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ADOMIAN'S METHOD FOR SOLVING A NONLINEAR EPIDEMIC MODEL

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Abstract

This paper solves the susceptible-infected-recovered (SIR) model by means of the Adomian decomposition method (ADM). The ADM provides series solutions for the infected and recovered individuals. Such series solutions transform to exact ones under certain constraints of the initial conditions. In addition, closed form solutions are obtained for the infected and recovered individuals by rearranging the components of the ADM series. The accuracy is examined via comparing our results with an accurate numerical method. Agreement between our results and those of the numerical method is achieved.

1. Introduction

Mathematical modeling of infectious diseases gained a considerable interest during the past two decades [1-3]. At the beginning of 2020, a special attention was given to the spread of COVID-19 [4-6]. Later, numerous mathematical models have been formulated in both classical and fractional forms to investigate the progress of COVID-19 in several countries [8]. The most common mathematical model describing the mechanism of Corona virus can be viewed as a consequence of the susceptible-infected-recovered (SIR) model. So, the SIR model is essential/fundamental to express the spread of various epidemics in terms of linear or nonlinear ordinary differential equations (ODEs). The simplest nonlinear SIR model was initially formulated in references [5, 6] via a system of ODEs, given by

$$\frac{dR}{d\tau} = I(\tau), \tag{1}$$

$$\frac{dI}{d\tau} = \sigma[1 - R(\tau) - I(\tau)]\, I(\tau) - I(\tau), \tag{2}$$

where $\tau = t/T$, t is the time in days and T is the time of transmission of the virus. The infected and the recovered individuals are represented by $I(t)$ and $R(t)$, respectively, where $S(t)$ denotes the susceptible individuals: $S(t) = 1 - R(t) - I(t)$ while σ is the transmission rate (physical contact number between susceptible and infected individuals). The model is subjected to the initial conditions (ICs) [6]:

$$R(0) = A, \quad I(0) = B. \tag{3}$$

The Adomian decomposition method (ADM) [9-18] and the homotopy perturbation method (HPM) [19-22] are well-known analytical approaches to solve linear/nonlinear ODEs. However, these methods give the same series solution if they are applied on a specific canonical form of an ODE. The main difference between the ADM and the HPM can be explained as follows. For an ODE, the ADM constructs the series solution through a single recurrence scheme while the HPM decomposes/divides the given ODE to a set of sequential ODEs by means of an auxiliary parameter, each ODE of such a set has to be solved independently. Therefore, the ADM is preferred in this work to directly deal with the system (1)-(3). The next section is devoted to apply the ADM on the current system. In a subsequent section, it will be shown that the series solution obtained by the ADM can be compacted in closed-form. The validity of the present approximations is examined via comparing our results with an accurate numerical method.

2. Application of the ADM

Integrating equations (1) and (2) once with respect to τ and using the ICs in equations (3), we obtain

$$R(\tau) = A + \int_0^\tau I(\mu)\,d\mu, \tag{4}$$

$$I(\tau) = B + \int_0^\tau [(\sigma - 1)I(\mu) - \sigma(R(\mu) + I(\mu))I(\mu)]\,d\mu. \tag{5}$$

The ADM assumes $R(\tau)$ and $I(\tau)$ in the forms:

$$R(\tau) = \sum_{n=0}^{\infty} R_n(\tau), \quad I(\tau) = \sum_{n=0}^{\infty} I_n(\tau). \tag{6}$$

Employing equations (6) into equations (4) and (5), we obtain the recurrence scheme:

$$R_0(\tau) = A, \quad I_0(\tau) = B, \tag{7}$$

$$R_n(\tau) = \int_0^\tau I_{n-1}(\mu)d\mu, \tag{8}$$

$$I_n(\tau) = \int_0^\tau \left[(\sigma - 1)I_{n-1}(\mu) - \sigma \sum_{k=0}^{n-1}(R_k(\mu) + I_k(\mu))I_{n-k-1}(\mu)\right]d\mu, \; n \geq 1. \tag{9}$$

Implementing this algorithm for $n = 1$ gives

$$R_1(\tau) = B\tau, \quad I_1(\tau) = -B\Omega\tau, \tag{10}$$

where Ω is the constant:

$$\Omega = 1 + \sigma(B + A - 1). \tag{11}$$

Hence, the first few components of R_n and $I_n(n \geq 0)$ can be summarized as

$$R_0(\tau) = A,$$

$$R_1(\tau) = B\tau,$$

$$R_2(\tau) = -B\Omega\frac{\tau^2}{2!},$$

$$R_3(\tau) = [B\Omega^2 + \sigma B^2(\Omega - 1)]\frac{\tau^3}{3!},$$

$$R_4(\tau) = [-B\Omega^3 - 4\sigma B^2\Omega(\Omega - 1) - \sigma^2 B^3(\Omega - 1)]\frac{\tau^4}{4!},$$

