Advances in Differential Equations and Control Processes
© 2024 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
http://dx.doi.org/10.17654/0974324324002
Volume 31, Number 1, 2024, Pages 15-26 **P-ISSN: 0974-3243**

DEVELOPING ANSATZ METHOD FOR SOLVING THE NEUTRON DIFFUSION SYSTEM UNDER GENERAL PHYSICAL CONDITIONS

Tami M. T. Alsubie, Abdelhalim Ebaid*, Amjad S. S. Albalawi, Saeed A. Alghamdi, Fawzi F. M. Alhamdi, Omar S. H. Alhamd, Salman M. M. Al-Anzi and Mona Aljoufi

Department of Mathematics
Faculty of Science
University of Tabuk
P.O. Box 741, Tabuk 71491, Saudi Arabia
e-mail: 441000104@stu.ut.edu.sa
aebaid@ut.edu.sa; halimgamil@yahoo.com
41000042@stu.ut.edu.sa
431000297@stu.ut.edu.sa
431000300@stu.ut.edu.sa
441000018@stu.ut.edu.sa
441000072@stu.ut.edu.sa
maljufi@ut.edu.sa; monaeljuofi@yahoo.com

Received: October 25, 2023; Accepted: December 13, 2023

2020 Mathematics Subject Classification: 34A99.

Keywords and phrases: neutron diffusion, partial differential equation, analytic solution, ansatz method.

*Corresponding author

How to cite this article: Tami M. T. Alsubie, Abdelhalim Ebaid, Amjad S. S. Albalawi, Saeed A. Alghamdi, Fawzi F. M. Alhamdi, Omar S. H. Alhamd, Salman M. M. Al-Anzi and Mona Aljoufi, Developing ansatz method for solving the neutron diffusion system under general physical conditions, Advances in Differential Equations and Control Processes 31(1) (2024), 15-26. http://dx.doi.org/10.17654/0974324324002

Published Online: February 12, 2024

Abstract

This paper develops an ansatz method to derive the analytic solution of the neutron flux system under general initial conditions. Explicit closed series forms are established for the neutron flux and the delayed neutron concentration in terms of exponential functions. Existing results in the literature are recovered as special cases.

1. Introduction

In this paper, we consider the following system:

$$\frac{1}{V}\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} + \left(-\sum_a + (1-\beta)\nu \sum_f\right)\phi(x, t) + \lambda C(x, t), \tag{1}$$

$$\frac{\partial C}{\partial t} = \beta\nu \sum_f \phi(x, t) - \lambda C(x, t), \tag{2}$$

where $\phi(x, t)$ and $C(x, t)$ stand for the neutron flux and the delayed neutron concentration, respectively. The parameters V, $\sum_a$, β, ν, $\sum_f$ and λ in the system (1)-(2) were well defined/described in [1, 2]. The system is governed by the boundary conditions (BCs):

$$\phi(0, t) = 0, \quad \phi(L, t) = 0, \quad t > 0, \tag{3}$$

and the following initial conditions (ICs) in general form

$$\phi(x, 0) = f(x), \quad C(x, 0) = g(x), \quad 0 < x < L, \tag{4}$$

where $f(x)$ and $g(x)$ are given real functions. The model (1)-(4) is of great importance in both reactor design and theoretical nuclear physics. Obtaining accurate solution for the above system is essential for the purpose of safety considerations. In the literature, many authors [1-8] employed different techniques to solve the present system under specific constant ICs. These authors implemented $f(x)$ and $g(x)$ as constant functions, given by [1-8]:

$$f(x) = \phi_0, \quad g(x) = \frac{\beta\nu\sum_f}{\lambda}\phi_0, \quad 0 < x < L. \tag{5}$$

