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# SIMULATION OF TWO-STEP ORDER 2 IMPLICIT STRONG METHOD FOR APPROXIMATING STRATONOVICH STOCHASTIC DIFFERENTIAL EQUATIONS

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## Abstract

This paper introduces a novel two-step order strong scheme to numerically solve Stratonovich Stochastic Differential Equations (SDEs) of order 2. The approach involves a unique technique that replaces stochastic integrals $J_\alpha$ with random variables, eliminating the need for their explicit calculation. The methodology combines the Stratonovich-Taylor expansion with the Runge-Kutta method to obtain approximate solutions with the desired order of accuracy. To validate

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the method's effectiveness, the paper includes experimental results that assess the approximation quality and quantify the associated errors.

## 1. Introduction

In this context, we are dealing with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$. Within this framework, there exists a $d$-dimensional Wiener martingale $W = (W_t)_{t\geq 0}$, with respect to this filtration. We are concerned with stochastic differential equations of the Stratonovich type. Its general formulation can be expressed as follows:

$$d\hat{x}_t = a(t, \hat{x}_t)dt + \sum_{j=1}^{d} b_j(t, \hat{x}_t) \circ dW_t^j, \quad \hat{x}_0 = \hat{x}, 0 \leq t \leq T. \tag{1.1}$$

The functions $a$ and $b$ involved in the equation are defined as Borel measurable functions on the domain $[0, \infty) \times \mathbb{R}^d$, with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, respectively. In our study, we adopt numerical methods based on equidistant time discretization, where the time points are given by $t_n = t_0 + nh$, and the step size is $h = \frac{T - t_0}{N}$ for positive integers $N$. We employ these methods within the specified time interval $[t_0, T]$ to approximate the trajectories of solutions. It is essential to work with discrete time steps because numerical methods for stochastic differential equations are inherently recursive in nature. We primarily focus on the strong convergence scheme in this paper. We can determine that a sequence of approximation processes, denoted as $(\hat{x}(jh))_{j=1}^{j=N}$, converges strongly with an order of $p$ to the solutions $\hat{x}^{(j)}$ of the stochastic differential equation (1.1) at time $T$ if certain conditions are met. Specifically, this convergence occurs when there exist a positive constant $C$ and a positive value $\delta_0$ such that the following holds for every $h \in (0, \delta_0]$:

$$(\mathbb{E}(\max_{j=1}^{N}|\ \hat{x}^{(j)} - \hat{x}(jh)\ |^2))^{\frac{1}{2}} \leq Ch^p. \tag{1.2}$$

For SDEs (1.1), the basic scheme which we have is that of Euler-Maruyama:

$$\hat{x}_i^{(j+1)} = \hat{x}_i^{(j)} + a_i(t_j,\ \hat{x}_i^{(j)})h + \sum_{k=1} b_{ik}(t,\ \hat{x}^{(j)})\Delta W_k^{(j)}. \tag{1.3}$$

In order to get Milstein scheme, we add the quadratic terms:

$$\hat{x}_i^{(j+1)} = \hat{x}_i^{(j)} + a_i(t_j,\ \hat{x}_i^{(j)})h + \sum_{k=1} b_{ik}(t,\ \hat{x}^{(j)})\Delta W_k^{(j)}$$

$$+ \sum_{k,\,l=1}^{d} \rho_{ikl}(t_j,\ \hat{x}^{(j)}) I_{kl}^{(j)}, \tag{1.4}$$

where

$$\Delta W_k^{(j)} = W_k((j+1)/h) - W_k(jh),$$

$$I_{kl}^{(j)} = \int_{jh}^{(j+1)h} \{W_k(t) - W_k(jh)\} dW_l(t)$$

and

$$\rho_{ikl}(t,\ \hat{x}) = \sum\nolimits_{m=1}^{q} b_{mk}(t,\ \hat{x}) \frac{\partial b_{il}}{\partial \hat{x}_m} b_{ml}(t,\ \hat{x})$$

[6]. In practice, obtaining analytical solutions for stochastic differential equations (SDEs), as described in equation (1.1), is often infeasible. Consequently, it becomes essential to develop stochastic numerical methods that can provide higher-order implicit two-step approximate solutions for such SDEs. The primary objective of this paper is to introduce a novel approach that is both computationally efficient and does not impose significant computational costs. One of the challenges in achieving higher-order approximations for SDEs is the need to include additional terms from the Stratonovich-Taylor expansion. This expansion adds complexity,

particularly, when dealing with the computation of iterated stochastic integrals $J_{\alpha}$. In recent years, various effective methods have been proposed to address this challenge, as documented in references such as, [1-9, 11-14]. However, these existing methods still do not fully meet the requirements for our specific goals. To overcome these limitations, we have incorporated perturbation and coupling techniques into our numerical scheme. This approach allows us to achieve a two-step higher-order approximation with an order of 2.

