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SOLVING A FRACTIONAL EVOLUTION EQUATION IN THE SENSE OF CAPUTO-HADAMARD WITH CAUCHY AND BOUNDARY CONDITIONS BY SBA METHOD

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Abstract

Exact solutions of fractional evolution equations in the sense of Caputo-Hadamard with Cauchy and boundary conditions are obtained by employing the SBA method.

1. Introduction

Mathematical modeling using fractional calculations is in use because integer derivative equations are not always well suited to explain certain phenomena efficiently. However, due to the non-local structure of these operators, linked to its memory effect, fractional derivation is a high storage of information. As a result, its use in the construction of mathematical models for a given phenomenon comes at a high cost in terms of numerical resolution. When using a discretization algorithm for non-integer derivatives, this structure must be taken into account, resulting in high algorithm complexity. Many classical numerical methods also encounter difficulties due to the complexity of their non-linear parts. Numerous attempts to solve these equations can be found in the literature. For example, in [12], the HPM, HPTM methods have been used, and the FNDM, NHPM methods in [17]. But most of these methods besides being complex only give an approximate solution to the problem. These can only solve Cauchy-type problems. These limitations motivate our interest in the developing a new method that takes these shortcomings into account.

In this article, we propose an iterative method known as the Some Blaise Abbo (SBA) method, capable of taking into account the complex structure of fractional derivation which easily handles non-linearity taking boundary conditions into account. Results obtained using the SBA method with Cauchy conditions can be found in the literature [2, 8, 9, 16].

2. Preliminaries

Most of the definitions and properties we employ for our work can be found in [1, 10, 13, 14, 20, 22, 24].

2.1. Fractional integral in Hadamard's sense

Definition 2.1. Let $a, b \in \mathbb{R}$, $0 < a < t < b < +\infty$ and $\alpha \in \mathbb{R}^+$. Then the *fractional integral in Hadamard's sense of order* α for a function $f \in L^1[a, b]$ is defined by

$$
{}^{H}\mathcal{I}_a^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} f(\tau)\frac{d\tau}{\tau}, & \text{for } \alpha > 0, \\ f(\tau), & \text{for } \alpha = 0, \end{cases} \quad t \in [a; b]. \tag{2.1}
$$

2.2. Fractional derivation in the sense of Caputo-Hadamard

Definition 2.2. Let $a, b \in \mathbb{R}$, $0 < a < t < b < +\infty$ and $\alpha \in \mathbb{R}^+$ with $n = [\alpha] + 1$ and $\delta = t\dfrac{d}{dt}$, $AC[a; b]$ be a space of absolutely continuous functions and $AC_\delta^n = \{f : [a; b] \to \mathbb{R};\ f, \delta^{n-1} \in AC[a; b]\}$. Then the *fractional derivative in the Caputo-Hadamard sense of order* α for a function $f \in AC_\delta^n[a, b]$ is defined by

$$
{}^{CH}\mathcal{D}_a^{\alpha} f(t)
$$

$$
= \begin{cases} {}^{H}\mathcal{I}_a^{n-\alpha}\delta^n f(t) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{n-\alpha-1} \left(\tau\frac{d}{d\tau}\right)^n f(\tau)\frac{d\tau}{\tau}, & \text{for } \alpha \notin \mathbb{N}, \\ \delta^n f(t), & \text{for } \alpha \in \mathbb{N}, \\ f(t), & \text{for } \alpha = 0. \end{cases}
$$

Proposition 2.1. *For* $a, b \in \mathbb{R}$, $0 < a < t < b < +\infty$ *and* $\alpha \in \mathbb{R}^{+}$ *with* $n = [\alpha] + 1$. *For* $f \in L^{1}[a; b]$,

$$ {}^{CH}\mathcal{D}_{a}^{\alpha}\, {}^{H}\mathcal{I}_{a}^{\alpha} f(t) = f(t), \tag{2.2}$$

$$ {}^{H}\mathcal{I}_{a}^{\alpha}\, {}^{CH}\mathcal{D}_{a}^{\alpha} f(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{j} f)(a)}{j!} \left[\ln\left(\frac{t}{a}\right)\right]^{j}. \tag{2.3}$$

3. Description and Convergence of the SBA Method

Here we use an improved version of the Some Blaise Abbo (SBA) method. The peculiarity of this version is that it simplifies the classical approach and sets a sufficient condition for the convergence of the series $u^{1} = \sum_{n=0}^{+\infty} u_{n}^{1}$, the approximate solution of the problem at the first iteration.