Although the authors [1-8] introduced different effective methods to solve the neutron diffusion system, such methods ignored the sense to obtain the solution in a direct manner. The preceding discussion formed the main purpose of the present work. Very recently, Al-Sharif et al. [9] developed ansatz method to solve the system (1)-(2) under the BCs (3) and the special case of the ICs (5). In [9], the authors showed that their ansatz method enjoyed many advantages over the previous analytical and numerical approaches [1-8]. As far as we know, the literature is rich of various methods to solve ordinary differential equations (ODEs) and partial differential equations (PDEs) such as the LT [10-18], differential transform method (DTM) [19, 20], the homotopy analysis method (HAM) [21, 22], the homotopy perturbation method (HPM) [23-27], and the Adomian decomposition method (ADM) [28-31]. However, most of the above methods need massive calculations to reach the desired accurate solution. The objective of this work is to introduce a simple procedure to obtain the solution of the system (1)-(2) under the BCs (3) and the general ICs (4).

2. Analytic Approach

Let us rewrite equations (1) and (2) as

$$\frac{\partial\phi}{\partial t} = VD\frac{\partial^2\phi}{\partial x^2} + \omega\phi(x, t) + \lambda VC(x, t), \tag{6}$$

$$\frac{\partial C}{\partial t} = \alpha\phi(x, t) - \lambda C(x, t), \tag{7}$$

where

$$\omega = V\left(-\sum_a + (1-\beta)\nu\sum_f\right), \quad \alpha = \beta\nu\sum_f. \tag{8}$$

Following Al-Sharif et al. [9], we assume $\phi(x, t)$ and $C(x, t)$ in the forms:

$$\phi(x, t) = \sum_{n=0}^{\infty} \sin(\gamma_n x)(A_n e^{\varepsilon_n t} + B_n e^{\delta_n t}), \tag{9}$$

$$C(x, t) = \sum_{n=0}^{\infty} \sin(\gamma_n x)(E_n e^{\varepsilon_n t} + F_n e^{\delta_n t}), \tag{10}$$

where $\gamma_n = (2n+1)\frac{\pi}{L}$. The coefficients A_n, B_n, E_n, ε_n, and δ_n are unknowns, to be evaluated later. It may be important here to mention that the ansatz/assumption (9) automatically satisfies the BCs (3). Substituting equations (9) and (10) into equations (6) and (7) yields the following system (see [9] for details):

$$(\varepsilon_n - \omega + VD\gamma_n^2)A_n - \lambda V E_n = 0, \tag{11}$$

$$(\delta_n - \omega + VD\gamma_n^2)B_n - \lambda V F_n = 0, \tag{12}$$

$$(\varepsilon_n + \lambda)E_n - \alpha A_n = 0, \tag{13}$$

$$(\delta_n + \lambda)F_n - \alpha B_n = 0. \tag{14}$$

Applying the first IC in (3) implies

$$\sum_{n=0}^{\infty} \sin(\gamma_n x)(A_n + B_n) = f(x). \tag{15}$$

Thus as in [18],

$$A_n + B_n = \frac{2}{L}\int_0^L f(x)\sin(\gamma_n x)\,dx, \tag{16}$$

or

$$A_n + B_n = I_f, \tag{17}$$

where I_f is defined by

$$I_f = \frac{2}{L}\int_0^L f(x)\sin(\gamma_n x)dx. \tag{18}$$

Similarly, the second IC in (3) leads to

$$\sum_{n=0}^{\infty} \sin(\gamma_n x)(E_n + F_n) = g(x), \tag{19}$$

and, hence on account of [18],

$$E_n + F_n = I_g, \tag{20}$$

where I_g is defined by

$$I_g = \frac{2}{L}\int_0^L g(x)\sin(\gamma_n x)dx. \tag{21}$$

Accordingly, we have the algebraic system:

