepsilon |u_\varepsilon|\phi_H\, dxdzdt - \varepsilon\int_0^\tau\int_\Omega g\rho_\varepsilon\phi_3\, dxdzdt. \tag{3.5}$$

In the following, we take $\mu(\rho) = \overline{\mu}\rho_\varepsilon$ and $\nu(\rho) = \overline{\nu}\rho_\varepsilon$, where $\overline{\mu}, \overline{\nu} > 0$.

Formally, multiplying the momentum equation $(1.3)_2$ by horizontal velocity u_ε, then by integrating by parts on Ω, we can deduce the following energy inequality:

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho_\varepsilon|u_\varepsilon|^2 + \rho_\varepsilon \ln\rho_\varepsilon - \rho_\varepsilon\right)dxdz + k\int_\Omega \rho_\varepsilon|u_\varepsilon|^3 dxdz$$

$$+\int_\Omega \rho_\varepsilon(2\overline{\mu}|D_x(u_\varepsilon)|^2 + \overline{\nu}|\partial_z u_\varepsilon|^2)dxdz \le 0. \tag{3.6}$$

In the same way, multiplying the momentum equation $(1.3)_3$ by vertical velocity v_ε, then by integrating by parts on Ω, we can deduce the following energy inequality:

$$\varepsilon^2\frac{d}{dt}\int_\Omega \rho_\varepsilon\frac{v_\varepsilon^2}{2}dxdz + \varepsilon^2\int_\Omega \rho_\varepsilon(2\overline{\mu}|D_x(v_\varepsilon)|^2 + \overline{\nu}|\partial_y v_\varepsilon|^2)dxdz$$

$$\le -\varepsilon g\int_\Omega v_\varepsilon\rho_\varepsilon dxdz \le g\int_\Omega \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}|v_\varepsilon|dxdz$$

$$\le \frac{\varepsilon}{2}g\left(\int_\Omega \rho_\varepsilon dxdz + \int_\Omega \rho_\varepsilon v_\varepsilon^2 dxdz\right), \tag{3.7}$$

where we used Cauchy's inequality in second to third line.

Combining (3.6) and (3.7), we obtain the energy inequality

$$\frac{d}{dt}\int_{\Omega}\left[\frac{1}{2}\rho_\varepsilon(|u_\varepsilon|^2+\varepsilon^2 v_\varepsilon^2)+\rho_\varepsilon\ln\rho_\varepsilon-\rho_\varepsilon\right]dxdz$$

$$+\int_{\Omega}\rho_\varepsilon(2\overline{\mu}|D_x(u_\varepsilon)|^2+\overline{\nu}|\partial_z u_\varepsilon|^2)dxdz$$

$$+\varepsilon^2\int_{\Omega}\rho_\varepsilon(2\overline{\mu}|D_x(v_\varepsilon)|^2+\overline{\nu}|\partial_z v_\varepsilon|^2)dxdz$$

$$\leq -k\int_{\Omega}\rho_\varepsilon|u_\varepsilon|^3 dxdz - g\varepsilon\int_{\Omega}\rho_\varepsilon v_\varepsilon dxdz$$

$$\leq \frac{g\varepsilon}{2}\left(\int_{\Omega}\rho_\varepsilon dxdz+\int_{\Omega}\rho_\varepsilon v_\varepsilon^2 dxdz\right)-k\int_{\Omega}\rho_\varepsilon|u_\varepsilon|^3 dxdz. \tag{3.8}$$

We assume that the initial data satisfies

$$\begin{cases} 0<\rho_0(x)\leq M<+\infty, & \rho_0\in L^1(\Omega), \\ \rho_0\ln\rho_0-\rho_0\in L^1(\Omega), & \nabla_x\sqrt{\rho_0}\in L^2(\Omega), \\ \dfrac{|m_0|^2}{\rho_0}\in L^1(\Omega). \end{cases} \tag{3.9}$$

3.2. Strong solution of CPE

The couple (r, ϕ) (where $\phi=(\phi_H, \phi_3)$) is a strong solution to the CPE system (1.6) in $\Omega\times(0, T)$, if it satisfies the equations in (1.6) with the boundary condition (1.4). Also, the solution satisfies following regularities:

$\sqrt{r}\in L^\infty(0, T; H^2(\Omega))$, $\partial_t\sqrt{r}\in L^\infty(0, T; H^1(\Omega))$, $r>0$ for all (t, x, z),

$\phi_H\in L^\infty(0, T; H^3(\Omega))\cap L^2(0, T; H^4(\Omega))$,

$\partial_t\phi_H\in L^2(0, T; H^2(\Omega))\cap L^\infty(0, T; H^1(\Omega))$,

with initial data $\sqrt{r_0}\in H^2(\Omega)$, $r_0>0$ and $\phi_0\in H^3(\Omega)$.

As the density is independent of z, we can obtain the following information of vertical velocity for the weak solution of CNS:

$$\rho v(t, x, z) = -div_x(\rho\tilde{u}) + z div_x(\rho\bar{u}), \text{ in the sense of } H^{-1}(\Omega), \tag{3.10}$$

where

$$\tilde{u}(t, x, z) = \int_0^z u(t, x, s)ds, \quad \bar{u}(t, x) = \int_0^1 u(t, x, z)dz.$$

3.3. Relative entropy inequality

Motivated by [16, 17], for any finite energy weak solution $(\mathbf{u}, \rho)$, where $\mathbf{u} = (u, v)$, to the CNS system (1.3), we introduce the relative energy functional

$$\begin{aligned}
&\xi([\rho, U] | [r, \phi]) \\
&= \int_\Omega \left(\frac{1}{2}\rho| u - \phi_H |^2 + \frac{\varepsilon^2}{2}\rho| v - \phi_3 |^2 + \rho\ln\rho - \rho - \rho\ln r + r \right) dxdz \\
&= \int_\Omega \left(\frac{1}{2}\rho| u |^2 + \frac{\varepsilon^2}{2}\rho| v |^2 + \rho\ln\rho - \rho \right) dxdz - \int_\Omega (\rho u\phi_H + \varepsilon^2\rho v\phi_3) dxdz \\
&\quad + \int_\Omega \left(\frac{1}{2}\rho| \phi_H |^2 + \frac{\varepsilon^2}{2}\rho| \phi_3 |^2 - \rho\ln r \right) dxdz + \int_\Omega \rho(r) dxdz,
\end{aligned} \tag{3.11}$$

where $(r, \phi) = (r, \phi_H, \phi_3)$ designs the local strong solution of the CPE (1.6) showed recently in [1], and r is a strictly positive function defined on $\Omega \times (0, T)$.

Lemma 3.1 (See [1, 19]). *A general function* $G = G(\rho)$ *can be decomposed into the essential and residual parts as*

$$G = G_{ess} + G_{res},$$

where

$$G_{ess} := \begin{cases} G & on\ \rho \in \left(\frac{1}{2}\underline{r}, 2\bar{r}\right), \\ 0 & otherwise. \end{cases} \tag{3.12}$$

Let us define

$$H(\rho) = \rho \int_1^{\rho} \frac{p(y)}{y} dy = \rho \ln \rho.$$

We would like to point out that the function

$$\rho \mapsto H(\rho) - H(r) - (\rho - r)H'(r) = \rho \ln \rho - \rho - \rho \ln r + r, \quad \rho,\ r > 0 \quad (3.13)$$

is strictly convex with minimum 0 attained at $\rho = r$. Therefore, for every $0 < \underline{r} < r < \overline{r} < \infty$, there is a positive constant c such that

$$H(\rho) - H(r) - (\rho - r)H'(r) > c(\rho - r)^2, \quad (3.14)$$

wherever $\rho \in \left(\frac{1}{2}\underline{r},\ 2\overline{r}\right)$ and

$$H(\rho) - H(r) - (\rho - r)H'(r) > c|\rho - r|, \quad (3.15)$$

wherever $\rho \in \mathbb{R}^+ \backslash \left(\frac{1}{2}\underline{r},\ 2\overline{r}\right)$.

Let us specify that $\underline{r} = \inf_{(0,T)\times\Omega} r$ and $\overline{r} = \sup_{(0,T)\times\Omega} r$.

