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AN EFFICIENT SCHEME TO SOLVE FOURTH ORDER NONLINEAR TRIPLY SINGULAR FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract

We present a novel mathematical model based on the fourth order multi-singular nonlinear functional differential equations. This designed nonlinear functional model has singularities at three points,

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making the model more complicated and harder in nature. The delayed and multi-prediction terms in the model clearly represent the functionality of the model. Three different variants of the novel nonlinear triply singular functional differential model have been presented and the numerical results of each variant are obtained by using a well-known spectral collocation technique. For the perfection and excellence of the designed mathematical nonlinear model, the obtained numerical results of each variant have been compared with the exact solutions.

1. Introduction

The study of functional differential (FD) equations have achieved a huge attention of researchers due to its variety of applications in biological, physical, engineering, social sciences, medical, finance and economics. Some of the wider range applications of FD equations are used in aggregate data problems [1], electrodynamics [2], simulation of population growth [3], HIV-1 infection model [4], tumor growth biological model [5], chemical kinetics system [6], infection hepatitis model of B-virus [7], gene regulations system [8] and viral infections system model [9]. Moreover, the singular differential equations are very important for the research community and play a vital role in the phenomena of astrophysics, spherical cloud of gas, science, engineering and technology. The singular study for the differential models is very interesting and experimental for researchers due to the singularity appearance at origin. Some wide-ranging applications of singular models are stellar structure model [10], thermal explosions system [11], model based on isothermal gas spheres [12], thermionic currents [13], oscillating magnetic regions [14], classical as well as quantum mechanics [15], isotropic continuous media [16], dusty fluid based models [17] and in the study of morphogenesis [18]. In recent few decades, the researcher's community is interested to solve and produce the numerical outcomes of FD and singular FD models due to the strength and importance of these models. There are many numerical existing techniques that have been used to solve the second order singular FD equations; some of them are a numerical approach explained by two mathematicians Kadalbajoo and Sharma [19, 20]. Mirzaee and Hoseini [21] used a collocation method with Fibonacci polynomials for solving singularly perturbed differential-difference equations. Sabir et al. [22] used differential transformation scheme for solving third-order nonlinear multi-singular functional differential equations. Xu and Jin [23] executed the boundary functions and fractional steps scheme for solving the model based on singular perturbed functional differential systems. Geng et al. [24] functioned a numerical scheme for presenting the solution of delay differential models using the singularly perturbation. The aim and task of the present study is to present the mathematical model named as fourth order nonlinear multi/triply-singular (MS) functional differential system (FDS), i.e., MS-FDS and present the numerical outcomes of the model by using a well-known numerical spectral collocation scheme. The general form of the novel model fourth order nonlinear MS-FDS is given as:

$$\frac{d^4}{d\chi^4}(\mathcal{Z}(\chi-\tau_1))+\frac{a_1}{\chi}\frac{d^3}{d\chi^3}(\mathcal{Z}(\chi+\tau_2))+\frac{a_2}{\chi^2}\frac{d^2}{d\chi^2}(\mathcal{Z}(\chi+\tau_3))$$

$$+\frac{a_3}{\chi^3}\frac{d}{d\chi}(\mathcal{Z}(\chi+\tau_3))+\chi\mathcal{Z}(\chi)=\mathcal{F}(\chi),$$

$$\mathcal{Z}(0)=\delta_1,\ \frac{d\mathcal{Z}(\chi)}{d\chi}|_{\chi=0}=\delta_2,\quad \frac{d^2\mathcal{Z}(\chi)}{d\chi^2}|_{\chi=0}=\delta_3,\quad \frac{d^3\mathcal{Z}(\chi)}{d\chi^3}|_{\chi=0}=\delta_4,$$

where τ_i, $i = 1, 2, 3, 4$ show the positive parameter values. Moreover, δ_i, $i = 1, 2, 3, 4$ are the positive constant values. The above novel designed nonlinear FD model has triply singularities at zero, that makes the model more harder and complicated in nature. The above fourth order nonlinear model is obtained by extending the impressive work of Sabir et al. [22], which is about to exploit the second and third orders nonlinear model based on nonlinear singular FD equation. For the clarification of the designed novel model, three different problems have been discussed and numerically solved by applying a famous numerical spectral collocation scheme.

The prime factors of the recent study are briefly provided as:

• The mathematical formulation of fourth order nonlinear triply singular FD model is developed effectively by extending the research work of Sabir et al. [22].