3.1. Description of the SBA method applied to fractional Caputo-Hadamard equations with Cauchy and boundary conditions

In this subsection, we show how to generalize the improved SBA method to find the solution of fractional equations with initial and/or boundary conditions. We describe this by taking a PDE in the sense of Caputo-Hadamard with the following initial and boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha} = L(u) + N(u), \\ (\delta^{j} u)(a, x) = f_{j}(x), \ j = 0, 1, 2, \dots, m-1, \\ u(t, l_{1}) = g(t), \\ u(t, l_{2}) = h(t), \end{cases}$$

$$ l_{1} \leq x \leq l_{2}, \ 0 < a < t < T, \tag{3.1}$$

where $u = u(t, x)$, $m = [\alpha] + 1$, $u_{x} = \frac{\partial u}{\partial x}$, in a suitable space, and L and N are linear and non-linear operators, respectively, such that $L(u) = L_{1}(u)$

$+ L_2(u_x, u_{xx}, u_{xxx}, \ldots)$ with $L_2 \neq 0$. In this description, we take the case where $Lu = u_x$. The technique generalizes to Lu defined above. Based on the above assumptions, the model (3.1) takes the form:

$$\begin{cases} {}^c\mathcal{D}_t^{\alpha} = u_x + N(u), \\ (\delta^j u)(a, x) = f_j(x), \; j = 0, 1, 2, \ldots, m-1, \\ u(t, l_1) = g(t), \\ u(t, l_2) = h(t), \end{cases}$$

$$l_1 \leq x \leq l_2, \; 0 < a < t < T,$$

$$u = u(t, x), \quad m = [\alpha] + 1, \quad u_x = \frac{\partial u}{\partial x}. \tag{3.2}$$

Integrating the first equality of (3.2) between l_1 and l_2, we obtain

$$u(t, l_2) - u(t, l_1) + \int_{l_1}^{l_2} N(u(t, s))ds - \int_{l_1}^{l_2} {}^c\mathcal{D}_t^{\alpha} u(t, s)ds = 0. \tag{3.3}$$

Similarly, by applying Hadamard's fractional integral ${}^H\mathcal{I}^{\alpha}$ of order α to the first equality of (3.2), we obtain

$$u(t, x) - \sum_{j=0}^{m-1} \frac{f_j(x)}{j!}\left[\ln\left(\frac{t}{a}\right)\right]^j - \mathcal{I}^{\alpha}(u_x) - \mathcal{I}^{\alpha}(Nu) = 0. \tag{3.4}$$

Equations (3.3) and (3.4) give

$$u(t, x) = \sum_{j=0}^{m-1} \frac{f_j(x)}{j!}\left[\ln\left(\frac{t}{a}\right)\right]^j + u(t, l_2) - u(t, l_1) + \mathcal{I}^{\alpha}(u_x)$$

$$+ \int_{l_1}^{l_2} N(u(t, s))ds - \int_{l_1}^{l_2} {}^c\mathcal{D}_t^{\alpha} u(t, s)ds + \mathcal{I}^{\alpha}(Nu).$$

Setting

$$\begin{cases} Ru = \mathcal{I}^{\alpha}(u_x), \\ \overline{N}u = -\int_{l_1}^{l_2} {}^{c}\mathcal{D}_t^{\alpha} u(t,\, s)ds + \int_{l_1}^{l_2} N(u(t,\, s))ds + \mathcal{I}^{\alpha}(Nu), \end{cases} \tag{3.5}$$

we obtain

$$u(t,\, x) = \sum_{j=0}^{m-1} \frac{f_j(x)}{j!}\left[\ln\left(\frac{t}{a}\right)\right]^j + u(t,\, l_2) - u(t,\, l_1) + Ru + \overline{N}u. \tag{3.6}$$

Applying the method of successive approximations, we obtain

$$u^k(t,\, x) = \sum_{j=0}^{m-1} \frac{f_j(x)}{j!}\left[\ln\left(\frac{t}{a}\right)\right]^j + u^k(t,\, l_2) - u^k(t,\, l_1) + Ru^k + \overline{N}u^{k-1}. \tag{3.7}$$