Due to the convexity of H and (3.12), we can deduce the following coercivity properties (see (3.14) and (3.15)):

$$\xi([\rho_\varepsilon, U_\varepsilon][r, \phi])$$

$$\geq C\int_\Omega (\rho|u - \phi_H|^2 + \varepsilon^2\rho|v - \phi_3|^2 + |\rho - r|_{ess}^2 + 1_{res} + \rho_{res})dxdz. \quad (3.16)$$

Thus, we deduce to (3.16) that

$$\begin{cases} \int_\Omega |\rho - r|_{ess}^2 dxdz = \int_\Omega \chi_{\left\{\frac{r}{2}<\rho<2\bar{r}\right\}} (\rho - r)^2 dxdz \\ \leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t), \\ \int_\Omega 1_{res} dxdz \leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t), \\ \int_\Omega \rho_{res} dxdz \leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t), \\ \int_\Omega \rho(|u - \phi_H|^2 + \varepsilon^2 |v - \phi_3|^2) dxdz \\ \leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t). \end{cases} \tag{3.17}$$

Moreover, from [16], we have

$$\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t) \in L^\infty(0, T),$$

$$\| \rho \|_{L^\alpha(\{\rho \geq 2\bar{r}\})} \leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])^{1/\alpha},$$

$$\| \rho^{\alpha/2} \|_{L^2(\{\rho \geq 2\bar{r}\})} \leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])^{1/2}, \ \alpha > 1. \tag{3.18}$$

The symbol C denotes a generic positive constant, which may vary from time to time.

Our main result is the following theorem:

Theorem 3.1. *Let* $T_{\max} > 0$ *be the life time of strong solution to CPE system* (1.6) *corresponding to initial data* $[r_0, \phi_0]$. *Let* $(\rho_\varepsilon, u_\varepsilon, v_\varepsilon)$ *be a sequence of dissipative weak solutions to the CNS system* (1.3) *from the initial data* (ρ_0, u_0, v_0) *depending on* ε *which satisfies* (3.9). *Suppose that*

$$\xi([\rho_0, U_0] | [r_0, \phi_0]) \to 0.$$

Then

$$ess \sup_{t \in (0, T_{\max})} \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]) \to 0, \tag{3.19}$$

when $\varepsilon \to 0$, *and where the couple* (r, ϕ) *satisfies the CPE system* (1.6) *on the time interval* $[0, T_{\max})$.

Remark 3.1. In order to see more clearly the sense of the limit above, we notice that (3.19) implies, for example,

$$\rho_\varepsilon \to r \text{ strongly in } L^\infty(0, T; L^1(\Omega)),$$

$$\sqrt{\rho_\varepsilon}\, U_\varepsilon \to \sqrt{r}\phi \text{ strongly in } L^\infty(0, T; L^2(\Omega)),$$

$$\rho_\varepsilon U_\varepsilon \to r\phi \text{ strongly in } L^\infty(0, T; L^2(\Omega)).$$

To establish the relative entropy inequality, we first take $\varphi = \frac{1}{2}|\phi_H|^2$, then $\varphi = \frac{\varepsilon^2}{2}|\phi_3|^2$ as a test function independently in weak formulation of the continuity equation (3.2). We obtain, by using identity $\partial_j(\phi_i\phi_i) = 2\phi_i\partial_j\phi_i$,

$$\frac{1}{2}\int_\Omega \rho_\varepsilon(\cdot, \tau)|\phi_H|^2(\cdot, \tau)dxdz$$

$$= \frac{1}{2}\int_\Omega \rho_0(\cdot)|\phi_H|^2(\cdot, 0)dxdz$$

$$+ \int_0^\tau\int_\Omega (\rho_\varepsilon\phi_H\partial_t\phi_H + \rho_\varepsilon u_\varepsilon\phi_H \cdot \nabla_x\phi_H + \rho_\varepsilon v_\varepsilon\phi_H\partial_z\phi_H)dxdzdt \quad (3.20)$$

and

$$\frac{\varepsilon^2}{2}\int_\Omega \rho_\varepsilon(\cdot, \tau)|\phi_3|^2(\cdot, \tau)dxdz$$

$$= \frac{\varepsilon^2}{2}\int_\Omega \rho_0(\cdot)|\phi_3|^2(\cdot, 0)dxdz$$

$$+ \varepsilon^2\int_0^\tau\int_\Omega (\rho_\varepsilon\phi_3\partial_t\phi_3 + \rho_\varepsilon u_\varepsilon\phi_3 \cdot \nabla_x\phi_3 + \rho_\varepsilon v_\varepsilon\phi_3\partial_z\phi_3)dxdzdt. \quad (3.21)$$

Second, with test function $\varphi = \ln r$, we have

$$\int_\Omega \rho_\varepsilon(\cdot,\ \tau)\ln r(\cdot,\ \tau)dxdz$$

$$= \int_\Omega \rho_0(\cdot)\ln r(\cdot,\ 0)dxdz$$

$$+ \int_0^\tau\int_\Omega\left(\rho_\varepsilon\frac{\partial_t r}{r} + \rho_\varepsilon u_\varepsilon\frac{\nabla_x r}{r} + \rho_\varepsilon v_\varepsilon\frac{\partial_z r}{r}\right)dxdzdt. \tag{3.22}$$

In the third step, we rewrite (3.5) as

$$\int_\Omega \rho_\varepsilon(u_\varepsilon\phi_H + \varepsilon^2 v_\varepsilon\phi_3)(\cdot,\ \tau)dxdz - \int_\Omega \rho_0(u_0\phi_{H,0} + \varepsilon^2 v_0\phi_{3,0})(\cdot)dxdz$$

$$- \int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon\partial_t\phi_H + \varepsilon^2 v_\varepsilon\partial_t\phi_3)dxdzdt$$

$$- \int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon v_\varepsilon\partial_z\phi_H + \varepsilon^2 v_\varepsilon^2\partial_z\phi_3)dxdzdt$$

$$+ 2\bar{\mu}\int_0^\tau\int_\Omega \rho_\varepsilon(D_x(u_\varepsilon) : \nabla_x\phi_H + \varepsilon^2 D_x(v_\varepsilon) : \nabla_x\phi_3)dxdzdt$$

$$- \int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon \otimes u_\varepsilon : \nabla_x\phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x\phi_3 dxdzdt)$$

$$+ \bar{\nu}\int_0^\tau\int_\Omega \rho_\varepsilon(\partial_z u_\varepsilon\partial_z\phi_H + \varepsilon^2\partial_z v_\varepsilon\partial_z\phi_3)dxdzdt$$

$$+ \int_0^\tau\int_\Omega k\rho_\varepsilon u_\varepsilon|\ u_\varepsilon\ |\phi_H\, dxdzdt - \int_0^\tau\int_\Omega p(\rho_\varepsilon)(div_x\phi_H + \partial_z\phi_3)dxdzdt$$

$$= -\varepsilon\int_0^\tau\int_\Omega g\rho_\varepsilon\phi_3 dxdzdt. \tag{3.23}$$

We multiply (3.22) and (3.23) by –1 and sum it up with (3.20) and the energy inequality (3.8) to deduce

$$\int_\Omega \left(\frac{1}{2}\rho_\varepsilon |u_\varepsilon - \phi_H|^2 + \frac{\varepsilon^2}{2}\rho_\varepsilon |v_\varepsilon - \phi_3|^2 + \rho_\varepsilon \ln\rho_\varepsilon - \rho_\varepsilon - \rho_\varepsilon \ln r \right)(\cdot, \tau)\, dxdz$$

$$- \int_\Omega \left(\frac{1}{2}\rho_\varepsilon |u_\varepsilon - \phi_H|^2 + \frac{\varepsilon^2}{2}\rho_\varepsilon |v_\varepsilon - \phi_3|^2 + \rho_\varepsilon \ln\rho_\varepsilon - \rho_\varepsilon - \rho_\varepsilon \ln r \right)(\cdot, 0)\, dxdz$$

$$+ \int_0^\tau \int_\Omega k\rho_\varepsilon |u_\varepsilon|^3\, dxdzdt + \int_0^\tau \int_\Omega \rho_\varepsilon (2\bar{\mu} |D_x(u_\varepsilon)|^2 + \bar{\nu} |\partial_z u_\varepsilon|^2)\, dxdzdt$$

$$+ \varepsilon^2 \int_0^\tau \int_\Omega \rho_\varepsilon (2\bar{\mu} |D_x(v_\varepsilon)|^2 + \bar{\nu} |\partial_z v_\varepsilon|^2)\, dxdzdt$$

$$+ \int_0^\tau \int_\Omega \rho_\varepsilon (u_\varepsilon \partial_t \phi_H + \varepsilon^2 v_\varepsilon \partial_t \phi_3)\, dxdzdt$$

$$+ \int_0^\tau \int_\Omega \rho_\varepsilon (u_\varepsilon v_\varepsilon \partial_z \phi_H + \varepsilon^2 v_\varepsilon^2 \partial_z \phi_3)\, dxdzdt$$

$$- 2\bar{\mu} \int_0^\tau \int_\Omega \rho_\varepsilon (D_x(u_\varepsilon) : \nabla_x \phi_H + \varepsilon^2 D_x(v_\varepsilon) \cdot \nabla_x \phi_3)\, dxdzdt$$