• The novel designed fourth order nonlinear triply singular FD model is solved numerically by using the spectral collocation scheme.

• For the clarification of the novel designed nonlinear triply singular FD model, the obtained numerical outcomes from the spectral collocation scheme are compared with the exact solution of each problem. The matching of the obtained and exact results designates the excellence of the designed novel model.

• Influence of the present scheme for solving the fourth order nonlinear triply singular FD model with higher accuracy and precision as well as outstanding reliability.

• The above fourth order FD model represented by equation (1) stated below is not easy to design and solve due to triply singularity, higher nonlinearity and functional nature. The spectral collocation scheme is a great selection and better choice to solve numerically such types of complex models.

Spectral methods [26-29], distinguished by their comprehensive and exponentially convergent characteristics, demonstrate superiority when compared to alternative numerical techniques. A key aspect common to all spectral methods is the representation of the solution to the problem as a finite series of various functions [30, 31]. Spectral methods encompass diverse techniques, including collocation [33], tau [34], Galerkin [35], and Petrov-Galerkin [36].

The remainder of the paper is supplied as follows: Section 2 provides the specifics of the developed methodology based on the spectral collocation scheme. Section 3 contains detailed numerical results. The last section includes a list of some closing thoughts and potential directions for future research.

2. Shifted Jacobi Collocation Method

For an accurate solution, the weighted residual collocation approach is essential [37-40]. This paper presents a numerical approach to solve a nonlinear singular second order coupled functional differential model by means of the shifted Jacobi collocation method:

$$\frac{d^4}{d\chi^4}(\mathcal{Z}(\chi-\tau_1))+\frac{a_1}{\chi}\frac{d^3}{d\chi^3}(\mathcal{Z}(\chi+\tau_2))+\frac{a_2}{\chi^2}\frac{d^2}{d\chi^2}(\mathcal{Z}(\chi+\tau_3))$$

$$+\frac{a_3}{\chi^3}\frac{d}{d\chi}(\mathcal{Z}(\chi+\tau_3))+\chi\mathcal{Z}(\chi)=\mathcal{F}(\chi),$$

$$\mathcal{Z}(0)=\delta_1,\ \frac{d\mathcal{Z}(\chi)}{d\chi}|_{\chi=0}=\delta_2,\quad \frac{d^2\mathcal{Z}(\chi)}{d\chi^2}|_{\chi=0}=\delta_3,\quad \frac{d^3\mathcal{Z}(\chi)}{d\chi^3}|_{\chi=0}=\delta_4,$$

$$0\le x\le \mathcal{L}, \tag{1}$$

where τ_i, $i=1, 2, 3, 4$ show the positive parameter values. Moreover, δ_i, $i=1, 2, 3, 4$ are the positive constant values. The solution of equation (1) is approximated as

$$\mathcal{Z}_{\mathcal{K}}(\chi)=\sum_{j=0}^{\mathcal{K}}\varsigma_j\mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(\chi)=\Delta_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(\chi). \tag{2}$$

The shifted Jacobi collocation method is used at $\chi_{\mathcal{L},\mathcal{K},j}^{(\rho,\sigma)}$ nodes to approximate the independent variable. The values of the dependent variable are assumed to be Jacobi-Gauss-Lobatto nodes. We estimate the necessary derivatives of the first and second orders of the approximate solutions:

$$\begin{aligned}
\mathcal{Z}'_{\mathcal{K}}(\chi+\tau_4) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \left(\mathcal{J}^{(\rho,\sigma)}_{\mathcal{L},j}(\chi+\tau_4)\right)' \\
&= \sum_{j=1}^{\mathcal{K}} \varsigma_j \, \frac{j+\rho+\sigma+1}{\mathcal{L}} \, \mathcal{P}^{(\rho+1,\sigma+1)}_{\mathcal{L},j-1}(\chi+\tau_4) \\
&= \wp^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_4}(\chi).
\end{aligned} \tag{3}$$

