Advances in Differential Equations and Control Processes
© 2023 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324323023
Volume 30, Number 4, 2023, Pages 413-429 **P-ISSN: 0974-3243**

AN INTEGRO-DIFFERENTIAL EQUATION IN COMPOUND POISSON RISK MODEL WITH VARIABLE THRESHOLD DIVIDEND PAYMENT STRATEGY TO SHAREHOLDERS AND TAIL DEPENDENCE BETWEEN CLAIMS AMOUNTS AND INTER-CLAIM TIME

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Received: October 6, 2023; Accepted: November 21, 2023
2020 Mathematics Subject Classification: 91B05, 62H05.
Keywords and phrases: Gerber-Shiu function, copula, integro-differential equation, ruin probability.
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How to cite this article: Kiswendsida Mahamoudou OUEDRAOGO, Delwendé Abdoul-Kabir KAFANDO, Francois Xavier OUEDRAOGO, Lassané SAWADOGO and Pierre Clovis NITIEMA, An integro-differential equation in compound Poisson risk model with variable threshold dividend payment strategy to shareholders and tail dependence between claims amounts and inter-claim time, Advances in Differential Equations and Control Processes 30(4) (2023), 413-429. http://dx.doi.org/10.17654/0974324323023

Published Online: December 2, 2023

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Abstract

This article is an extension of the compound Poisson risk model with variable threshold dividend payment strategy to shareholders and a dependence between claims amounts and inter-claim times via Spearman copula. We find the integro-differential equation associated to this risk model.

1. Introduction

In the modeling of financial risks, especially in insurance, we have long assumed independence between the random variables involved in the risk model [12, 18, 23]. However, in certain practical contexts, this assumption is inadequate and too restrictive. For instance, in flood insurance, the occurrence of multiple floods in a short period can lead to significant damages and claim amounts due to the accumulation of water. In earthquake insurance, it is the opposite, as in a high-risk area, the longer the time between two earthquakes, the more significant the second earthquake due to the accumulation of energy.

To address this limitation, many works integrate dependence between certain random variables, particularly claim amounts and inter-claim times, into the risk model, using Farlie-Gumbel-Morgenstern copula [5, 7, 10, 11, 13, 19, 20, 22]. Although this copula is commonly used in the literature, it has certain limitations and cannot model tail dependencies [2-4].

To overcome the insufficiency of the Farlie-Gumbel-Morgenstern copula, while taking into account the reality of insurance companies, we consider in this work a compound Poisson risk model with dependence between claim amounts and inter-claim times via Spearman copula and a variable threshold dividend payment strategy to shareholders of equation $b_t = at + b_0$, where $0 < u \leq b_0$; $0 \leq a < c$.

The risk model with affine threshold dividend payment strategy was proposed to compensate the non-optimality of the risk model with constant threshold dividend payment strategy [6]. In this model, when the surplus process reaches the variable threshold barrier b_t, the premium are paid to shareholders at a constant rate $c - a$. Denoting by $U_b(t)$ the surplus process in the presence of the threshold dividend barrier b_t (with $U_b(0) = u$), the model follows the following dynamics:

$$dU_b(t) = \begin{cases} cdt - dS(t) & \text{if } U_b(t) < b_t \\ adt - dS(t) & \text{if } U_b(t) = b_t, \end{cases} \tag{1.1}$$

where

• $U_b(t)$ is the surplus process in the presence of the dividend barrier of threshold b_t (the initial surplus $U_b(0) = u$, where $0 < u \leq b$).

• c is a constant rate of premium collected by the insurer per unit of time, $0 \leq a < c$.

• $S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregated loss process with a compound Poisson distribution, where

∗ $\{N(t), t \geq 0\}$ is the total number of recorded claims up to time t, following a Poisson process with intensity $\lambda > 0$. (Note that $S(t) = 0$ if $N(t) = 0$).

∗ $\{X_i, i \geq 1\}$ is a sequence of random variables representing the individual claim amount with a common density function f and cumulative distribution function F assumed to follow an exponential distribution with parameter β.

- The inter-claim times $\{V_i, i \geq 1\}$ form a sequence of random variables following an exponential distribution with the parameter λ, the probability density function $k(t) = \lambda e^{-\lambda t}$, and the cumulative distribution $K(t) = 1 - e^{-\lambda t}$.