$$- \bar{\nu} \int_0^\tau \int_\Omega \rho_\varepsilon (\partial_z u_\varepsilon \partial_z \phi_H + \varepsilon^2 \partial_z v_\varepsilon \partial_z \phi_3)\, dxdzdt$$

$$- \int_0^\tau \int_\Omega k\rho_\varepsilon u_\varepsilon |u_\varepsilon| \phi_H\, dxdzdt$$

$$+ \int_0^\tau \int_\Omega \rho_\varepsilon (u_\varepsilon \otimes u_\varepsilon : \nabla_x \phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x \phi_3)\, dxdzdt$$

$$\leq \int_0^\tau \int_\Omega (\rho_\varepsilon \phi_H \partial_t \phi_H + \rho_\varepsilon u_\varepsilon \phi_H \cdot \nabla_x \phi_H + \rho_\varepsilon v_\varepsilon \phi_H \partial_z \phi_H)\, dxdzdt$$

$$+ \varepsilon^2 \int_0^\tau \int_\Omega (\rho_\varepsilon \phi_3 \partial_t \phi_3 + \rho_\varepsilon u_\varepsilon \phi_3 \nabla_x \phi_3 + \rho_\varepsilon v_\varepsilon \phi_3 \partial_z \phi_3)\, dxdzdt$$

$$- \int_0^\tau \int_\Omega \left(\rho_\varepsilon \frac{\partial_t r}{r} + \rho_\varepsilon u_\varepsilon \frac{\nabla_x r}{r} + \rho_\varepsilon v_\varepsilon \frac{\partial_z r}{r} \right) dxdzdt$$

$$+ \varepsilon\int_0^\tau\int_\Omega g\rho_\varepsilon\phi_3\,dxdzdt - \varepsilon\int_0^\tau\int_\Omega g\rho_\varepsilon v_\varepsilon dxdzdt$$

$$- \int_0^\tau\int_\Omega p(\rho_\varepsilon)(div_x\phi_H + \partial_z\phi_3)dxdzdt. \tag{3.24}$$

We have

$$\int_\Omega (\rho_\varepsilon\phi_H\partial_t\phi_H + \varepsilon^2\rho_\varepsilon\phi_3\partial_t\phi_3 - \rho_\varepsilon(u_\varepsilon\partial_t\phi_H + \varepsilon^2 v_\varepsilon\partial_t\phi_3))dxdz$$

$$= \int_\Omega \rho_\varepsilon(\phi_H - u_\varepsilon)\partial_t\phi_H\,dxdz + \varepsilon^2\int_\Omega \rho_\varepsilon(\phi_3 - v_\varepsilon)\partial_t\phi_3\,dxdz. \tag{3.25}$$

By a direct computation, we have

$$\int_\Omega div_x(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon)\phi_H = -\int_\Omega \rho_\varepsilon u_\varepsilon(u_\varepsilon\nabla_x\phi_H).$$

So

$$-\int_\Omega \rho_\varepsilon(u_\varepsilon \otimes u_\varepsilon : \nabla_x\phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x\phi_3)dxdz$$

$$- \int_0^\tau\int_\Omega \rho_\varepsilon(u_\varepsilon v_\varepsilon\partial_z\phi_H + \varepsilon^2 v_\varepsilon^2\partial_z\phi_3)dxdzdt$$

$$+ \int_\Omega (\rho_\varepsilon u_\varepsilon\phi_H \cdot \nabla_x\phi_H + \rho_\varepsilon v_\varepsilon\phi_H\partial_z\phi_H + \varepsilon^2\rho_\varepsilon u_\varepsilon\phi_3\nabla_x\phi_3 + \varepsilon^2\rho_\varepsilon v_\varepsilon\phi_3\partial_z\phi_3)dxdz$$

$$= \int_\Omega \rho_\varepsilon(u_\varepsilon\nabla_x\phi_H + v_\varepsilon\partial_z\phi_H)\phi_H dxdzdt$$

$$- \int_\Omega \rho_\varepsilon(u_\varepsilon \cdot \nabla_x\phi_H + v_\varepsilon\partial_z\phi_H)u_\varepsilon\,dxdzdt$$

$$+ \varepsilon^2\int_\Omega \rho_\varepsilon(u_\varepsilon\nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3)dxdz$$

$$- \varepsilon^2\int_\Omega \rho_\varepsilon(u_\varepsilon \cdot \nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3)v_\varepsilon\,dxdz$$

$$= \int_{\Omega} \rho_\varepsilon (u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H)(\phi_H - u_\varepsilon) dxdzdt$$

$$+ \varepsilon^2 \int_{\Omega} \rho_\varepsilon (u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3)(\phi_3 - v_\varepsilon) dxdz. \tag{3.26}$$

The sum of (3.25) and (3.26) gives

$$\int_{\Omega} \rho_\varepsilon (\partial_t \phi_H + u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H)(\phi_H - u_\varepsilon) dxdzdt$$

$$+ \varepsilon^2 \int_{\Omega} \rho_\varepsilon (\partial_t \phi_3 + u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3)(\phi_3 - v_\varepsilon) dxdz.$$

Moreover, we add the following equality to (3.24):

$$\int_{\Omega} r(\cdot, \tau) dxdz - \int_{\Omega} r(\cdot, 0) dxdz = \int_0^\tau \int_{\Omega} \partial_t r dxdzdt$$

$$= \int_0^\tau \int_{\Omega} \partial_t p(r) dxdzdt. \tag{3.27}$$

Thus, the right-hand of (3.24) is given by

$$\partial_t r - \left(\rho_\varepsilon \frac{\partial_t r}{r} + \rho_\varepsilon u_\varepsilon \frac{\nabla_x r}{r} + \rho_\varepsilon v_\varepsilon \frac{\partial_z r}{r} \right) = (r - \rho_\varepsilon) \frac{\partial_t r}{r} - \rho_\varepsilon U_\varepsilon \frac{\nabla r}{r}. \tag{3.28}$$

Furthermore, considering that ϕ satisfies no-slop boundary conditions, we have

$$\int_{\Omega} (\phi \cdot \nabla r + r div \phi) dxdz = \int_{\Omega} div(r\phi) dxdz$$

$$= \int_{\partial\Omega} r\phi \cdot \mathbf{n} dS = 0. \tag{3.29}$$

Thus, we can add the term $\int_{\Omega} (\phi \cdot \nabla r + r div \phi) dxdz$ to the right-hand side of (3.24) to obtain

$$-\rho_\varepsilon div \phi + \phi \cdot \nabla r + r div \phi = (r - \rho_\varepsilon) div \phi + r\phi \frac{\nabla r}{r}. \tag{3.30}$$

Hence, the sum of (3.28) and (3.30) gives

$$(r-\rho_\varepsilon)\frac{\partial_t r}{r} - \rho_\varepsilon U_\varepsilon \frac{\nabla r}{r} + (r-\rho_\varepsilon) div\phi + r\phi\frac{\nabla r}{r}$$

$$= (r-\rho_\varepsilon)\frac{\partial_t r}{r} + (r-\rho_\varepsilon) div\phi + \frac{\nabla r}{r}\cdot(r\phi - \rho_\varepsilon U_\varepsilon). \tag{3.31}$$

Furthermore, we have

$$2\bar{\mu}\rho_\varepsilon(|D_x(u_\varepsilon)|^2 + \varepsilon^2|D_x(v_\varepsilon)|^2)$$

$$-2\bar{\mu}\rho_\varepsilon(D_x(u_\varepsilon):\nabla_x\phi_H + \varepsilon^2 D_x(v_\varepsilon)\cdot\nabla_x\phi_3)$$

$$= 2\bar{\mu}\rho_\varepsilon(D_x(u_\varepsilon):\nabla_x u_\varepsilon + \varepsilon^2 D_x(v_\varepsilon)\cdot\nabla_x v_\varepsilon)$$

$$-2\bar{\mu}\rho_\varepsilon(D_x(u_\varepsilon):\nabla_x\phi_H + \varepsilon^2 D_x(v_\varepsilon):\nabla_x\phi_3)$$

$$= 2\bar{\mu}\rho_\varepsilon(D_x(u_\varepsilon):\nabla_x(u_\varepsilon - \phi_H) + \varepsilon^2\nabla_x(v_\varepsilon)\cdot\nabla_x(v_\varepsilon - \phi_3))$$

and

$$\bar{\nu}\rho_\varepsilon(|\partial_z u_\varepsilon|^2 + \varepsilon^2|\partial_z v_\varepsilon|^2) - \bar{\nu}\rho_\varepsilon(\partial_z u_\varepsilon\partial_z\phi_H + \varepsilon^2\partial_z v_\varepsilon\partial_z\phi_3)$$

$$= \bar{\nu}\rho_\varepsilon(\partial_z u_\varepsilon\partial_z(u_\varepsilon - \phi_H) + \varepsilon^2\partial_z v_\varepsilon\partial_z(v_\varepsilon - \phi_3)).$$