Also, we get

$$\begin{aligned}
\mathcal{Z}''_{\mathcal{K}}(\chi+\tau_3) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \left(\mathcal{J}^{(\rho,\sigma)}_{\mathcal{L},j}(\chi+\tau_3)\right)'' \\
&= \sum_{j=2}^{\mathcal{K}} \varsigma_j \, \frac{\Gamma(j+\rho+\sigma+3)}{\mathcal{L}^2\Gamma(j+\rho+\sigma+1)} \, \mathcal{P}^{(\rho+2,\sigma+2)}_{\mathcal{L},j-2}(\chi+\tau_3) \\
&= \Xi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_3}(\chi),
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathcal{Z}'''_{\mathcal{K}}(\chi+\tau_2) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \left(\mathcal{J}^{(\rho,\sigma)}_{\mathcal{L},j}(\chi+\tau_2)\right)''' \\
&= \sum_{j=3}^{\mathcal{K}} \varsigma_j \, \frac{\Gamma(j+\rho+\sigma+4)}{\mathcal{L}^3\Gamma(j+\rho+\sigma+1)} \, \mathcal{P}^{(\rho+3,\sigma+3)}_{\mathcal{L},j-3}(\chi+\tau_2) \\
&= \Upsilon^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_2}(\chi),
\end{aligned} \tag{5}$$

$$\begin{aligned}
\mathcal{Z}^{(iv)}_{\mathcal{K}}(\chi-\tau_1) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \left(\mathcal{J}^{(\rho,\sigma)}_{\mathcal{L},j}(\chi-\tau_1)\right)^{(iv)} \\
&= \sum_{j=4}^{\mathcal{K}} \varsigma_j \, \frac{\Gamma(j+\rho+\sigma+5)}{\mathcal{L}^4\Gamma(j+\rho+\sigma+1)} \, \mathcal{P}^{(\rho+4,\sigma+4)}_{\mathcal{L},j-4}(\chi-\tau_1) \\
&= \Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_1}(\chi).
\end{aligned} \tag{6}$$

Then, we can estimated the residual of (1) as

$$\Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_1}(\chi) + \frac{a_1}{\chi}\Upsilon^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_2}\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i} + \frac{a_2}{\chi^2}\Xi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_3}\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}$$

$$+ \frac{a_3}{\chi^3}\wp^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_4}\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i} = \mathcal{F}(\chi). \tag{7}$$

At the $\mathcal{K} - 3$ points, the residual (7) is

$$\Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_1}(\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) + \frac{a_1}{\chi}\Upsilon^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_2}(\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) + \frac{a_2}{\chi^2}\Xi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_3}(\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i})$$

$$+ \frac{a_3}{\chi^3}\wp^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},\tau_4}(\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) = \mathcal{F}(\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) \tag{8}$$

and

$$\Delta^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(\mathcal{L}) = \delta_1, \quad \wp^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},0}(0) = \delta_2,$$

$$\Xi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},0}(\mathcal{L}) = \delta_3, \quad \Upsilon^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},0}(0) = \delta_4. \tag{9}$$

For unknown coefficients ς_j, $j = 0, ..., \mathcal{K}$, the system of nonlinear algebraic equations derived from equations (8) and (9) can be solved.

3. Results and Discussions

Here, the numerical solutions of the three nonlinear examples based on the designed model MS-FDS are presented. The study of nonlinear equations is considered very significant and has many applications.

3.1. Problem I

The fourth order MS-FDS involving trigonometric functions is given by

$$\frac{d^4}{d\chi^4}(\mathcal{Z}(\chi-1)) + \frac{1}{\chi}\frac{d^3}{d\chi^3}(\mathcal{Z}(\chi+1)) + \frac{2}{\chi^2}\frac{d^2}{d\chi^2}(\mathcal{Z}(\chi+2))$$

$$+ \frac{3}{\chi^3}\frac{d}{d\chi}(\mathcal{Z}(\chi+3)) + \chi\mathcal{Z}(\chi) = \mathcal{F}(\chi),$$

$$\mathcal{Z}(0) = 1, \quad \frac{d\mathcal{Z}(\chi)}{d\chi}|_{\chi=0} = 0, \quad \frac{d^2\mathcal{Z}(\chi)}{d\chi^2}|_{\chi=0} = -1, \quad \frac{d^3\mathcal{Z}(\chi)}{d\chi^3}|_{\chi=0} = 0.$$