The purpose of this work is to determine the integro-differential equation of Gerber-Shiu function in the risk model defined by equation (1.1). To achieve this, the rest of the article is organized as follows: In Section 2, we present the preliminaries related to the risk model defined by equation (1.1). In Section 3, we introduce the tail dependence structure. In Section 4, we determine the integro-differential equation satisfied by the Gerber-Shiu function in the risk model defined by equation (1.1).

2. Preliminaries

2.1. Instant of ruin

The instant of ruin T_b of the insurance company is defined by

$$T_b = \inf\{t \geq 0, U(t) < 0\}. \tag{2.1}$$

When the probability of ruin is always zero, by convention, we denote $T_b = \infty$, and in this case

$$U(t) \geq 0, \quad \forall t \geq 0.$$

2.2. Expected discounted penalty function of Gerber-Shiu

The expected discounted penalty function of Gerber-Shiu, first appeared in the work of Gerber and Shiu [12]. Nowadays, this function is of significant interest in research.

Its analysis remains a central question both in insurance and finance, as it is a valuable tool not only for studying the probability of ruin but also calculating retirement and reinsurance premiums, option pricing, and more.

It is defined by

$$\phi_b(u) = E[e^{-\delta T_b} w(U_{(T_b^-)}, |U_b(T_b)|) I(\tau < \infty) | U(0) = u], \tag{2.2}$$

where

- T_b is the instant of ruin defined by equation (2.1).
- T_b^- is the instant just before ruin.
- δ is an interest force.
- The penalty function $w(x, y)$ is a positive function of the surplus just before ruin, $U_b(T_b^-)$ and the deficit at ruin $|U_b(T_b)|$, $\forall x, y \geq 0$.
- 1_A is an indicator function, which equals 1 if event 1_A occurs and 0 otherwise.

3. Model of Dependence Based on Spearman Copula

Copulas introduced by Abe Sklar in 1998, are an innovative and relevant tool for introducing dependence between multiple random variables. Given the marginal distribution functions of several random variables, copulas allow us to establish their joint distribution function. Nowadays, they are fundamental in modeling multivariate distributions in finance, insurance and hydrology. Key references on copulas theory include Joe [8] and Nelsen [9]. In this article, the structure of dependence is ensured by the Spearman copulas. It is defined for all $(u, v) \in [0, 1]^2$ and $\alpha \in [0, 1]$ as follow:

$$C_\alpha(u, v) = (1-\alpha)C_I(u, v) + \alpha C_M(u, v), \tag{3.1}$$

where $C_I(u, v) = uv$; $C_M(u, v) = \min(u, v)$ and α is a dependency parameter.

The Spearman copulas allow for the introduction of positive dependence as well as tail dependencies in many situations. It also includes independence when $\alpha = 0$. Using formula (3.1), the random vectors of claim amounts and inter-claim times (X, V) possess the joint distribution function given by

$$F_{X,V}(x, t) = C_\alpha(F_X(x), F_V(t)) = (1-\alpha)F_I(x, t) + \alpha F_M(x, t), \tag{3.2}$$

where F_X and F_V are the marginal distributions of the random variables *X* and *V*, respectively.

In the risk model defined by equation (1.1), the Gerber Shiu function $\phi_b(u)$ takes the following form (see [2], [3] or [4]):

$$\phi_b(u) = (1-\alpha)(I_{b,1}(u) + I_{b,2}(u)) + \alpha(I_{b,3}(u) + I_{b,4}(u)), \tag{3.3}$$

where

- $I_{b,1}(u) = \int_0^\infty \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_I(x, t),$

- $I_{b,2}(u) = \int_0^\infty \int_{u+ct}^\infty e^{-\delta t}w(u+ct, x-u-ct)dF_I(x, t),$

- $I_{b,3}(u) = \int_0^\infty \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_M(x, t),$

- $I_{b,4}(u) = \int_0^\infty \int_{u+ct}^\infty e^{-\delta t}w(u+ct, x-u-ct)dF_M(x, t).$

4. Integro-differential Equation Satisfied by the Gerber-Shiu Function $\phi_b(u)$

To obtain integro-differential equation satisfied by Gerber-Shiu function $\phi_b(u)$ in the risk model defined by equation (1.1), we follow the approach described below:

- The first claim occurs at time *t* before the surplus process reaches the barrier b_t $\left(\text{i.e., } t < \frac{b_0 - u}{c - a}\right)$. Its amount *x* satisfies $x < u + ct$.