Putting these together, we get the relative entropy inequality defined by

$$\xi([\rho_\varepsilon, U_\varepsilon]|[r,\phi])(\cdot,\tau)$$

$$+\int_0^\tau\int_\Omega 2\bar{\mu}\rho_\varepsilon(D_x u_\varepsilon:\nabla_x(u_\varepsilon-\phi_H) + \varepsilon^2\nabla_x v_\varepsilon:\nabla_x(v_\varepsilon-\phi_3))\,dxdzdt$$

$$+\int_0^\tau\int_\Omega \bar{\nu}\rho_\varepsilon(\partial_z u_\varepsilon\partial_z(u_\varepsilon-\phi_H) + \varepsilon^2\partial_z v_\varepsilon\partial_z(v_\varepsilon-\phi_3))\,dxdzdt$$

$$\le \xi([\rho_\varepsilon, U_\varepsilon]|[r,\phi])(\cdot,0)$$

$$+\int_0^\tau\int_\Omega \rho_\varepsilon(\partial_t\phi_H + u_\varepsilon\nabla_x\phi_H + v_\varepsilon\partial_z\phi_H)(\phi_H - u_\varepsilon)\,dxdzdt$$

$$+ \varepsilon^2 \int_0^\tau \int_\Omega \rho_\varepsilon (\partial_t \phi_3 + u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3)(\phi_3 - v_\varepsilon) dxdzdt$$

$$+ \varepsilon \int_0^\tau \int_\Omega g\rho_\varepsilon (\phi_3 - v_\varepsilon) dxdzdt + \int_0^\tau \int_\Omega k\rho_\varepsilon u_\varepsilon | u_\varepsilon | (\phi_H - u_\varepsilon) dxdzdt$$

$$+ \int_0^\tau \int_\Omega \left((r - \rho_\varepsilon) \frac{\partial_t r}{r} + (r - \rho_\varepsilon) div\phi + \frac{\nabla r}{r} \cdot (r\phi - \rho_\varepsilon U_\varepsilon) \right) dxdzdt. \quad (3.32)$$

With $2\overline{\mu} = \overline{\nu} = 1,$ we rewrite (3.32) as

$$\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \rho_\varepsilon (Du_\varepsilon : \nabla(u_\varepsilon - \phi_H) + \varepsilon^2 | \nabla v_\varepsilon |^2) dxdzdt$$

$$\leq \int_0^\tau \int_\Omega \rho_\varepsilon (\partial_t \phi_H + u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H + k\phi_H | \phi_H |)(\phi_H - u_\varepsilon) dxdzdt$$

$$+ \varepsilon^2 \int_0^\tau \int_\Omega \rho_\varepsilon (\partial_t \phi_3 + u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3)(\phi_3 - v_\varepsilon) dxdzdt$$

$$+ \varepsilon^2 \int_0^\tau \int_\Omega \nabla v_\varepsilon \cdot \nabla \phi_3 dxdzdt + \varepsilon \int_0^\tau \int_\Omega g\rho_\varepsilon (\phi_3 - v_\varepsilon) dxdzdt$$

$$+ \int_0^\tau \int_\Omega k\rho_\varepsilon (u_\varepsilon | u_\varepsilon | - \phi_H | \phi_H |)(\phi_H - u_\varepsilon) dxdzdt$$

$$+ \int_0^\tau \int_\Omega \left((r - \rho_\varepsilon) \frac{\partial_t r}{r} + (r - \rho_\varepsilon) div\phi + \frac{\nabla r}{r} \cdot (r\phi - \rho_\varepsilon U_\varepsilon) \right) dxdzdt, \quad (3.33)$$

where $Du = (D_x u, \partial_z u)$ and we add

$$\rho_\varepsilon (k\phi_H | \phi_H | (\phi_H - u_\varepsilon) - k\phi_H | \phi_H | (\phi_H - u_\varepsilon))$$

in right-hand of (3.32).

4. Convergence

In this section, we prove our main result. The proof of Theorem 3.1 is strongly based on the relative energy inequality (3.33) by considering the strong solution (r, ϕ), where $\phi = (\phi_H, \phi_3)$ as a test function in the relative entropy (3.11).

As in [1], we have the following remark:

Remark 4.1. We assume that the strong solution satisfies (1.6) pointwise and all the terms in this formulation are well defined. In particular, $r \in C^1(\Omega \times (0, T))$, $\phi \in C^1(\Omega \times (0, T))^3$ with $\nabla U \in C^1(\Omega \times (0, T))^{3\times 3}$. It is worth to mention that the proof presented below works also for weak solutions with sufficient regularity.

Next, in three steps, we prove our main result.

Step 1. Estimate on the relative energy inequality (3.33).

We begin by

$$\int_\Omega \rho_\varepsilon u_\varepsilon(\phi_H - u_\varepsilon) \cdot \nabla_x \phi_H \, dxdz = \int_\Omega \rho_\varepsilon(u_\varepsilon - \phi_H)(\phi_H - u_\varepsilon) \cdot \nabla_x \phi_H \, dxdz$$

$$+ \int_\Omega \rho_\varepsilon \phi_H (\phi_H - u_\varepsilon) \cdot \nabla_x \phi_H \, dxdz. \quad (4.1)$$

As $[r, \phi_H, \phi_3]$ is a strong solution, concerning the first term of (4.1), we have (recall that $\nabla_x \phi_H$ is a bounded function)

$$\int_\Omega \rho_\varepsilon(u_\varepsilon - \phi_H)(\phi_H - u_\varepsilon) \cdot \nabla_x \phi_H \, dxdz \le \int_\Omega |\rho_\varepsilon(u_\varepsilon - \phi_H)(\phi_H - u_\varepsilon) \cdot \nabla_x \phi_H| dxdz$$

$$\le C \int_\Omega \rho_\varepsilon |\phi_H - u_\varepsilon|^2 dxdz$$

$$\le C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]), \quad (4.2)$$

where we used (3.17).

Again, since $[r, \phi_H, \phi_3]$ is a strong solution of (1.6), we can rearrange the momentum equation $(1.6)_2$ as follows:

$$r\partial_t\phi_H + \phi_H\partial_t r + \phi_H div_x(r\phi_H) + r\phi_H\cdot\nabla_x\phi_H + r\phi_3\partial_z\phi_H$$

$$+ \phi_H\partial_z(r\phi_3) + kr\phi_H|\phi_H| + \nabla_x r$$

$$= div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H)$$

$$\Rightarrow r[\partial_t\phi_H + \phi_H\cdot\nabla_x\phi_H + \phi_3\partial_z\phi_H + k\phi_H|\phi_H|]$$

$$+ \phi_H[\partial_t r + div_x(r\phi_H) + \partial_z(r\phi_3)]$$

$$= div_x(rD_x\phi_H) + \partial_x(r\partial_z\phi_H) - \nabla_x r,$$

which by means of the continuity equation reduces into

$$\partial_t\phi_H + \phi_H\cdot\nabla_x\phi_H + \phi_3\partial_z\phi_H + k\phi_H|\phi_H|$$

$$= \frac{1}{r}(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H) - \nabla_x r).$$

So we rewrite

$$\rho_\varepsilon(\partial_t\phi_H + u_\varepsilon\nabla_x\phi_H + v_\varepsilon\partial_z\phi_H + k\phi_H|\phi_H|)$$

$$= \rho_\varepsilon(\partial_t\phi_H + \phi_H\nabla_x\phi_H + \phi_3\partial_z\phi_H + k\phi_H|\phi_H|$$

$$+ (u_\varepsilon - \phi_H)\nabla_x\phi_H + (v_\varepsilon - \phi_3)\partial_z\phi_H)$$

$$= \frac{\rho_\varepsilon}{r}(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H) - \nabla_x r)$$

$$+ \rho_\varepsilon(u_\varepsilon - \phi_H)\nabla_x\phi_H + \rho_\varepsilon(v_\varepsilon - \phi_3)\partial_z\phi_H.$$

Hence,

$$\int_0^\tau\int_\Omega \rho_\varepsilon(\partial_t\phi_H + u_\varepsilon\nabla_x\phi_H + v_\varepsilon\partial_z\phi_H + k\phi_H|\phi_H|)(\phi_H - u_\varepsilon)dxdzdt$$

$$= \int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(div_x(rD_x\phi_H) + \partial_z(r\phi_H) - \nabla_x r)(\phi_H - u_\varepsilon)dxdzdt$$

$$+ \int_0^\tau\int_\Omega \rho_\varepsilon(v_\varepsilon - \phi_3)(\phi_H - u_\varepsilon)\partial_z\phi_H dxdzdt - \int_0^\tau\int_\Omega \rho_\varepsilon(\phi_H - u_\varepsilon)^2 dxdzdt$$

$$\leq \int_0^\tau \int_\Omega \frac{\rho_\varepsilon}{r}(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H) - \nabla_x r)(\phi_H - u_\varepsilon)dxdzdt$$

$$+ \int_0^\tau \int_\Omega \rho_\varepsilon(v_\varepsilon - \phi_3)(\phi_H - u_\varepsilon)\partial_z\phi_H dxdzdt + C\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]), \qquad (4.3)$$

where we have used (4.2).