$$R_5(\tau) = [B\Omega^4 + 11\sigma B^2\Omega^2(\Omega - 1)$$

$$+ \sigma^2 B^3(\Omega - 1)(11\Omega - 4) + \sigma^3 B^4(\Omega - 1)]\frac{\tau^5}{5!}, \tag{12}$$

and

$$I_0(\tau) = B,$$

$$I_1(\tau) = -B\Omega\tau,$$

$$I_2(\tau) = [B\Omega^2 + \sigma B^2(\Omega - 1)]\frac{\tau^2}{2!},$$

$$I_3(\tau) = [-B\Omega^3 - 4\sigma B^2\Omega(\Omega - 1) - \sigma^2 B^3(\Omega - 1)]\frac{\tau^3}{3!},$$

$$I_4(\tau) = [B\Omega^4 + 11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4) + \sigma^3 B^4(\Omega - 1)]\frac{\tau^4}{4!}. \tag{13}$$

In view of the above components, we obtain the following series solutions for $R(\tau)$ and $I(\tau)$:

$$R(\tau) = A + B\tau - B\Omega\frac{\tau^2}{2!} + [B\Omega^2 + \sigma B^2(\Omega - 1)]\frac{\tau^3}{3!} + [-B\Omega^3 - 4\sigma B^2\Omega(\Omega - 1) - \sigma^2 B^3(\Omega - 1)]\frac{\tau^4}{4!} + [B\Omega^4 + 11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4) + \sigma^3 B^4(\Omega - 1)]\frac{\tau^5}{5!} + \ldots, \tag{14}$$

and

$$I(\tau) = B - B\Omega\tau + [B\Omega^2 + \sigma B^2(\Omega - 1)]\frac{\tau^2}{2!} + [-B\Omega^3 - 4\sigma B^2\Omega(\Omega - 1) - \sigma^2 B^3(\Omega - 1)]\frac{\tau^3}{3!} + [B\Omega^4 + 11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4) + \sigma^3 B^4(\Omega - 1)]\frac{\tau^4}{4!} + \ldots. \tag{15}$$

Although the higher-order components are available via a software, we can observe that the terms involved in the components (12) and (13) follow certain patterns. Hence, the sum of these components can be emerged in two certain entire functions and this is the subject of the next section.

3. Closed Form Solution

In this section, we show that a closed form solution can be obtained for the current nonlinear COVID-19 model. Before launching the main target of this section, it may be suitable to address some observations about the characteristics of the obtained components. We observe from equations (12) that the first three components $R_n(\tau)\,(n = 0, 1, 2)$ consist of only one term while the higher-order components $R_n(\tau)\,(n = 3, 4, 5, \ldots)$ contain $n-1$ terms. Suppose that $SR_f(\tau)$ is the sum of $R_n(\tau)\,(n = 0, 1, 2)$ in addition to the first term taken from each component $R_n(\tau)\,(n = 3, 4, 5, \ldots).$ Then

$$SR_f(\tau) = A + B\tau - B\Omega\frac{\tau^2}{2!} + B\Omega^2\frac{\tau^3}{3!} - B\Omega^3\frac{\tau^4}{4!} + B\Omega^4\frac{\tau^5}{5!} + \ldots, \quad (16)$$

which can be written as

$$SR_f(\tau) = A - \frac{B}{\Omega}\left[-\frac{B\Omega}{1!} + \frac{(B\Omega)^2}{2!} - \frac{(B\Omega)^3}{3!} + \frac{(B\Omega)^4}{4!} - \frac{(B\Omega)^5}{5!} + \ldots\right]. \quad (17)$$

Thus

$$SR_f(\tau) = A + \frac{B}{\Omega}(1 - e^{-\Omega\tau}). \quad (18)$$

Also, assume that $SR_l(\tau)$ is the sum of the last terms taken from each component $R_n(\tau)\,(n = 3, 4, 5, \ldots).$ Then

$$SR_l(\tau) = \sigma B^2(\Omega - 1)\frac{\tau^3}{3!} - \sigma^2 B^3(\Omega - 1)\frac{\tau^4}{4!} + \sigma^3 B^4(\Omega - 1)\frac{\tau^5}{5!} - \ldots, \quad (19)$$

i.e.,

$$SR_l(\tau) = \frac{\Omega - 1}{\sigma^2 B}\left[\frac{(\sigma B\tau)^3}{3!} - \frac{(\sigma B\tau)^4}{4!} + \frac{(\sigma B\tau)^5}{5!} - \dots\right]$$

$$= \frac{1 - \Omega}{\sigma^2 B}\left[-\frac{(\sigma B\tau)^3}{3!} + \frac{(\sigma B\tau)^4}{4!} - \frac{(\sigma B\tau)^5}{5!} + \dots\right]$$

$$= \frac{1 - \Omega}{\sigma^2 B}\left[e^{-\sigma B\tau} - 1 + \sigma B\tau - \frac{1}{2}\sigma^2 B^2\tau^2\right]. \tag{20}$$