- The first claim occurs at time *t* before the surplus process reaches the barrier b_t $\left(\text{i.e., } t < \frac{b_0 - u}{c - a}\right)$. Its amount *x* satisfies $x > u + ct$.

- The first claim occurs at time t after the surplus process reaches the barrier b_t $\left(\text{i.e., } t > \dfrac{b_0 - u}{c - a}\right)$. Its amount x satisfies $x < at + b_0$.

- The first claim occurs at time t after the surplus process reaches the barrier b_t $\left(\text{i.e., } t > \dfrac{b_0 - u}{c - a}\right)$. Its amount x satisfies $x > at + b_0$.

By conditioning on the time and the amount of the first claim and considering the different scenarios mentioned above, the integrals $I_{b,1}(u)$ and $I_{b,2}(u)$ in relation (3.3) become:

- $$I_{b,1}(u) = \int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-\delta t} \phi_b(u + ct - x) dF_I(x, t)$$
$$+ \int_{\frac{b_0-u}{c-a}}^{\infty} \int_0^{at+b_0} e^{-\delta t} \phi_b(at + b_0 - x) dF_I(x, t).$$

- $$I_{b,2}(u) = \int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^{\infty} e^{-\delta t} w(u + ct, x - u - ct) dF_I(x, t)$$
$$+ \int_{\frac{b_0-u}{c-a}}^{\infty} \int_{at+b_0}^{\infty} e^{-\delta t} w(at + b_0, x - at - b_0) dF_I(x, t).$$

By defining $I_b(u) = I_{b,1}(u) + I_{b,2}(u)$, we have

$$I_b(u) = \int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-\delta t} \phi_b(u + ct - x) dF_I(x, t)$$
$$+ \int_{\frac{b_0-u}{c-a}}^{\infty} \int_0^{at+b_0} e^{-\delta t} \phi_b(at + b_0 - x) dF_I(x, t)$$
$$+ \int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^{\infty} e^{-\delta t} w(u + ct, x - u - ct) dF_I(x, t)$$
$$+ \int_{\frac{b_0-u}{c-a}}^{\infty} \int_{at+b_0}^{\infty} e^{-\delta t} w(at + b_0, x - at - b_0) dF_I(x, t),$$

$$I_b(u) = \lambda \int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-(\delta+\lambda)t} \phi_b(u+ct-x) f_X(x) dx dt$$

$$+ \lambda \int_{\frac{b_0-u}{c-a}}^{\infty} \int_0^{at+b_0} e^{-(\delta+\lambda)t} \phi_b(at+b_0-x) f_X(x) dx dt$$

$$+ \lambda \int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^{\infty} e^{-(\delta+\lambda)t} w(u+ct,\ x-u-ct) f_X(x) dx dt$$

$$+ \lambda \int_{\frac{b_0-u}{c-a}}^{\infty} \int_{at+b_0}^{\infty} e^{-(\delta+\lambda)t} w(at+b_0,\ x-at-b_0) f_X(x) dx dt. \tag{4.1}$$

To simplify the notation of equation (4.1), we introduce

$$\omega(u) = \int_u^{\infty} w(u,\ x-u) f(x) dx;$$

$$\sigma_b(u) = \int_0^u \phi_b(u-x) f(x) dx + \omega(u). \tag{4.2}$$

Equation (4.1) becomes

$$I_b(u) = \lambda \int_0^{\frac{b_0-u}{c-a}} e^{-(\delta+\lambda)t} \sigma_b(u+ct) dt$$

$$+ \lambda \int_{\frac{b_0-u}{c-a}}^{\infty} e^{-(\delta+\lambda)t} \sigma_b(at+b_0) dt. \tag{4.3}$$

The relation (4.3) can be put in the form:

$$I_b(u) = \lambda \int_0^{\infty} e^{-(\delta+\lambda)t} \sigma_b((u+ct) \wedge (at+b_0)) dt, \tag{4.4}$$

where $u \wedge v = \min(u, v)$.