Moreover, we have

$$\partial_t\phi_3 + u_\varepsilon\nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3 = \partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3$$
$$+ (u_\varepsilon - \phi_H)\nabla_x\phi_3 + (v_\varepsilon - \phi_3)\partial_z\phi_3.$$

Thus, using Cauchy's inequality and (3.17), we get

$$\varepsilon^2\int_0^\tau \int_\Omega \rho_\varepsilon(\partial_t\phi_3 + u_\varepsilon\nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3)(\phi_3 - v_\varepsilon)dxdzdt$$

$$\leq \int_0^\tau \int_\Omega \rho_\varepsilon(\phi_3 - v_\varepsilon)^2 dxdzdt + \varepsilon^4\int_0^\tau \int_\Omega \rho_\varepsilon(\partial_t\phi_3 + u_\varepsilon\nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3)^2 dxdzdt$$

$$\leq \xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]) + \varepsilon^4\int_0^\tau \int_\Omega \rho_\varepsilon(\partial_t\phi_3 + u_\varepsilon\nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3)^2 dxdzdt$$

$$\leq \xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]) + \varepsilon^4\int_0^\tau \int_\Omega \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdzdt$$

$$+ \varepsilon^4\int_0^\tau \int_\Omega \rho_\varepsilon((u_\varepsilon - \phi_H)\nabla_x\phi_3 + (v_\varepsilon - \phi_3)\partial_z\phi_3)^2 dxdzdt. \qquad (4.4)$$

We decompose $\int_\Omega \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$ into three parts as follows:

$$\int_\Omega \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$= \int_\Omega \chi_{\left\{\frac{r}{2}<\rho_\varepsilon<2\bar{r}\right\}} \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$+\int_\Omega \chi_{\left\{\rho_\varepsilon \le \frac{r}{2}\right\}} \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$+\int_\Omega \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}} \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$\le C\int_\Omega \chi_{\left\{\frac{r}{2}<\rho_\varepsilon<2\bar{r}\right\}} (\rho_\varepsilon - r)(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$+\int_\Omega \chi_{\left\{\rho_\varepsilon \le \frac{r}{2}\right\}} r(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$+\int_\Omega \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}} \rho_\varepsilon(\partial_t\phi_3 + \phi_H\nabla_x\phi_3 + \phi_3\partial_z\phi_3)^2 dxdz$$

$$\le C\int_\Omega \chi_{\left\{\frac{r}{2}<\rho_\varepsilon<2\bar{r}\right\}} (\rho_\varepsilon - r)^2 dxdz + C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])$$

$$+ C\int_\Omega \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}} \rho_\varepsilon dxdz + C$$

$$\le C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]) + C. \tag{4.5}$$

Putting (4.5) into (4.4), we have

$$\varepsilon^2\int_0^\tau\int_\Omega \rho_\varepsilon(\partial_t\phi_3 + u_\varepsilon\nabla_x\phi_3 + v_\varepsilon\partial_z\phi_3)(\phi_3 - v_\varepsilon) dxdzdt$$

$$\le C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]) + o(\varepsilon^2). \tag{4.6}$$

Moreover, we apply again the inequality of Cauchy as

$$\varepsilon^2\int_0^\tau\int_\Omega \nabla v_\varepsilon \cdot \nabla\phi_3 dxdzdt \le \varepsilon^2\left(\int_0^\tau\int_\Omega \frac{|\nabla v_\varepsilon|^2}{2} dxdzdt + \int_0^\tau\int_\Omega \frac{|\nabla\phi_3|^2}{2} dxdzdt\right)$$

$$\le \frac{\varepsilon^2}{2}\int_0^\tau\int_\Omega |\nabla v_\varepsilon|^2 dxdzdt + o(\varepsilon^2). \tag{4.7}$$

Thus, the entropy relative inequality (3.33) becomes

$$\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])|_{t=0}^{t=r} + \int_0^\tau\int_\Omega \rho_\varepsilon (Du_\varepsilon : \nabla(u_\varepsilon - \phi_H) + \frac{\varepsilon^2}{2}|\nabla v_\varepsilon|^2)\, dxdzdt$$

$$\le C\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]) + \int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)\, dxdzdt$$

$$- \int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(\phi_H - u_\varepsilon)\nabla_x r dxdzdt$$

$$+ \int_0^\tau\int_\Omega \rho_\varepsilon(v_\varepsilon - \phi_3)(\phi_H - u_\varepsilon)\partial_z\phi_H\, dxdzdt$$

$$+ \varepsilon\int_0^\tau\int_\Omega g\rho_\varepsilon(\phi_3 - v_\varepsilon)\, dxdzdt$$

$$+ \int_0^\tau\int_\Omega k\rho_\varepsilon(u_\varepsilon|u_\varepsilon| - \phi_H|\phi_H|)(\phi_H - u_\varepsilon)\, dxdzdt$$

$$+ \int_0^\tau\int_\Omega \left((r-\rho_\varepsilon)\frac{\partial_t r}{r} + (r-\rho_\varepsilon)div\phi + \frac{\nabla r}{r}\cdot(r\phi - \rho_\varepsilon U_\varepsilon)\right)dxdzdt + o(\varepsilon^2). \quad (4.8)$$

Step 2. We estimate the non-linear term

$$\int_0^\tau\int_\Omega \rho_\varepsilon(v_\varepsilon - \phi_3)(\phi_H - u_\varepsilon)\partial_z\phi_H\, dxdzdt$$

of (4.8). This estimate is the major difficulty in our analysis. We have

$$\int_0^\tau\int_\Omega \rho_\varepsilon(v_\varepsilon - \phi_3)(\phi_H - u_\varepsilon)\partial_z\phi_H\, dxdzdt$$

$$= \int_0^\tau\int_\Omega \rho_\varepsilon v_\varepsilon(\phi_H - u_\varepsilon)\partial_z\phi_H\, dxdzdt$$

$$- \int_0^\tau\int_\Omega \rho_\varepsilon\phi_3(\phi_H - u_\varepsilon)\partial_z\phi_H\, dxdzdt. \quad (4.9)$$

In a similar way to [16, 19, 24], we decompose $\int_\Omega \rho_\varepsilon \phi_3 (\phi_H - u_\varepsilon) \partial_z \phi_H \, dxdz$ into three parts. We have

$$
\begin{aligned}
&\int_\Omega \rho_\varepsilon \phi_3 (\phi_H - u_\varepsilon) \partial_z \phi_H \, dxdzdt \\
&= \int_\Omega \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} \rho_\varepsilon \phi_3 (\phi_H - u_\varepsilon) \partial_z \phi_H dxdz \\
&\quad + \int_\Omega \chi_{\left\{\rho_\varepsilon \le \frac{r}{2}\right\}} \rho_\varepsilon \phi_3 (\phi_H - u_\varepsilon) \partial_z \phi_H dxdz \\
&\quad + \int_\Omega \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}} \rho_\varepsilon \phi_3 (\phi_H - u_\varepsilon) \partial_z \phi_H dxdz \\
&\le C \| \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} (\rho_\varepsilon - r) \|_{L^2(\Omega)} \| \phi_3 \partial_z \phi_H \|_{L^3(\Omega)} \| \phi_H - u_\varepsilon \|_{L^6(\Omega)} \\
&\quad + \| \chi_{\left\{\rho_\varepsilon \le \frac{r}{2}\right\}} 1 \|_{L^2(\Omega)} \| r \|_{L^\infty(\Omega)} \| \phi_3 \partial_z \phi_H \|_{L^3(\Omega)} \| \phi_H - u_\varepsilon \|_{L^6(\Omega)} \\
&\quad + \int_\Omega \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}} \rho_\varepsilon \phi_3 (\phi_H - u_\varepsilon) \partial_z \phi_H dxdz \\
&\le C \int_\Omega \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} (\rho_\varepsilon - r)^2 dxdz + C \int_\Omega \chi_{\left\{\rho_\varepsilon \le \frac{r}{2}\right\}} 1 dxdz \\
&\quad + C \int_\Omega \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}} \rho_\varepsilon dxdz + \lambda \| \phi_H - u_\varepsilon \|^2_{L^6(\Omega)} \\
&\le C \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]) + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)} \\
&\quad + \lambda \| \partial_z \phi_H - \partial_z u_\varepsilon \|^2_{L^2(\Omega)}. \qquad (4.10)
\end{aligned}
$$

Note that in the last inequality, we have used Lemma 2.1.