$$(\varepsilon_n - \omega + VD\gamma_n^2)A_n - \lambda VE_n = 0, \tag{22}$$

$$(\delta_n - \omega + VD\gamma_n^2)B_n - \lambda VF_n = 0, \tag{23}$$

$$(\varepsilon_n + \lambda)E_n - \alpha A_n = 0, \tag{24}$$

$$(\delta_n + \lambda)F_n - \alpha B_n = 0, \tag{25}$$

$$A_n + B_n = I_f, \tag{26}$$

$$E_n + F_n = I_g. \tag{27}$$

This system must be solved in order to determine the unknowns A, B, E, F, ε_n and δ_n. In [9], the authors showed that ε_n and δ_n can be obtained as

$$\varepsilon_n = \frac{a_n + \sqrt{a_n^2 + 4b_n}}{2}, \quad \delta_n = \frac{a_n - \sqrt{a_n^2 + 4b_n}}{2}, \tag{28}$$

where

$$a_n = \omega - \lambda - VD\gamma_n^2, \quad b_n = \lambda(\omega + \alpha V - VD\gamma_n^2). \tag{29}$$

From (22) and (23), we have

$$A_n = \frac{\lambda V E_n}{\varepsilon_n - \Omega_n}, \quad B_n = \frac{\lambda V F_n}{\delta_n - \Omega_n}, \tag{30}$$

where

$$\Omega_n = \omega - VD\gamma_n^2. \tag{31}$$

Substituting (30) into (26), we get

$$\frac{\lambda V E_n}{\varepsilon_n - \Omega_n} + \frac{\lambda V F_n}{\delta_n - \Omega_n} = I_f. \tag{32}$$

On solving (32) and (27) for E_n and simplifying, we obtain

$$E_n = \frac{(\varepsilon_n - \Omega_n)[(\delta_n - \Omega_n)I_f - \lambda V I_g]}{\lambda V(\delta_n - \varepsilon_n)}, \quad \delta_n \neq \varepsilon_n. \tag{33}$$

We can prove that $(\varepsilon_n - \Omega_n)(\delta_n - \Omega_n) = -\alpha\lambda V$ (see Appendix in [9]). Therefore, equation (33) becomes

$$E_n = \frac{\alpha I_f + (\varepsilon_n - \Omega_n)I_g}{\varepsilon_n - \delta_n}, \quad \delta_n \neq \varepsilon_n. \tag{34}$$

Employing (34) in (27),

$$F_n = \frac{-\alpha I_f + (\Omega_n - \delta_n)I_g}{\varepsilon_n - \delta_n}, \quad \delta_n \neq \varepsilon_n. \tag{35}$$

From (34) and (35), it is obvious that the sum $E_n + F_n$ equals I_g which is equivalent to equation (27). Inserting (34) and (35) into equation (30), we obtain A_n and B_n as

$$A_n = \frac{\lambda V[\alpha I_f + (\varepsilon_n - \Omega_n)I_g]}{(\varepsilon_n - \Omega_n)(\varepsilon_n - \delta_n)} \tag{36}$$

and

$$B_n = \frac{\lambda V[(\Omega_n - \delta_n) I_g - \alpha I_f]}{(\delta_n - \Omega_n)(\varepsilon_n - \delta_n)}, \tag{37}$$

respectively. From the equality $(\varepsilon_n - \Omega_n)(\delta_n - \Omega_n) = -\alpha\lambda V$, we have $\varepsilon_n - \Omega_n = -\alpha\lambda V/(\delta_n - \Omega_n)$ and hence can get A_n in the form:

$$A_n = \frac{\lambda V I_g - (\delta_n - \Omega_n) I_f}{\varepsilon_n - \delta_n}. \tag{38}$$

Similarly, B_n takes the form

$$B_n = \frac{-\lambda V I_g + (\varepsilon_n - \Omega_n) I_f}{\varepsilon_n - \delta_n}. \tag{39}$$

It may be noted from equations (38) and (39) that $A_n + B_n = I_f$ which is equivalent to equation (26).