Now, we determine the integrals $I_{b,3}(u)$ and $I_{b,4}(u)$ in relation (3.3).

The support of copula C_M is $D = \{(u, v) \in [0, 1]^2 : u = v\}$.

On the domain $[0, 1]^2 \backslash D$, $\frac{\partial^2 C_M}{\partial u \partial v} = 0$ and on D, C_M follows a uniform distribution.

Since the structure of dependence between the claim amounts and inter-claim times is described by the copula C_M, these are monotonous and there almost surely exists an increasing function l, such that $X = l(V)$ (see Nelsen [9, P. 27]). We deduce that (see [2], [3] or [4]):

$$l(t) = \frac{\lambda}{\beta} t. \tag{4.5}$$

The joint distribution $F_{X,V}(x, t)$ of the random vector (X, V) is singular, and its support is the domain

$$D' = \{(x, t) : F_X(x) = F_V(t)\} = \{(x, t) : x = l(t)\}.$$

Its distribution is $G(t) = F_M(l(t), t) = 1 - e^{-\lambda t}$ on the domain

$$D' = \left\{(x, t) : x = \frac{\lambda}{\beta} t\right\}.$$

The integral $I_{b,3}(u)$ becomes

$$I_{b,3}(u) = \int_0^{\frac{b_0 - u}{c-a}} \int_0^{u+ct} e^{-\delta t} \phi_b(u + ct - x) dF_M(x; t)$$

$$+ \int_{\frac{b_0-u}{c-a}}^{\infty} \int_0^{at+b_0} e^{-\delta t} \phi_b(at + b_0 - x) dF_M(x; t)$$

$$= \int_K e^{-\delta t} \phi_b(u + ct - x) dG(t)$$

$$+ \int_J e^{-\delta t} \phi_b(at + b_0 - x) dG(t), \tag{4.6}$$

where

$$K = \left\{t \in \mathbb{R}^{+} : 0 \leq t \leq \frac{b_0 - u}{c - a} \text{ and } 0 \leq x = \frac{\lambda}{\beta} t \leq u + ct\right\}$$

$$= \left\{t \in \mathbb{R}^{+} : 0 \leq t \leq \frac{b_0 - u}{c - a} \text{ and } \left(\frac{\lambda}{\beta} - c\right) t \leq u\right\}$$

$$= \left\{t \in \mathbb{R}^{+} : 0 \leq t \leq \frac{b_0 - u}{c - a} \text{ and } t \in \mathbb{R}^{+}\right\}.$$

Because $c > \frac{\lambda}{\beta}$ and $u > 0$ (solvency condition: $\mathbb{E}[cV - X] > 0$) and $u \geq 0$.

Therefore,

$$K = \left[0; \frac{b_0 - u}{c - a}\right], \tag{4.7}$$

$$J = \left\{t \in \mathbb{R}^{+} : t \geq \frac{b_0 - u}{c - a} \text{ and } x = \frac{\lambda}{\beta} t \leq at + b_0\right\}$$

$$= \left\{t \in \mathbb{R}^{+} : t \geq \frac{b_0 - u}{c - a} \text{ and } \left(\frac{\lambda}{\beta} - a\right) t \leq b_0\right\},$$

$$\frac{\lambda}{\beta} - a < 0;\ u > 0 \text{ and } t \geq 0 \Rightarrow \left\{t \in \mathbb{R}^{+} \text{ et } \left(\frac{\lambda}{\beta} - a\right) t \leq b_0\right\} = \mathbb{R}_{+}.$$

Thus

$$J = \left[\frac{b_0 - u}{c - a}; +\infty\right[. \tag{4.8}$$

Using relations (4.7) and (4.8), the integral $I_{b,3}(u)$ can be written as:

$$I_{b,3}(u) = \int_0^{\frac{b_0 - u}{c - a}} e^{-\delta t} \phi_b(u + ct - x) dG(t)$$

$$+ \int_{\frac{b_0 - u}{c - a}}^{\infty} e^{-\delta t} \phi_b(at + b_0 - x) dG(t)$$

$$= \lambda \int_0^{\frac{b_0 - u}{c-a}} e^{-(\delta+\lambda)t} \phi_b\left(u + ct - \frac{\lambda}{\beta} t\right) dt$$

$$+ \lambda \int_{\frac{b_0 - u}{c-a}}^{\infty} e^{-(\delta+\lambda)t} \phi_b\left(at + b_0 - \frac{\lambda}{\beta} t\right) dt. \tag{4.9}$$