The crucial and difficult part of our proof is the analysis of the first term of the right-hand of (4.9). Using (3.10), we have

$$\int_{\Omega} \rho_\varepsilon v_\varepsilon (\phi_H - u_\varepsilon)\partial_z \phi_H dxdz$$

$$= \int_{\Omega} (-div_x(\rho_\varepsilon \tilde{u}_\varepsilon) + z div_x(\rho_\varepsilon \tilde{u}_\varepsilon))(\phi_H - u_\varepsilon)\partial_z \phi_H dxdz$$

$$= \int_{\Omega} (\rho_\varepsilon \tilde{u}_\varepsilon - z\rho_\varepsilon \tilde{u}_\varepsilon)(\phi_H - u_\varepsilon)\partial_z \nabla_x \phi_H dxdz$$

$$+ \int_{\Omega} (\rho_\varepsilon \tilde{u}_\varepsilon - z\rho_\varepsilon \tilde{u}_\varepsilon)\partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz. \tag{4.11}$$

In the following, we estimate the terms on the right-hand side of (4.11). Firstly, we deal with $\int_{\Omega} \rho_\varepsilon \tilde{u}_\varepsilon (\phi_H - u_\varepsilon)\partial_z \nabla_x \phi_H \, dxdz$ in the following:

$$\int_{\Omega} \rho_\varepsilon \tilde{u}_\varepsilon (\phi_H - u_\varepsilon)\partial_z \nabla_x \phi_H \, dxdz = \int_{\Omega} \rho_\varepsilon (\tilde{u}_\varepsilon - \tilde{\phi}_H)\partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon) dxdz$$

$$+ \int_{\Omega} \rho_\varepsilon \tilde{\phi}_H \partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon) dxdz$$

$$= I_1 + I_2, \tag{4.12}$$

where

$$\tilde{\phi}_H = \int_0^z \phi_H(x, s, t) ds.$$

Using Cauchy's inequality, it follows that

$$I_1 = \int_{\Omega} \rho_\varepsilon (\tilde{u}_\varepsilon - \tilde{\phi}_H)\partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon) dxdz$$

$$\leq \frac{1}{2} \| \partial_z \nabla_x \phi_H \|_{L^\infty(\Omega)} \left(\int_{\Omega} \rho_\varepsilon |\tilde{u}_\varepsilon - \tilde{\phi}_H|^2 dxdz + \int_{\Omega} \rho_\varepsilon |u_\varepsilon - \phi_H|^2 dxdz \right)$$

$$\leq C \int_{\Omega} \rho_\varepsilon \left| \int_0^z (u_\varepsilon - \phi_H)(s) ds \right|^2 dxdz + \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])$$

$$\leq C \int_{\Omega} \rho_\varepsilon \left(\int_0^1 | u_\varepsilon - \phi_H |^2 (s) ds \right) dxdz + \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])$$

$$\leq C\int_0^1\int_\Omega \rho_\varepsilon | u_\varepsilon - \phi_H |^2(s)\,dxdzds + \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])$$

$$\leq C\int_\Omega \rho_\varepsilon | u_\varepsilon - \phi_H |^2(s)\,dxdz + \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])$$

$$\leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]). \tag{4.13}$$

Similar to the above analysis (4.10), we decompose the term I_2 into three parts

$$I_2 = \int_\Omega \rho_\varepsilon \tilde{\phi}_H \partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon)\,dxdz$$

$$= \int_\Omega \chi_{\left\{\rho_\varepsilon \leq \frac{r}{2}\right\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon)\,dxdz$$

$$+ \int_\Omega \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon)\,dxdz$$

$$+ \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \nabla_x \phi_H \cdot (\phi_H - u_\varepsilon)\,dxdz$$

$$\leq \| \chi_{\left\{\rho_\varepsilon \leq \frac{r}{2}\right\}} 1 \|_{L^2(\Omega)} \| r \|_{L^\infty(\Omega)} \| \tilde{\phi}_H \partial_z \nabla_x \phi_H \|_{L^3(\Omega)} \| \phi_H - u_\varepsilon \|_{L^6(\Omega)}$$

$$+ C \| \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} (\rho_\varepsilon - r) \|_{L^2(\Omega)} \| \tilde{\phi}_H \partial_z \nabla_x \phi_H \|_{L^3(\Omega)} \| \phi_H - u_\varepsilon \|_{L^6(\Omega)}$$

$$+ \| \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \|_{L^2(\Omega)}^2 \| \tilde{\phi}_H \partial_z \nabla_x \phi_H \|_{L^3(\Omega)} \| \phi_H - u_\varepsilon \|_{L^6(\Omega)}$$

$$\leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t) + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|_{L^2(\Omega)}^2$$

$$+ \lambda \| \partial_z \phi_H - \partial_z u_\varepsilon \|_{L^2(\Omega)}^2. \tag{4.14}$$

Let us consider another non-linear term:

$$\int_\Omega \rho_\varepsilon \tilde{u}_\varepsilon \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon)\,dxdz.$$

Then

$$\int_\Omega \rho_\varepsilon \tilde{u}_\varepsilon \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$= \int_\Omega \chi_{\rho_\varepsilon < 2\bar{r}} \rho_\varepsilon \tilde{u}_\varepsilon \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$+ \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \tilde{u}_\varepsilon \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz, \tag{4.15}$$

where the first term on the right side of (4.15) gets split into two parts as

$$\int_\Omega \chi_{\rho_\varepsilon < 2\bar{r}} \rho_\varepsilon \tilde{u}_\varepsilon \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$= \int_\Omega \chi_{\rho_\varepsilon < 2\bar{r}} \rho_\varepsilon (\tilde{u}_\varepsilon - \tilde{\phi}_H) \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$+ \int_\Omega \chi_{\rho_\varepsilon < 2\bar{r}} \rho_\varepsilon \tilde{\phi}_H \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$= \int_\Omega \chi_{\rho_\varepsilon < 2\bar{r}} \rho_\varepsilon (\tilde{u}_\varepsilon - \tilde{\phi}_H) \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$+ \int_\Omega \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$+ \int_\Omega \chi_{\left\{\rho_\varepsilon \leq \frac{r}{2}\right\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$\leq \| \chi_{\left\{\rho_\varepsilon \leq \frac{r}{2}\right\}} 1 \|_{L^2(\Omega)} \| r \|_{L^\infty(\Omega)} \| \tilde{\phi}_H \partial_z \phi_H \|_{L^\infty(\Omega)} \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|_{L^2(\Omega)}$$

$$+ \| \chi_{\left\{\frac{r}{2} < \rho_\varepsilon < 2\bar{r}\right\}} \rho_\varepsilon \|_{L^2(\Omega)} \| \tilde{\phi}_H \partial_z \phi_H \|_{L^\infty(\Omega)} \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|_{L^2(\Omega)}$$

$$+ \| \chi_{\rho_\varepsilon < 2\bar{r}} \sqrt{\rho_\varepsilon} \|_{L^\infty(\Omega)} \| \partial_z \phi_H \|_{L^\infty(\Omega)}$$

$$\times \| \sqrt{\rho_\varepsilon} (\tilde{u}_\varepsilon - \tilde{\phi}_H) \|_{L^2(\Omega)} \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|_{L^2(\Omega)}$$

$$\leq C \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t) + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)}. \tag{4.16}$$

The decomposition of remainder of (4.15) is identical to the above as:

$$\int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \tilde{u}_\varepsilon \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$= \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon (\tilde{u}_\varepsilon - \tilde{\phi}_H) \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$+ \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$= J_1 + J_2. \tag{4.17}$$

We have, by using (3.18),

$$J_2 = \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \tilde{\phi}_H \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$\leq \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon^{\alpha/2} \tilde{\phi}_H \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$\leq \| \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon^{\alpha/2} \|_{L^2(\Omega)} \| \tilde{\phi}_H \partial_z \phi_H \|_{L^\infty(\Omega)} \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|_{L^2(\Omega)}$$

$$\leq C \| \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon^{\alpha/2} \|^2_{L^2(\Omega)} + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)}$$

$$\leq C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t) + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)}. \tag{4.18}$$