2.1. The analytic solution

From equation (9), we obtain $\phi(x, t)$ as

$$\phi(x, t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) u_1(t), \tag{40}$$

where

$$\begin{aligned} u_1(t) &= A_n e^{\varepsilon_n t} + B_n e^{\varepsilon_n t} \\ &= \frac{1}{(\varepsilon_n - \delta_n)} [(\lambda V I_g - (\delta_n - \Omega_n) I_f) e^{\varepsilon_n t} \\ &\quad + (-\lambda V I_g + (\varepsilon_n - \Omega_n) I_f) e^{\delta_n t}]. \end{aligned} \tag{41}$$

Also, from equation (10), we obtain $C(x, t)$ as

$$C(x, t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) u_2(t), \tag{42}$$

where

$$u_2(t) = E_n e^{\varepsilon_n t} + F_n e^{\delta_n t}$$

$$= \frac{1}{(\varepsilon_n - \delta_n)}[(\alpha I_f + (\varepsilon_n - \Omega_n) I_g) e^{\varepsilon_n t}$$

$$+ (-\alpha I_f + (\Omega_n - \delta_n) I_g) e^{\delta_n t}]. \quad (43)$$

3. Verification at Special Case

In Section 2, the coefficients A_n, B_n, E_n and F_n are obtained in terms of I_f and I_g as

$$A_n = \frac{\lambda V I_g - (\delta_n - \Omega_n) I_f}{\varepsilon_n - \delta_n}, \quad B_n = \frac{-\lambda V I_g + (\varepsilon_n - \Omega_n) I_f}{\varepsilon_n - \delta_n}, \quad (44)$$

$$E_n = \frac{\alpha I_f + (\varepsilon_n - \Omega_n) I_g}{\varepsilon_n - \delta_n}, \quad F_n = \frac{-\alpha I_f + (\Omega_n - \delta_n) I_g}{\varepsilon_n - \delta_n}. \quad (45)$$

The main aim of this section is to show that the corresponding coefficients in [9] can be recovered as a special case of the current ones. In [9], the authors considered the ICs (5) as

$$\phi(x, 0) = f(x) = \phi_0, \;\; C(x, t) = g(x) = h\phi_0, \;\; 0 < x < L, \quad (46)$$

where h is defined by

$$h = \frac{\beta \nu \sum_f}{\lambda} = \frac{\alpha}{\lambda}. \quad (47)$$

In this case, we can use equation (18) to calculate I_f as

$$I_f = \frac{2}{L}\int_0^L \phi_0 \sin(\gamma_n x) dx = \frac{2\phi_0}{\gamma_n L}[1 - \cos(\gamma_n L)]. \quad (48)$$

From the definition $\gamma_n = (2n+1)\frac{\pi}{L}$, we can find that $\cos(\gamma_n L) = -1, \;\; \forall n.$ Hence, equation (48) reads

$$I_f = \frac{4\phi_0}{\gamma_n L}. \tag{49}$$

By similar analysis, we can obtain I_g from equation (21) as

$$I_g = \frac{4h\phi_0}{\gamma_n L}, \tag{50}$$

i.e., $I_g = hI_f$. Inserting (51) and (52) into (44)-(47), we obtain

$$A_n = \frac{4\phi_0}{\gamma_n L}\left[\frac{\alpha V - \delta_n + \Omega_n}{\varepsilon_n - \delta_n}\right], \quad B_n = \frac{4\phi_0}{\gamma_n L}\left[\frac{-\alpha V + \varepsilon_n - \Omega_n}{\varepsilon_n - \delta_n}\right], \tag{51}$$

$$E_n = \frac{4\phi_0}{\gamma_n L}\left[\frac{\alpha + h(\varepsilon_n - \Omega_n)}{\varepsilon_n - \delta_n}\right], \quad F_n = \frac{4\phi_0}{\gamma_n L}\left[\frac{-\alpha + h(\Omega_n - \delta_n)}{\varepsilon_n - \delta_n}\right]. \tag{52}$$

4. Conclusion

In this paper, a general ansatz method was developed to determine the analytic solution of the neutron flux system under general initial conditions. The neutron flux and the delayed neutron concentration were obtained in explicit forms. The proposed approach is straightforward and also simpler in contrast to other methods in the literature. Moreover, the results in the literature were recovered as a special case of the current ones. Furthermore, the developed approach may deserve a possible extension in future to include complex systems related to neutron diffusion.