Let $SR_r(\tau)$ denote the sum of the rest of terms of $R(\tau)$. Then

$$SR_r(\tau) = -4\sigma B^2\Omega(\Omega - 1)\frac{\tau^4}{4!}$$

$$+ [11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4)]\frac{\tau^5}{5!} + \dots, \tag{21}$$

and accordingly, we can write

$$R(\tau) = SR_f + SR_l + SR_r, \tag{22}$$

which leads to the following closed form solution:

$$R(\tau) = A + \frac{B}{\Omega}(1 - e^{-\Omega\tau})$$

$$+ \frac{1 - \Omega}{\sigma^2 B}\left[e^{-\sigma B\tau} - 1 + \sigma B\tau - \frac{1}{2}\sigma^2 B^2\tau^2\right] - 4\sigma B^2\Omega(\Omega - 1)\frac{\tau^4}{4!}$$

$$+ [11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4)]\frac{\tau^5}{5!}. \tag{23}$$

Similarly, the first two components in equations (13) for $I_n(\tau)(n = 0, 1)$ consist of only one term while $I_n(\tau)(n = 2, 3, 4, \dots)$ contain n terms. Let $SI_f(\tau)$ be the sum of the first two components $I_n(\tau)(n = 0, 1)$ in addition to the first term taken from each higher-order component $I_n(\tau)(n = 2, 3, 4, ...)$. Then

$$SI_f(\tau) = B - B\Omega\tau + B\Omega^2 \frac{\tau^2}{2!} - B\Omega^3 \frac{\tau^3}{3!} + B\Omega^4 \frac{\tau^4}{4!} - \dots, \tag{24}$$

which is

$$SI_f(\tau) = Be^{-\Omega\tau}. \tag{25}$$

Beside, the sum $SI_l(\tau)$ of the last terms selected from the higher-order component $I_n(\tau)(n = 2, 3, 4, ...)$ gives

$$\begin{aligned} SI_l(\tau) &= (\Omega - 1)\left[\sigma B^2 \frac{\tau^2}{2!} - \sigma^2 B^3 \frac{\tau^3}{3!} + \sigma^3 B^4 \frac{\tau^4}{4!} - \dots\right] \\ &= \frac{\Omega - 1}{\sigma}\left[\frac{(\sigma B\tau)^2}{2!} - \frac{(\sigma B\tau)^3}{3!} + \frac{(\sigma B\tau)^4}{4!} - \dots\right] \\ &= \frac{\Omega - 1}{\sigma}[e^{-\sigma B\tau} - 1 + \sigma B\tau]. \end{aligned} \tag{26}$$

The sum $SI_r(\tau)$ of the rest of terms of $I(\tau)$ is as follows:

$$\begin{aligned} SI_r(\tau) = &-4\sigma B^2\Omega(\Omega - 1)\frac{\tau^3}{3!} \\ &+ [11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4)]\frac{\tau^4}{4!} + \dots. \end{aligned} \tag{27}$$

Hence,

$$I(\tau) = SI_f + SI_l + SI_r, \tag{28}$$

and this gives $I(\tau)$ in the following closed form solution:

$$\begin{aligned} I(\tau) = Be^{-\Omega\tau} &+ \frac{\Omega - 1}{\sigma}[e^{-\sigma B\tau} - 1 + \sigma B\tau] - 4\sigma B^2\Omega(\Omega - 1)\frac{\tau^3}{3!} \\ &+ [11\sigma B^2\Omega^2(\Omega - 1) + \sigma^2 B^3(\Omega - 1)(11\Omega - 4)]\frac{\tau^4}{4!}. \end{aligned} \tag{29}$$

It may be important to refer the expression (29) for the infected individuals $I(\tau)$ which is actually the derivative of the recovered individuals $R(\tau)$ obtained in equation (23). In addition, the present closed form solutions satisfy the ICs (3). Furthermore, the above closed solutions have not been reported in the relevant literature for the current nonlinear model of COVID-19. Moreover, the accuracy of the current analysis is examined in the subsequent subsection.