Due to Hölder and Cauchy inequalities, it follows that

$$J_1 = \int_\Omega \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon (\tilde{u}_\varepsilon - \tilde{\phi}_H) \partial_z \phi_H \cdot \nabla_x (\phi_H - u_\varepsilon) dxdz$$

$$\leq \| \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} \rho_\varepsilon \|_{L^4(\Omega)} \| \chi_{\{\rho_\varepsilon \geq 2\bar{r}\}} (\tilde{u}_\varepsilon - \tilde{\phi}_H) \|_{L^4(\Omega)}$$

$$\times \| \partial_z \phi_H \|_{L^\infty(\Omega)} \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|_{L^2(\Omega)}$$

$$
\begin{aligned}
&\le \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}\rho_\varepsilon \|^2_{L^4(\Omega)} \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|^2_{L^4(\Omega)} + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)} \\
&\le \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}\rho_\varepsilon \|^2_{L^4(\Omega)} \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|_{L^3(\Omega)} \\
&\quad \times \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\nabla \tilde{u}_\varepsilon - \nabla \tilde{\phi}_H) \|_{L^2(\Omega)} + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)} \\
&\le \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}\rho_\varepsilon \|^4_{L^4(\Omega)} \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|^2_{L^3(\Omega)} + \lambda \| \nabla_x \tilde{u}_\varepsilon - \nabla_x \tilde{\phi}_H \|^2_{L^2(\Omega)} \\
&\quad + \lambda \| \partial_z \tilde{u}_\varepsilon - \partial_z \tilde{\phi}_H \|^2_{L^2(\Omega)} + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)} \\
&\le \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}\rho_\varepsilon \|^4_{L^4(\Omega)} \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|_{L^2(\Omega)} \\
&\quad \times \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|_{H^1(\Omega)} + \lambda \| \nabla_x \tilde{u}_\varepsilon - \nabla_x \tilde{\phi}_H \|^2_{L^2(\Omega)} \\
&\quad + \lambda \| \partial_z \tilde{u}_\varepsilon - \partial_z \tilde{\phi}_H \|^2_{L^2(\Omega)} + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)} \\
&\quad \le \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}\rho_\varepsilon \|^8_{L^4(\Omega)} \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|^2_{L^2(\Omega)} \\
&\quad + \lambda \| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H) \|^2_{L^2(\Omega)} + \lambda \| \nabla_x \tilde{u}_\varepsilon - \nabla_x \tilde{\phi}_H \|^2_{L^2(\Omega)} \\
&\quad + \lambda \| \partial_z \tilde{u}_\varepsilon - \partial_z \tilde{\phi}_H \|^2_{L^2(\Omega)} + \lambda \| \nabla_x \phi_H - \nabla_x u_\varepsilon \|^2_{L^2(\Omega)},
\end{aligned}
\tag{4.19}
$$

where we have used Lemma 2.2:

$$
\| f \|^2_{L^4} \le \| \nabla f \|_{L^2} \| f \|_{L^3} \quad \text{and} \quad \| f \|^2_{L^3} \le \| f \|_{L^2} \| f \|_{H^1}.
$$

From (3.18), we have

$$
\| \chi_{\{\rho_\varepsilon \ge 2\bar{r}\}}\rho_\varepsilon \|^8_{L^4(\Omega)} = \left(\int_{\rho_\varepsilon \ge 2\bar{r}} \rho_\varepsilon^4 dxdz \right)^{8/4} \le C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])^2. \tag{4.20}
$$

Recalling (4.13), we have

$$\|\chi_{\{\rho_\varepsilon \geq 2\bar{r}\}}(\tilde{u}_\varepsilon - \tilde{\phi}_H)\|^2_{L^2(\Omega)} = \int_{\rho_\varepsilon \geq 2\bar{r}} |\tilde{u}_\varepsilon - \tilde{\phi}_H|^2 dxdz$$

$$= \int_{\rho_\varepsilon \geq 2\bar{r}} \frac{1}{\rho_\varepsilon} \rho_\varepsilon |\tilde{u}_\varepsilon - \tilde{\phi}_H|^2 dxdz$$

$$\leq \frac{1}{\|r\|_{L^\infty(\Omega)}} \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]). \tag{4.21}$$

As in (4.19), we have

$$\|\nabla_x \tilde{u}_\varepsilon - \nabla_x \tilde{\phi}_H\|^2_{L^2(\Omega)} \leq \|\nabla_x u_\varepsilon - \nabla_x \phi_H\|^2_{L^2(\Omega)},$$

$$\|\partial_z \tilde{u}_\varepsilon - \partial_z \tilde{\phi}_H\|^2_{L^2(\Omega)} \leq \|\partial_z u_\varepsilon - \partial_z \phi_H\|^2_{L^2(\Omega)}. \tag{4.22}$$

Combining the estimates (4.20), (4.21) and (4.22), we arrive at the conclusion that

$$\int_0^\tau J_1 dt \leq C \int_0^\tau h(t) \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]) dt$$

$$+ \lambda \int_0^\tau \|\nabla_x u_\varepsilon - \nabla_x \phi_H\|^2_{L^2(\Omega)} dt$$

$$+ \lambda \int_0^\tau \|\partial_z u_\varepsilon - \partial_z \phi_H\|^2_{L^2(\Omega)} dt, \tag{4.23}$$

where $h(t) \in L^1(0, T)$.

The estimate of remainder in (4.11) can be completed by the analogous method. Therefore, we can summarize what we have proved as follows:

$$\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])|_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \rho_\varepsilon(Du_\varepsilon : \nabla(u_\varepsilon - \phi_H) + \varepsilon^2|\nabla v_\varepsilon|^2)dxdzdt$$

$$\leq \int_0^\tau h(t)\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])(t)dt$$

$$+ \lambda\int_0^\tau(\|\nabla_x u_\varepsilon - \nabla_x\phi_H\|^2_{L^2(\Omega)} + \|\partial_z u_\varepsilon - \partial_z\phi_H\|^2_{L^2(\Omega)})dt$$

$$+ \int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)dxdzdt$$

$$- \int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(\phi_H - u_\varepsilon)\nabla_x r dxdzdt + \int_0^\tau\int_\Omega \varepsilon g\rho_\varepsilon(\phi_3 - v_\varepsilon)dxdzdt$$

$$+ \int_0^\tau\int_\Omega k\rho_\varepsilon(u_\varepsilon|u_\varepsilon| - \phi_H|\phi_H|)(\phi_H - u_\varepsilon)dxdzdt$$

$$+ \int_0^\tau\int_\Omega\left((r - \rho_\varepsilon)\frac{\partial_t r}{r} + (r - \rho_\varepsilon)div\phi + \frac{\nabla r}{r}\cdot(r\phi - \rho_\varepsilon U_\varepsilon)\right)dxdzdt + o(\varepsilon^2). \tag{4.24}$$

Then we deduce that

$$\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])|_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \rho_\varepsilon(D_x(u_\varepsilon - \phi_H) : \nabla_x(u_\varepsilon - \phi_H)$$

$$+ |\partial_z(u_\varepsilon - \phi_H)|^2 + \varepsilon^2|\nabla v_\varepsilon|^2)dxdzdt$$

$$- \int_0^\tau\int_\Omega (div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)dxdzdt$$

$$+ \int_0^\tau\int_\Omega \rho_\varepsilon(D_x\phi_H : \nabla_x(u_\varepsilon - \phi_H) + \partial_z\phi_H\cdot\partial_z(u_\varepsilon - \phi_H))dxdzdt$$

$$\leq \int_0^\tau h(t)\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])(t)dt$$

$$+ \lambda\int_0^\tau (\| \nabla_x u_\varepsilon - \nabla_x \phi_H \|^2_{L^2(\Omega)} + \| \partial_z u_\varepsilon - \partial_z \phi_H \|^2_{L^2(\Omega)})dt$$

$$+ \int_0^\tau\int_\Omega \left(\frac{\rho_\varepsilon}{r} - 1\right)(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)dxdzdt$$

$$+ \int_0^\tau\int_\Omega k\rho_\varepsilon(u_\varepsilon| u_\varepsilon | - \phi_H| \phi_H |)(\phi_H - u_\varepsilon)dxdzdt$$

$$+ \int_0^\tau\int_\Omega \varepsilon g\rho_\varepsilon(\phi_3 - v_\varepsilon)dxdzdt$$

$$- \int_0^\tau\int_\Omega \left((\rho_\varepsilon - r)\frac{\partial_t r}{r} + (\rho_\varepsilon - r)div\phi + \frac{\nabla r}{r}\cdot(\rho_\varepsilon U_\varepsilon - r\phi)\right)dxdzdt$$

$$- \int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(\phi_H - u_\varepsilon)\nabla_x r dxdzdt + o(\varepsilon^2). \tag{4.25}$$