Acknowledgment

The authors extend their appreciation to the Deanship of Research and Graduate Studies at University of Tabuk for funding this work through Research no. 0002-1444-S.

References

[1] C. Ceolin, M. T. Vilhena, S. B. Leite and C. Z. Petersen, An analytical solution of the one-dimensional neutron diffusion kinetic equation in Cartesian geometry, International Nuclear Atlantic Conference – INAC 2009, Janerio, Brazil, 2009.

[2] S. M. Khaled, Exact solution of the one-dimensional neutron diffusion kinetic equation with one delayed precursor concentration in Cartesian geometry, AIMS Mathematics 7(7) (2022), 12364-12373. doi: 10.3934/math.2022686.

[3] A. A. Nahla and M. F. Al-Ghamdi, Generalization of the analytical exponential model for homogeneous reactor kinetics equations, J. Appl. Math. Volume 2012, Article ID 282367. https://doi.org/10.1155/2012/282367.

[4] F. Tumelero, C. M. F. Lapa, B. E. J. Bodmann and M. T. Vilhena, Analytical representation of the solution of the space kinetic diffusion equation in a one-dimensional and homogeneous domain, Braz. J. Radiat. Sci. 7 (2019), 1-13. https://doi.org/10.15392/bjrs.v7i2B.389.

[5] A. A. Nahla, F. Al-Malki and M. Rokaya, Numerical techniques for the neutron diffusion equations in the nuclear reactors, Adv. Stud. Theor. Phys. 6 (2012), 649-664.

[6] S. M. Khaled, Power excursion of the training and research reactor of Budapest University, Int. J. Nucl. Energy Sci. Technol. 3 (2007), 42-62. https://doi.org/10.1504/IJNEST.2007.012440.

[7] S. M. Khaled and F. A. Mutairi, The influence of different hydraulics models in treatment of some physical processes in super critical states of light water reactors, Int. J. Nucl. Energy Sci. Technol. 8 (2014), 290-309. https://doi.org/10.1504/IJNEST.2014.064940.

[8] S. Dulla, P. Ravetto, P. Picca and D. Tomatis, Analytical benchmarks for the kinetics of accelerator driven systems, Joint International Topical Meeting on Mathematics and Computation and Supercomputing in Nuclear Applications, Monterey-California, on CD-ROM, 2007.

[9] M. A. Al-Sharif, A. Ebaid, H. S. Alrashdi, A. H. Alenazy and N. E. Kanaan, A novel ansatz method for solving the neutron diffusion system in Cartesian geometry, Journal of Advances in Mathematics and Computer Science 37(11) (2022), 90-99.

[10] Abdulrahman F. Aljohani, Abdelhalim Ebaid, Emad H. Aly, Ioan Pop, Ahmed O. M. Abubaker and Dalal J. Alanazi, Explicit solution of a generalized mathematical model for the solar collector/photovoltaic applications using nanoparticles, Alexandria Engineering Journal 67 (2023), 447-459. https://doi.org/10.1016/j.aej.2022.12.044.

[11] S. M. Khaled, The exact effects of radiation and Joule heating on magnetohydrodynamic Marangoni convection over a flat surface, Therm. Sci. 22 (2018), 63-72.

[12] Mona D. Aljoufi, Exact solution of a solar energy model using four different kinds of nanofluids: advanced application of Laplace transform, Case Studies in Thermal Engineering 50 (2023), 103396. https://doi.org/10.1016/j.csite.2023.103396.

[13] A. Ebaid, W. Alharbi, M. D. Aljoufi and E. R. El-Zahar, The exact solution of the falling body problem in three-dimensions: comparative study, Mathematics 8(10) (2020), 1726. https://doi.org/10.3390/math8101726.