3.1. Special case: Exact solution

Here, we show that the closed form solutions (23) and (29) reduce to exact forms at a prescribed physical condition. Assume that the sum of the initial values of the susceptible and infected individuals is unity, i.e., $A + B = 1$. It corresponds to the zero initial susceptible individuals $S(0) = 0$ (since $S(0) = 1 - R(0) - I(0) = 1 - A - B$). Thus equation (11) implies $\Omega = 1$. Employing this value of Ω in equations (23) and (29), respectively, yields

$$R(\tau) = A + B(1 - e^{-\tau}) \tag{30}$$

and

$$I(\tau) = Be^{-\tau}. \tag{31}$$

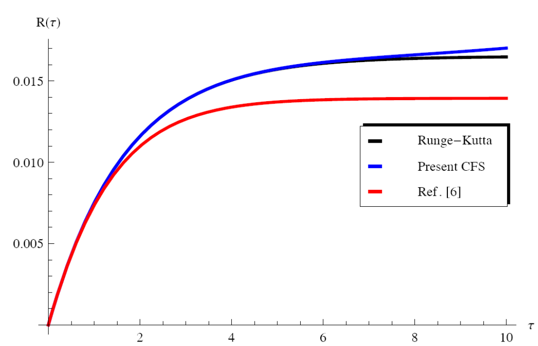

Figure 1. Comparison between the CFS (present, equation (23)), the numerical solution (Runge-Kutta), and the HPM (reference [6]) at $A = 0$, $B = 0.01$ and $\sigma = 0.4$ for $R(\tau)$.

It can be noted from equations (30) and (31) that the exact solutions at the considered special case are independent of the contact number σ. Also, such exact solutions can be derived when σ vanishes. This means that the special case $A + B = 1$ is identical to the case $\sigma = 0$.

4. Validations

The objective of this section is to extract numerical results for the purpose of performing comparisons with two analytical and numerical approaches in the literature. In reference [6], the following approximations were determined via the HPM

$$
\begin{aligned}
R(t) = {} & A - Be^{-\tau} + B \\
& + \sigma B\left[- B(- \tau e^{-\tau} - e^{-\tau}) - \frac{1}{2} Be^{-2\tau} + Be^{-\tau} - \tau e^{-\tau} - e^{-\tau}\right] \\
& - \sigma B\left(\frac{3}{2} B - 1\right),
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
I(t) = {} & Be^{-\tau} + e^{-\tau}[- \sigma B(B\tau - Be^{-\tau} - \tau) - \sigma B^2] \\
& - \frac{1}{2} \sigma Be^{-\tau}(4\sigma B^2 - 2\sigma B + 2B) \\
& - \frac{1}{2} \sigma Be^{-\tau}[4\sigma B^2 \tau e^{-\tau} + 6\sigma B^2 e^{-\tau} - 2\sigma B^2 e^{-2\tau} \\
& - \sigma B^2 \tau^2 - 2\sigma B^2 \tau - 2\sigma Be^{-\tau} + 2\sigma B\tau \\
& + 2\sigma B\tau^2 - 4\sigma B\tau e^{-\tau} + 2B\tau - 2Be^{-\tau} - \sigma\tau^2].
\end{aligned} \tag{33}
$$

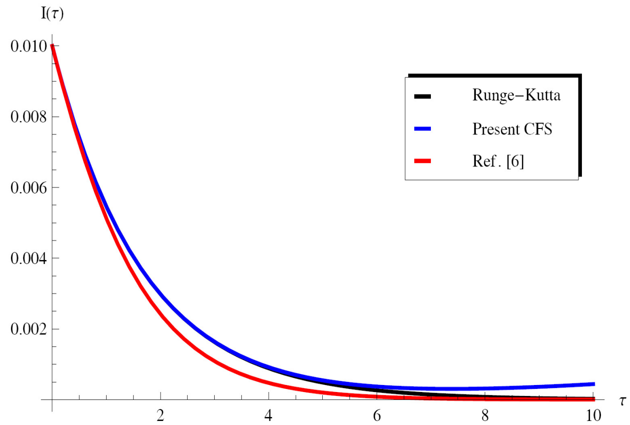

Figure 2. Comparison between the CFS (present, equation (29)), the numerical solution (Runge-Kutta), and the HPM (reference [6]) at $A = 0$, $B = 0.01$ and $\sigma = 0.4$ for $I(\tau)$.

Figure 1 shows the comparison between the CFS (present, equation (23)), the numerical solution using MATHEMATICA (Runge-Kutta), and the HPM (reference [6]) at $A = 0$, $B = 0.01$ and $\sigma = 0.4$ for $R(\tau)$. Also, Figure 2 displays the comparison between the CFS (present, equation (29)), the numerical solution, and the HPM (reference [6]) at the same values of parameters for $I(\tau)$. These figures reveal that our analysis enjoys better accuracy when compared with the HPM [6].

5. Conclusions

This paper solved the susceptible-infected-recovered (SIR) model by applying the ADM. The obtained series solutions for the infected and recovered individuals were constructed in closed-forms. These closed-forms are transformed to exact solutions under certain conditions of the given parameters. The accuracy of the obtained results was validated via comparison with an accurate numerical method. In addition, the comparison between our results and those through other analytical methods in the literature reveals the advantages of the present analysis.

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