By analogy, we have

$$I_{b,4}(u) = \int_0^{\frac{b_0 - u}{c-a}} \int_{u+ct}^{\infty} e^{-\delta t} w(u + ct, x - u - ct) F_M(x; t)$$

$$+ \int_{\frac{b_0 - u}{c-a}}^{\infty} \int_{at+b_0}^{\infty} e^{-\delta t} w(at + b_0, x - at - b_0) F_M(x; t)$$

$$= \int_{K'} e^{-\delta t} w(u + ct, x - u - ct) dG(t)$$

$$+ \int_{J'} e^{-\delta t} w(at + b_0, x - at - b_0) dG(t), \tag{4.10}$$

where

$$K' = \left\{ t \in \mathbb{R}^+ : t \leq \frac{b_0 - u}{c - a} \text{ and } 0 \leq x = \frac{\lambda}{\beta} t \geq u + ct \right\}$$

$$= \left\{ t \in \mathbb{R}^+ : t \leq \frac{b_0 - u}{c - a} \text{ and } \left(\frac{\lambda}{\beta} - c\right) t \geq u \right\}$$

or $\left\{ t \in \mathbb{R}^+ \text{ and } \left(\frac{\lambda}{\beta} - c\right) t \geq u \right\} = \varnothing.$

Therefore,

$$K' = \varnothing, \tag{4.11}$$

$$J' = \left\{ t \in \mathbb{R}^+ : t \geq \frac{b_0 - u}{c - a} \text{ and } x = \frac{\lambda}{\beta} t \geq at + b_0 \right\}$$

$$= \left\{ t \in \mathbb{R}^+ : t \geq \frac{b_0 - u}{c - a} \text{ and } \left(\frac{\lambda}{\beta} - a\right) t \geq b_0 \right\}.$$

Since

$$\left\{t \in \mathbb{R}^+ \text{ and } \left(\frac{\lambda}{\beta} - a\right)t \geq b_0\right\} = \varnothing,$$

we have

$$J' = \varnothing. \tag{4.12}$$

By injecting relations (4.11) and (4.12) into (4.10), we obtain

$$I_{b,4}(u) = 0. \tag{4.13}$$

Assume

$$I_b^\star(u) = I_{b,3}(u) + I_{b,4}(u).$$

Then using the relations (4.9) and (4.13), we have

$$I_b^\star(u) = \lambda \int_0^{\frac{b_0-u}{c-a}} e^{-(\delta+\lambda)t} \phi_b\left(u + ct - \frac{\lambda}{\beta}t\right) dt$$

$$+ \lambda \int_{\frac{b_0-u}{c-a}}^{\infty} e^{-(\delta+\lambda)t} \phi_b\left(at + b_0 - \frac{\lambda}{\beta}t\right) dt. \tag{4.14}$$

The relation (4.14) can be put in the form:

$$I_b^\star(u) = \lambda \int_0^{\infty} e^{-(\delta+\lambda)t} \phi_b\left(\left(u + ct - \frac{\lambda}{\beta}t\right) \wedge \left(at + b_0 - \frac{\lambda}{\beta}t\right)\right) dt. \tag{4.15}$$

From the relations (3.3), (4.4) and (4.15), the Gerber-Shiu function $\phi_b(u)$ can be put into the form:

$$\phi_b(u) = \lambda(1-\alpha) \int_0^{\infty} e^{-(\delta+\lambda)t} \sigma_b((u + ct) \wedge (at + b_0)) dt$$

$$+ \alpha\lambda \int_0^{\infty} e^{-(\delta+\lambda)t} \phi_b\left(\left(u + ct - \frac{\lambda}{\beta}t\right) \wedge \left(at + b_0 - \frac{\lambda}{\beta}t\right)\right) dt. \tag{4.16}$$