Step 3. Now, we estimate the remaining terms in the relative energy inequality (4.25). We have

$$-\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(\phi_H - u_\varepsilon)\nabla_x r + (\rho_\varepsilon - r)\frac{\partial_t r}{r}$$

$$+ (\rho_\varepsilon - r)div\phi + \frac{\nabla r}{r}\cdot(\rho_\varepsilon U_\varepsilon - r\phi)dxdzdt$$

$$= -\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}\phi_H\nabla_x r - \frac{\rho_\varepsilon}{r}u_\varepsilon\nabla_x r + \frac{(\rho_\varepsilon - r)}{r}(\partial_t r + rdiv\phi)$$

$$- \frac{\nabla r}{r}r\phi + \frac{\nabla_x r}{r}\rho_\varepsilon u_\varepsilon + \frac{\partial_z r}{r}\rho_\varepsilon v_\varepsilon dxdzdt$$

$$= -\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}\phi_H\nabla_x r - \frac{(\rho_\varepsilon - r)}{r}\phi\nabla r - \frac{\nabla r}{r}r\phi + \frac{\partial_z r}{r}\rho_\varepsilon v_\varepsilon dxdzdt$$

$$= -\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}\phi_H\nabla_x r - \frac{\rho_\varepsilon}{r}\phi_H\nabla_x r - \frac{\rho_\varepsilon}{r}\phi_3\partial_z r + \frac{\partial_z r}{r}\rho_\varepsilon v_\varepsilon dxdzdt$$

$$= -\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(v_\varepsilon - \phi_3)\partial_z r dxdzdt = 0, \tag{4.26}$$

using the fact that $\partial_t r + r div\phi + \phi\nabla r = 0$ and the periodic boundary condition. Moreover, using the inequality of Cauchy, we get from (3.17) that

$$\int_\Omega \varepsilon\rho_\varepsilon(\phi_3 - v_\varepsilon)dxdz = \int_\Omega \varepsilon\sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}(\phi_3 - v_\varepsilon)dxdz$$

$$\leq \int_\Omega \frac{1}{2}\rho_\varepsilon dxdz + \int_\Omega \frac{\varepsilon^2}{2}\rho_\varepsilon|\phi_3 - v_\varepsilon|^2 dxdz$$

$$\leq C\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]). \tag{4.27}$$

Furthermore, using again Cauchy's inequality, we have

$$k\int_\Omega \rho_\varepsilon(u_\varepsilon|u_\varepsilon| - \phi_H|\phi_H|)(\phi_H - u_\varepsilon)dxdz$$

$$\leq k\int_\Omega \frac{1}{2}\rho_\varepsilon(u_\varepsilon|u_\varepsilon| - \phi_H|\phi_H|)^2 dxdz$$

$$+ k\int_\Omega \frac{1}{2}\rho_\varepsilon|\phi_H - u_\varepsilon|^2 dxdz$$

$$\leq k\int_\Omega \frac{1}{2}\rho_\varepsilon(u_\varepsilon|u_\varepsilon| - \phi_H|\phi_H|)^2 dxdz + C\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]). \tag{4.28}$$

We decompose the term

$$\int_\Omega \left(\frac{\rho_\varepsilon}{r} - 1\right)(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)dxdz$$

of (4.25) into two parts and we use the regularity of the strong solution to get

$$\left|\int_\Omega \frac{1}{r}(\rho_\varepsilon - r)(div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)dxdzdt\right|$$

$$\leq C\int_\Omega |\rho_\varepsilon - r|_{ess}|\phi_H - u_\varepsilon|dxdz$$

$$+ C\int_\Omega |\rho_\varepsilon - r|_{res}|\phi_H - u_\varepsilon|dxdz = K_1 + K_2. \tag{4.29}$$

According to (3.16) using the inequality of Cauchy, we have

$$K_1 \le C\int_\Omega \frac{1}{\sqrt{\rho_\varepsilon}} | \rho_\varepsilon - r |_{ess} \sqrt{\rho_\varepsilon} | \phi_H - u_\varepsilon | dxdz$$

$$\le C\int_\Omega (| \rho_\varepsilon - r |_{ess}^2 + \rho_\varepsilon | \phi_H - u_\varepsilon |^2) dxdz$$

$$\le C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]). \tag{4.30}$$

We split K_2 as follows:

$$K_2 = C\int_{\rho_\varepsilon > 2\overline{r}} | \rho_\varepsilon - r || \phi_H - u_\varepsilon | dxdz$$

$$+ C\int_{\rho_\varepsilon < \frac{1}{2}\underline{r}} | \rho_\varepsilon - r || \phi_H - u_\varepsilon | dxdz,$$

where the first integral may be treated in the same way as K_1 and second integral is estimated with the help of the inequality of Poincaré as follows:

$$C\int_{\rho_\varepsilon < \frac{1}{2}\underline{r}} | \rho_\varepsilon - r || \phi_H - u_\varepsilon | dxdz$$

$$\le C\int_\Omega | \phi_H - u_\varepsilon | dxdz$$

$$\le C\int_\Omega 1_{res}^2 dxdz + \lambda \int_\Omega | \phi_H - u_\varepsilon |^2 dxdz$$

$$\le C\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi]) + \lambda \| \nabla_x u_\varepsilon - \nabla_x \phi_H \|_{L^2(\Omega)}^2$$

$$+ \lambda \| \partial_z u_\varepsilon - \partial_z \phi_H \|_{L^2(\Omega)}^2. \tag{4.31}$$

Furthermore, as $D\phi_H$ is bounded,

$$-\int_\Omega (div_x(rD_x\phi_H) + \partial_z(r\partial_z\phi_H))(\phi_H - u_\varepsilon)dxdz$$

$$+\int_\Omega \rho_\varepsilon(D_x\phi_H : \nabla_x(u_\varepsilon - \phi_H) + \partial_z\phi_H \cdot \partial_z(u_\varepsilon - \phi_H))dxdz$$

$$= \int_\Omega rD_x\phi_H : \nabla_x(\phi_H - u_\varepsilon) + r\partial_z\phi_H(\partial_z\phi_H - \partial_z u_\varepsilon)dxdz$$

$$+\int_\Omega \rho_\varepsilon(D_x\phi_H : \nabla_x(u_\varepsilon - \phi_H) + \partial_z\phi_H \cdot \partial_z(u_\varepsilon - \phi_H))dxdz$$

$$= \int_\Omega (\rho_\varepsilon - r)D\phi_H : \nabla(u_\varepsilon - \phi_H)dxdz$$

$$\leq \frac{1}{2}\| D\phi_H \|_{L^\infty(\Omega)}\left(\int_\Omega (\rho_\varepsilon - r)^2 dxdz + \int_\Omega |\nabla(u_\varepsilon - \phi_H)|^2 dxdz\right)$$

$$\leq C\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi]) + \lambda\| \nabla_x u_\varepsilon - \nabla_x\phi_H \|^2_{L^2(\Omega)}$$

$$+ \lambda\| \partial_z u_\varepsilon - \partial_z\phi_H \|^2_{L^2(\Omega)}. \tag{4.32}$$

Therefore, the relative entropy inequality can be written as

$$\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])(\tau) + C\int_0^\tau \left(\| \rho_\varepsilon^{\frac{1}{2}}(Du_\varepsilon - D\phi_H)\|^2_{L^2} + \varepsilon^2\| \rho_\varepsilon^{\frac{1}{2}}\nabla v_\varepsilon \|^2_{L^2}\right)dt$$

$$\leq \int_0^\tau h_\varepsilon(t)\xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])(t)dt + \xi([\rho_\varepsilon, U_\varepsilon]|[r, \phi])(0)$$

$$+ \lambda\int_0^\tau \| \nabla u_\varepsilon - \nabla\phi_H \|^2_{L^2} dt$$

$$+ \int_0^\tau\int_\Omega \frac{k}{2}\rho_\varepsilon(u_\varepsilon|u_\varepsilon| - \phi_H|\phi_H|)^2 dxdzdt + o(\varepsilon^2). \tag{4.33}$$

When ε tends towards 0 in (4.33), we get

$$\lim_{\varepsilon\to 0} \xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(\tau) \leq \lim_{\varepsilon\to 0} C\int_0^\tau h_\varepsilon(t)\xi([\rho_\varepsilon, U_\varepsilon] | [r, \phi])(t)\,dt. \quad (4.34)$$

Then applying the Gronwall's inequality, the proof of Theorem 3.1 is complete.

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