By using $\mathcal{Z}(\chi) = \cos(\chi)$, the $\mathcal{F}(\chi)$ is selected to provide the precise answer. Table 1 contains a list of Problem I's maximum absolute errors along with more precise results. Figure 1 displays the perfect matching between the approximate and exact solutions. Taking $\rho = \sigma = -\frac{1}{2}$, we obtain the numerical solution of Problem I as:

$$\begin{aligned}\mathcal{Z}_{15}(\chi) = 1 &- 1.15757 \times 10^{-17} x - 0.5x^2 - 2.89346 \times 10^{-17} x^3 \\ &+ 0.0416666x^4 + 2.62497 \times 10^{-8} x^5 \\ &- 0.00138888x^6 - 2.5714 \times 10^{-10} x^7 \\ &+ 0.000024801x^8 - 6.41827 \times 10^{-11} x^9 \\ &- 2.75609 \times 10^{-7} x^{10} + 1.20075 \times 10^{-12} x^{11} \\ &+ 2.08911 \times 10^{-9} x^{12} + 7.11339 \times 10^{-12} x^{13} \\ &- 1.49738 \times 10^{-11} x^{14} + 6.42075 \times 10^{-13} x^{15}. \qquad (10)\end{aligned}$$

Table 1. Maximum absolute errors $\mathcal{M}_E$ of Problem I

$\mathcal{K}$	(0, 0)	$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$
5	2.34671×10^{-3}	2.27211×10^{-3}	2.31724×10^{-3}	2.44649×10^{-3}	2.41107×10^{-3}
10	3.08203×10^{-8}	3.00505×10^{-8}	2.88573×10^{-8}	3.33498×10^{-3}	3.15241×10^{-3}
15	8.95676×10^{-8}	6.33278×10^{-8}	1.08041×10^{-7}	1.33052×10^{-7}	1.18489×10^{-7}

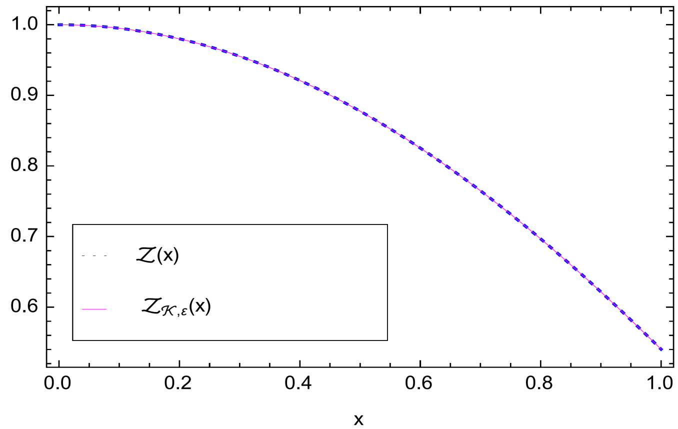

Figure 1. Curves of the exact and numerical solutions $(\mathcal{Z}$ and $\mathcal{Z}_{\mathcal{K}})$ of Problem I, where $\rho = \sigma = -\frac{1}{2}$ and $\mathcal{K} = 15.$

3.2. Problem II

Consider the nonlinear fourth order MS-FDS:

$$\frac{d^4}{d\chi^4}(\mathcal{Z}(\chi-1)) + \frac{1}{\chi}\frac{d^3}{d\chi^3}(\mathcal{Z}(\chi+1)) + \frac{2}{\chi^2}\frac{d^2}{d\chi^2}(\mathcal{Z}(\chi+2)) + \frac{3}{\chi^3}\frac{d}{d\chi}(\mathcal{Z}(\chi+3)) + \chi\mathcal{Z}(\chi) = \mathcal{F}(\chi),$$

$$\mathcal{Z}(0) = 1, \quad \frac{d\mathcal{Z}(\chi)}{d\chi}|_{\chi=0} = 1, \quad \frac{d^2\mathcal{Z}(\chi)}{d\chi^2}|_{\chi=0} = 0, \quad \frac{d^3\mathcal{Z}(\chi)}{d\chi^3}|_{\chi=0} = 0.$$