We look for u^k as a series of the form

$$u^k = \sum_{n=0}^{+\infty} u_n^k. \tag{3.8}$$

By (3.8) and (3.7), we obtain the following SBA algorithm:

$$\begin{cases} u_0^k = \sum_{j=0}^{m-1} \frac{f_j(x)}{j!}\left[\ln\left(\frac{t}{a}\right)\right]^j + u^k(t,\, l_2) - u^k(t,\, l_1) + \overline{N}u^{k-1}, \\ u_{n+1}^k = Ru_n^k. \end{cases} \tag{3.9}$$

The above algorithm consists in first calculating the terms of the sequence $(u_n^k)_n$ for fixed $k \geq 1$, and deduce u^k if the series $\sum_{n=0}^{+\infty} u_n^k$ converges.

So for the first iteration, $k = 1$, we choose u^0 such that $Nu^0 = 0$. Calculating the terms of the sequence $(u_n^1)_n$, we deduce $u^1 = \sum_{n=0}^{+\infty} u_n^1$. Then we evaluate Nu^1. If $Nu^1 = 0$, then u^1 is the general solution of the problem. Otherwise, if possible, we replace the initial problem by an equivalent transformation, with the new non-linear term $\overline{N}$, so that by repeating the algorithm, we can obtain $\overline{N}u^1 = 0$.

3.2. Convergence of the SBA method

The proof of convergence can be found in our previous article [9].

4. Examples

Example 4.1. Consider the following nonlinear fractional evolution equation in the Caputo-Hadamard sense:

$$\begin{cases} {}^{CH}\mathcal{D}^{\alpha}u = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial^2 u}{\partial x^2}\right)^2, \\ u(0,\ x) = x^2, \\ u(t,\ 0) = 0, \\ u(t,\ 1) = \dfrac{\left[\ln\left(\dfrac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}, \end{cases} \tag{4.1}$$

$0 \le x \le 1$; $0 < a < t < b < +\infty$; $0 < \alpha \le 1$; $u = u(t,\ x)$; $u \in C_{\delta}^{1}([a;\ b])$; ${}^{CH}\mathcal{D}_a^{\alpha}(\cdot)$ the derivative in the Caputo-Hadamard sense; ${}^{H}\mathcal{I}_a^{\alpha}(\cdot)$ the integral in the Hadamard sense.

From the above description, we obtain the following SBA algorithm:

$$\begin{cases} u_0^k = u^k(0,\ x) + u^k(t,\ 1) - u^k(t,\ 0) + Nu^{k-1}, \\ u_{n+1}^k = Ru_n^k \end{cases} \tag{4.2}$$

with

$$\begin{cases} Ru^k = {}^H\mathcal{I}^{\alpha}\left(\dfrac{\partial u^k}{\partial x}\right), \\ Nu^{k-1} = -\displaystyle\int_0^1 ({}^H\mathcal{D}^{\alpha} u^{k-1}(t, s))ds + \dfrac{1}{2}\, {}^H\mathcal{I}^{\alpha}\left(\dfrac{\partial^2 u^{k-1}(t, x)}{\partial x^2}\right)^2 \\ \qquad\qquad + \dfrac{1}{2}\displaystyle\int_0^1 \left(\dfrac{\partial^2 u^{k-1}(t, s)}{\partial s^2}\right)^2 ds. \end{cases}$$

First iteration, $k = 1$.

The algorithm (4.2) for $k = 1$ taking u^0 such that $Nu^0 = 0$ gives:

$$\begin{cases} u_0^1 = x^2 + \dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}, \\ u_1^1 = 2x\dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2^1 = 2\dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_n^1 = 0,\ n \geq 3. \end{cases} \tag{4.3}$$

So,

$$\begin{aligned} u^1 &= u_0^1 + u_1^1 + u_2^1 \\ &= x^2 + (1+2x)\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