[14] A. Ebaid, C. Cattani, A. S. Al. Juhani and E. R. El-Zahar, A novel exact solution for the fractional Ambartsumian equation, Adv. Differ. Equ. Volume 2021, Article number 88. https://doi.org/10.1186/s13662-021-03235-w.

[15] A. Ebaid and H. K. Al-Jeaid, The Mittag-Leffler functions for a class of first-order fractional initial value problems: dual solution via Riemann-Liouville fractional derivative, Fractal Fract. 6 (2022), 85. https://doi.org/10.3390/fractalfract6020085.

[16] A. F. Aljohani, A. Ebaid, E. A. Algehyne, Y. M. Mahrous, C. Cattani and H. K. Al-Jeaid, The Mittag-Leffler function for re-evaluating the chlorine transport model: comparative analysis, Fractal Fract. 6 (2022), 125. https://doi.org/10.3390/fractalfract6030125.

[17] A. F. Aljohani, A. Ebaid, E. A. Algehyne, Y. M. Mahrous, P. Agarwal, M. Areshi and H. K. Al-Jeaid, On solving the chlorine transport model via Laplace transform, Sci. Rep. 12 (2022), 12154. https://doi.org/10.1038/s41598-022-14655-3.

[18] R. Haberman, Elementary Applied Partial Differential Equations, 2nd ed., Prentice-Hall, Inc., USA, 1987.

[19] A. Ebaid, Approximate periodic solutions for the non-linear relativistic harmonic oscillator via differential transformation method, Commun. Nonlinear Sci. Numer. Simul. 15 (2010), 1921-1927.

[20] Y. Liu and K. Sun, Solving power system differential algebraic equations using differential transformation, IEEE Trans. Power Syst. 35 (2020), 2289-2299. https://doi.org/10.1109/TPWRS.2019.2945512.

[21] S. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, CRC Press, Boca Raton, FL, USA, 2003.

[22] A. Chauhan and R. Arora, Application of homotopy analysis method (HAM) to the non-linear KdV equations, Commun. Math. 31 (2023), 205-220. https://doi.org/10.46298/cm.10336.

[23] A. Ebaid, Remarks on the homotopy perturbation method for the peristaltic flow of Jeffrey fluid with nano-particles in an asymmetric channel, Comput. Math. Appl. 68 (2014), 77-85.

[24] M. Nadeem and F. Li, He-Laplace method for nonlinear vibration systems and nonlinear wave equations, J. Low Freq. Noise Vib. Act. Control. 38 (2019), 1060-1074.

[25] A. Ebaid, A. F. Aljohani and E. H. Aly, Homotopy perturbation method for peristaltic motion of gold-blood nanofluid with heat source, Int. J. Numer. Meth. Heat Fluid Flow 30 (2020), 3121-3138. https://doi.org/10.1108/HFF-11-2018-0655.

[26] M. Nadeem, S. A. Edalatpanah, I. Mahariq and W. H. F. Aly, Analytical view of nonlinear delay differential equations using Sawi iterative scheme, Symmetry 14 (2022), 2430. https://doi.org/10.3390/sym14112430.

[27] Z. Ayati and J. Biazar, On the convergence of Homotopy perturbation method, J. Egyptian Math. Soc. 23 (2015), 424-428.

[28] J. S. Duan and R. Rach, A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations, Appl. Math. Comput. 218 (2011), 4090-4118.

[29] A. Ebaid, A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method, J. Comput. Appl. Math. 235 (2011), 1914-1924.

[30] S. Bhalekar and J. Patade, An analytical solution of Fisher's equation using decomposition method, Am. J. Comput. Appl. Math. 6 (2016), 123-127.

[31] A. H. S. Alenazy, A. Ebaid, E. A. Algehyne and H. K. Al-Jeaid, Advanced study on the delay differential equation $y'(t) = ay(t) + by(ct)$, Mathematics 10 (2022), 4302. https://doi.org/10.3390/math10224302.