The formula $\mathcal{F}(\chi) = 1 + \chi + \chi^4$ yields the precise answer. Table 2 displays the maximum absolute errors of the method. Figure 2 shows the optimal matching of approximate and exact solutions achieved by the method. Absolute errors of Problem II, where $\rho = \sigma = 0$ and $K = 4,$ is shown in Figure 3. With $\rho = \sigma = 0,$ the numerical solution to Problem II is obtained:

$$\mathcal{Z}_4(\chi) = 1 + x - 2.22045 \times 10^{-16} x^2 + x^4. \tag{11}$$

Table 2. Maximum absolute errors $\mathcal{M}_E$ of Problem II at $K = 4$

(ρ, σ)	$\mathcal{M}_\mathcal{E}$	(ρ, σ)	$\mathcal{M}_\mathcal{E}$
$\left(\frac{1}{2}, \frac{1}{2}\right)$	4.4964×10^{-15}	$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	8.32667×10^{-16}
$\left(-\frac{1}{2}, 0\right)$	7.77156×10^{-16}	$(1, 0)$	2.44249×10^{-15}
$(0, 0)$	7.21645×10^{-16}	$\left(0, \frac{1}{2}\right)$	8.88178×10^{-16}

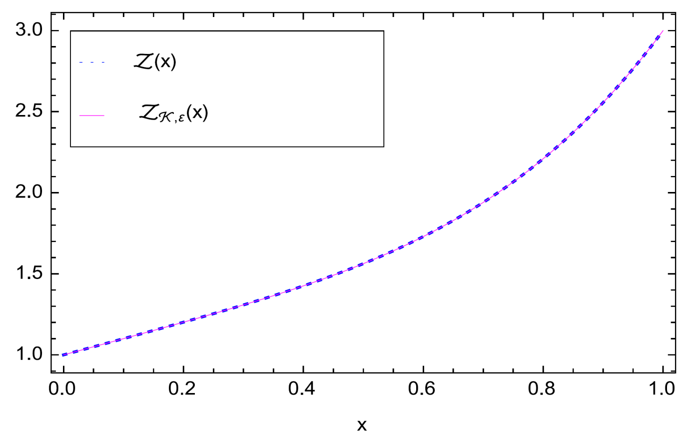

Figure 2. Curves of the exact and numerical solutions $(\mathcal{Z}$ and $\mathcal{Z}_\mathcal{K})$ of Problem II, where $\rho = \sigma = 0$ and $\mathcal{K} = 4$.

3.3. Problem III

Here, we test the fourth order MS-FDS:

$$\frac{d^4}{d\chi^4}(\mathcal{Z}(\chi - 1)) + \frac{1}{\chi}\frac{d^3}{d\chi^3}(\mathcal{Z}(\chi + 1)) + \frac{2}{\chi^2}\frac{d^2}{d\chi^2}(\mathcal{Z}(\chi + 2))$$

$$+ \frac{3}{\chi^3}\frac{d}{d\chi}(\mathcal{Z}(\chi + 3)) + \chi\mathcal{Z}(\chi) = \mathcal{F}(\chi),$$

$$\mathcal{Z}(0) = 1, \quad \frac{d\mathcal{Z}(\chi)}{d\chi}|_{\chi=0} = 1, \quad \frac{d^2\mathcal{Z}(\chi)}{d\chi^2}|_{\chi=0} = 0, \quad \frac{d^3\mathcal{Z}(\chi)}{d\chi^3}|_{\chi=0} = 0.$$

The $\mathcal{F}(\chi)$ is chosen such that the exact solution is given by $\mathcal{Z}(\chi) = 1 + \chi^5$. Table 3 appears the accurate results for the maximum absolute errors $\mathcal{M}_E$ of our method. Also, we see the prefect matching of the approximate and exact solutions in Figure 4. Moreover, the absolute error of Problem II is sketched in Figure 5. Taking $\rho = -\frac{1}{2}$, $\sigma = \frac{1}{2}$, we obtain the numerical solution of Problem II as:

$$\mathcal{Z}_5(\chi) = 1 + 8.88178 \times 10^{-16} x^3 - 3.10862 \times 10^{-15} x^4 + x^5. \tag{12}$$

Table 3. Maximum absolute errors $\mathcal{M}_E$ of Problem III at $K = 5$

(ρ, σ)	$\mathcal{M}_\mathcal{E}$	(ρ, σ)	$\mathcal{M}_\mathcal{E}$
$\left(-\frac{1}{2}, \frac{1}{2}\right)$	1.77636×10^{-15}	$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	2.60902×10^{-15}
$\left(\frac{1}{2}, \frac{1}{2}\right)$	1.08885×10^{-13}	$(1, 0)$	3.33067×10^{-14}
$(0, 0)$	1.08885×10^{-13}	$\left(0, \frac{1}{2}\right)$	2.57572×10^{-14}