Evaluating Nu^1, we have

$$Nu^1 = -\int_0^1 ({}^H\mathcal{D}^\alpha u^1(t, s))ds + \frac{1}{2}\,{}^H\mathcal{I}^\alpha\left(\frac{\partial^2 u^1(t, x)}{\partial x^2}\right)^2 + \frac{1}{2}\int_0^1\left(\frac{\partial^2 u^1(t, s)}{\partial s^2}\right)^2 ds$$

$$= 2\left(-1 - \frac{\left[\ln\left(\frac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)} + 1 + \frac{\left[\ln\left(\frac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)}\right)$$

$$= 0.$$

We deduce the solution to the problem as:

$$u = x^2 + (1+2x)\frac{\left[\ln\left(\frac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)} + 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{4.4}$$

Example 4.2. Consider the following nonlinear fractional evolution equation in the Caputo-Hadamard sense:

$$\begin{cases} {}^{CH}\mathcal{D}^\alpha u = \dfrac{\partial u}{\partial x} + \dfrac{1}{4}\left(\dfrac{\partial^2 u}{\partial x^2}\right)^3, \\ u(0, x) = x(1-x), \\ u(t, 0) = -\dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)} - 2\dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u(t, 1) = -3\dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)} - 2\dfrac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases} \tag{4.5}$$

$0 \le x \le 1$; $0 < a < t < b < +\infty$; $0 < \alpha \le 1$; $u = u(t, x)$; $u \in C_\delta^1([a; b])$; ${}^{CH}\mathcal{D}_a^\alpha(\cdot)$ the derivative in the Caputo-Hadamard sense; ${}^H\mathcal{I}_a(\cdot)$ the integral in the Hadamard sense.

Proceeding as above, we obtain the following SBA algorithm:

$$\begin{cases} u_0^k = u^k(0,\ x) + u^k(t,\ 1) - u^k(t,\ 0) + Nu^{k-1}, \\ u_{n+1}^k = Ru_n^k \end{cases} \tag{4.6}$$

with

$$\begin{cases} Ru^k = {}^H\mathcal{I}^\alpha\left(\dfrac{\partial u^k}{\partial x}\right), \\ Nu^{k-1} = -\displaystyle\int_0^1 ({}^H\mathcal{D}^\alpha u^{k-1}(t,\ s))ds + \dfrac{1}{4}\,{}^H\mathcal{I}^\alpha\left(\dfrac{\partial^2 u^{k-1}(t,\ x)}{\partial x^2}\right)^3 \\ \qquad\qquad + \dfrac{1}{4}\displaystyle\int_0^1\left(\dfrac{\partial^2 u^{k-1}(t,\ s)}{\partial s^2}\right)^3 ds. \end{cases}$$

First iteration, $k = 1$.

The algorithm (4.6) for $k = 1$ taking u^0 such that $Nu^0 = 0$, is as follows:

$$\begin{cases} u_0^1 = x(1-x) - 2\dfrac{\left[\ln\left(\dfrac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)}, \\ u_1^1 = (1-2x)\dfrac{\left[\ln\left(\dfrac{t}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)}, \\ u_2^1 = -2\dfrac{\left[\ln\left(\dfrac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_n^1 = 0,\ n \geq 3. \end{cases} \tag{4.7}$$

So,

$$u^1 = u_0^1 + u_1^1 + u_2^1$$

$$= x(1-x) - 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} + (1-2x)\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$= x(1-x) + (-1-2x)\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Evaluating Nu^1, we have

$$Nu^1 = -\int_0^1 ({}^{H}\mathcal{D}^{\alpha} u^1(t, s))ds + \frac{1}{4}\,{}^{H}\mathcal{I}^{\alpha}\left(\frac{\partial^2 u^1(t, x)}{\partial x^2}\right)^3 + \frac{1}{4}\int_0^1 \left(\frac{\partial^2 u^1(t, s)}{\partial s^2}\right)^3 ds$$

$$= 2 + 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} - 2 - 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}$$

$$= 0.$$

We deduce the solution to the problem as:

$$u = x(1-x) + (-1-2x)\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{\left[\ln\left(\frac{t}{a}\right)\right]^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{4.8}$$

5. Conclusion

The results obtained in this article show that the SBA method is suitable for the numerical resolution of fractional Caputo-Hadamard-type evolutions with Cauchy and boundary conditions. The peculiarity of this method is that it does not discretize like classical numerical methods, and consequently the physical properties of the modeled phenomena are preserved. What is more, it uses an algorithm that bypasses the computation of Adomian polynomials and efficiently handles the non-linear part involving these equations.

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