Setting $s = u + ct$; $s = u + ct - \frac{\lambda}{\beta} t$ in the relation (4.16), we have

$$\phi_b(u) = \frac{\lambda}{c}(1-\alpha)\int_u^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)} \sigma_b\left(s \wedge \left(a\frac{s-u}{c} + b_0\right)\right)ds$$

$$+ \frac{\alpha\beta\lambda}{\beta c - \lambda}\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)} \phi_b\left(s \wedge \left(\beta\left(\frac{a - \frac{\lambda}{\beta}}{\beta c - \lambda}\right)(s-u)\right) + b_0\right)ds. \tag{4.17}$$

Theorem 4.1. *The Gerber-Shiu function* $\phi_b(u)$ *satisfies the following integro-differential equation*:

$$\left(\mathcal{D} - \frac{\beta(\delta+\lambda)}{\beta c - \lambda}\ell\right)\left(\mathcal{D} - \frac{\delta+\lambda}{c}\ell\right)\phi_b(u)$$

$$= \left(\frac{\beta\lambda(\delta+\lambda)(1-\alpha)}{c(\beta c - \lambda)}\ell - \frac{\lambda}{c}(1-\alpha)\mathcal{D}\right)\sigma_b(u)$$

$$+ \left(\frac{\alpha\beta\lambda(\beta c(\delta+\lambda) - \lambda)}{c(\beta c - \lambda)^2}\ell - \frac{\alpha\beta\lambda}{\beta c - \lambda}\mathcal{D}\right)\phi_b(u), \tag{4.18}$$

where $\mathcal{D}$ *and* ℓ *are the differentiation and identity operators*, *respectively*.

Proof. Differentiating the function $\phi_b(u)$ in the relation (4.17) with respect to *u*, we have

$$\phi_b'(u) = \frac{\lambda}{c}(1-\alpha)\left(\frac{\delta+\lambda}{c}\right)\int_u^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)} \sigma_b\left(s \wedge \left(a\frac{s-u}{c} + b_0\right)\right)ds$$

$$+ \frac{\alpha\beta^2\lambda(\delta+\lambda)}{(\beta c - \lambda)^2}\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}$$

$$\times \phi_b\left(s \wedge \left(\beta\left(\frac{a-\frac{\lambda}{\beta}}{\beta c-\lambda}\right)(s-u)\right)+b_0\right)ds$$

$$-\frac{\lambda}{c}(1-\alpha)\sigma_b(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(u). \tag{4.19}$$

Using the differentiation and identity operators $\mathcal{D}$ and ℓ, we calculate $g(u)=\left(\mathcal{D}-\frac{\delta+\lambda}{c}\ell\right)\phi_b(u)$. We have

$$g(u)=-\frac{\lambda}{c}(1-\alpha)\sigma_b(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(u)+\frac{\alpha\beta\lambda}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)$$

$$\times\int_u^\infty e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(\beta\left(\frac{a-\frac{\lambda}{\beta}}{\beta c-\lambda}\right)(s-u)\right)+b_0\right)ds. \tag{4.20}$$

Differentiating the function $g(u)$ in the relation (4.20) with respect to u, we have

$$g'(u)=-\frac{\lambda}{c}(1-\alpha)\sigma_b'(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b'(u)$$

$$+\frac{\alpha\beta^2\lambda}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)$$

$$\times\int_u^\infty e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(\beta\left(\frac{a-\frac{\lambda}{\beta}}{\beta c-\lambda}\right)(s-u)\right)+b_0\right)ds$$

$$-\frac{\alpha\beta\lambda}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\phi_b(u). \tag{4.21}$$

Using the differentiation and identity operators $\mathcal{D}$ and ℓ, we calculate $h(u)=\left(\mathcal{D}-\frac{\beta(\delta+\lambda)}{\beta c-\lambda}\ell\right)g(u)$. We have

$$h(u) = \frac{\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}(1-\alpha)\sigma_b(u) - \frac{\lambda}{c}(1-\alpha)\sigma_b'(u) - \frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b'(u)$$

$$+ \frac{\alpha\beta\lambda(\beta c(\delta+\lambda)-\lambda)}{c(\beta c-\lambda)^2}\phi_b(u). \tag{4.22}$$

From the relations (4.20) and (4.22), we deduce (4.18). □

5. Conclusion

In this article, we have determined the integro-differential equation of Gerber-Shiu function in the compound Poisson risk model with variable threshold dividend payment strategy to shareholders and a dependence between claim amounts and inter-claim times via the Spearman copula. Determination of the ultimate ruin probability is our future goal.

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