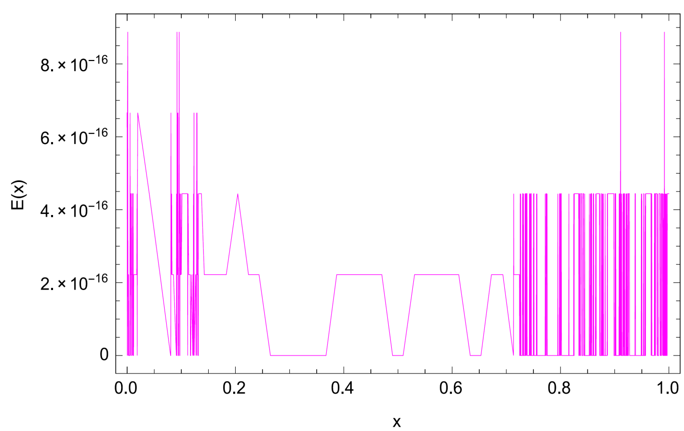

Figure 3. Absolute errors of Problem II, where $\rho = \sigma = 0$ and $\mathcal{K} = 4$.

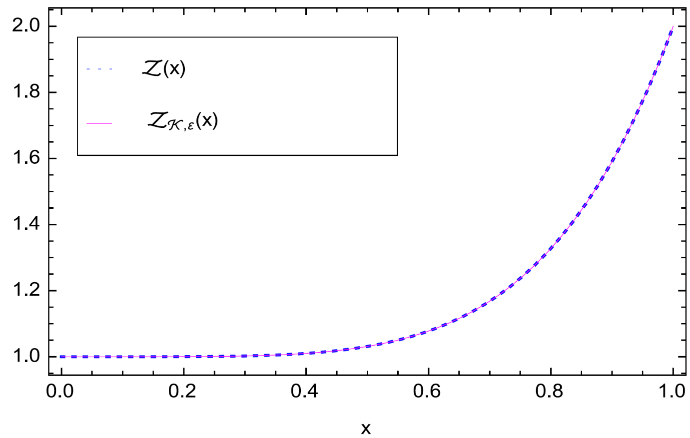

Figure 4. Curves of the exact and numerical solutions $(\mathcal{Z}$ and $\mathcal{Z}_{\mathcal{K}})$ of Problem III, where $\rho = -\frac{1}{2}$, $\sigma = \frac{1}{2}$ and $\mathcal{K} = 5$.

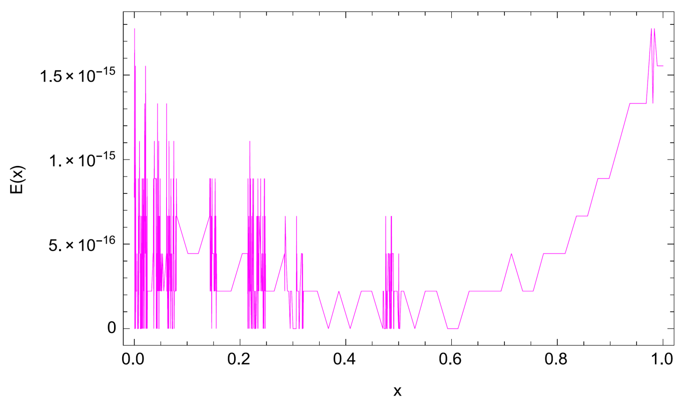

Figure 5. Absolute errors of Problem III, where $\rho = -\frac{1}{2}$, $\sigma = \frac{1}{2}$ and $\mathcal{K} = 5$.

4. Conclusion

The task to design a novel mathematical model based on the fourth order nonlinear triply singular FD equation along with its modeled equations was not easy. However, this model is presented successfully and solved by using the spectral collocation scheme numerically. For the perfection of the designed model, the obtained numerical results using the spectral collocation scheme are compared with the exact solution for each problem. The basic traditional and conventional approaches fail to solve such a complicated and complex designed model because of triple singularity, higher nonlinearity and functional nature. The spectral collocation approach is a great technique and better selection to solve this complex, complicated, nonlinear and triply singular systems. Consequently, the implemented approach is not only effective but appropriate too. The spectral collocation scheme is a fast track convergent scheme, which can be implemented effectively to any type of functional, linear, homogeneous, nonlinear, non-homogeneous or multitype singularities. In future, system of second order, third order and fourth order multi-singular will be designed and can be verified by applying a proposed spectral collocation scheme.

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