### 1.1. Perturbation method

The fundamental concept behind this method is to address the challenge of simulating a random variable that comprises two independent random variables, especially when generating either of them is challenging. To illustrate this idea, consider the scenario where we aim to simulate $U = X + \varepsilon Y$. Here, $X$ and $Y$ are independent random variables, with $X$ having a smooth probability density, and $\varepsilon$ being a small parameter. Generating $U$ directly by generating both $X$ and $Y$ can be difficult in cases where $Y$ is hard to generate. A more practical alternative is to introduce another random variable, let us call it $Z$, which is easy to generate and independent of $X$. The key idea here is that $Z$ should have moments up to order $m-1$ identical to those of $Y$. In other words, $E(Z^k) = E(Y^k)$ for all $k$ from 1 to $m-1$. Consequently, it can be demonstrated that $V = X + \varepsilon Z$ serves as an approximation to $U$, with an error on the order of $\varepsilon^m$ [6]. To provide some justification for this approximation, consider the probability density functions. If $f_X$ represents the density of $X$ and so on, then we can express $f_U(\hat{x})$ as $E[f_X(\hat{x} - \varepsilon Y)]$. This leads to

$$f_U(\hat{x}) = f(\hat{x}) + \sum_{k=1}^{m-1} \frac{-\varepsilon}{k!} f^{(k)}(\hat{x}) E(Y^k) + O(\varepsilon^m).$$

Because $Z$ shares the same moments as $Y$, we obtain an analogous expression for $f_V(\hat{x})$. As a result, the difference between $f_U(\hat{x})$ and $f_V(\hat{x})$ is of the order $O(\varepsilon^m)$. This approximation is also reflected in the Wasserstein distance, which is estimated to have the same order, namely $W(U, V) = O(\varepsilon^m)$. This implies that there exists a coupling between $U$ and $V$ for which $E(U - V)^2 = O(\varepsilon^{2m})$ [6].

### 1.2. Coupling

When we refer to coupling probability measures, we are essentially creating a connection between two distinct probability spaces. Consider two probability spaces $(X, \mu)$ and $(Y, \nu)$. The act of coupling $\mu$ and $\nu$ involves the creation of two random variables called $\overline{X}$ and $\overline{Y}$, introduced on a shared probability space $(\Omega, P)$. The essential requirement for this coupling is that the distribution, or law, of $\overline{X}$ corresponds to $\mu$, while the law of $\overline{Y}$ corresponds to $\nu$. Together, this pair of random variables, $(\overline{X}, \overline{Y})$, forms a coupling of $(\mu, \nu)$. To simplify the language, we also refer to the distribution of $(\overline{X}, \overline{Y})$ as a coupling of $(\mu, \nu)$ [10].

## 2. Implicit Two-step Order 2 Strong Scheme for Stratonovich Stochastic Differential Equations

To achieve higher accuracy when solving Stratonovich stochastic differential equations (SDEs), it is often necessary to include additional stochastic integral terms through stochastic Taylor expansion. While this leads to more accurate methods, it also results in an increased number of partial derivatives in the equations, making numerical solutions computationally challenging. As a result, there is a growing demand for the development of derivative-free schemes to address this issue, typically involving the replacement of partial derivatives in stochastic Taylor approximations with difference quotients. The following $L^p$ version of Theorem 4 [6] plays a crucial role in establishing the error bound for the modified variable $Y$.

**Theorem 1.** *Assume that the matrix* $(b_{ik})$ *has rank q. Suppose that the random variables* $L_\alpha$ *have all moments finite and that* $\mathbb{E}(K_{\alpha_1} \dots K_{\alpha_r}) = \mathbb{E}(L_{\alpha_1} \dots L_{\alpha_r})$, *whenever* $\alpha_1, \dots, \alpha_r \in \mathcal{M}_m$ *satisfy* $\sum_{k=1}^{r}(l(\alpha_k) - 1) \le m - 1$. *Then for* $p \ge 2$, $\mathbb{W}_p(Y, \overline{Y}) \le C\varepsilon^{m+1}$ [6].

**Proof.** To prove Theorem 1, we use the same argument as Sandy has mentioned in the second paragraph of Section 4 in [7] by making the argument for $p$. □

When it comes to computing numerical solutions, multi-step methods offer increased efficiency compared to one-step methods. This is primarily because multi-step methods necessitate just one evaluation of the right-hand side to compute the differential equation. Additionally, multi-step methods can exhibit better stability, particularly for larger time steps, although it is worth noting that unstable multi-step methods do exist. For Stratonovich stochastic differential equations like the one in (1.1), we can refer to the formula provided in [12].

### 2.1. Moment calculation

To calculate the moments, we have used the following three lemmas:

**Lemma 1** (Lemma 5, [6]). *Let* $\beta = (jj \cdots j)$ *with length* $l \ge 2$. *Then* (i) *if* $j = 0$, *then* $K_\beta = \frac{1}{l!}$, *and* (ii) *if* $j > 0$, *then* $K_\beta = 0$ *if l is odd*, *while* $K_\beta = \frac{(-1)^r}{2^r r!}$ *if* $l = 2r$.

**Lemma 2** (Lemma 6, [6]). *If* $\beta_1, \dots, \beta_s \in \mathcal{M}$ *and if some* $j \ge 1$ *occurs an odd number of times in the concatenated multi-index* $\beta_1 \dots \beta_s$, *then* $\mathbb{E}(K_{\beta_1} \dots K_{\beta_s}) = 0$.

**Lemma 3** (Lemma 7, [6]). (i) *If* $0 \le k < l$, *then* $K_{lk} = -K_{kl}$ *and* $\mathbb{E}(K_{kl}^2) = \frac{1}{12}$.

(ii) *If* $k > 0$, *then* $\mathbb{E}K_{0kk} = \mathbb{E}K_{k0k} = \mathbb{E}K_{kk0} = -\frac{1}{6}$.

$K_{lk} = -K_{kl}$ can be proved by using integration by parts as follows:

$$\int_0^1 B_k dB_l = B_1 B_1 - \int_0^1 B_l dB_k = 0 - \int_0^1 B_l dB_k = -\int_0^1 B_l dB_k,$$

where $B_k$ and $B_l$ are independent.

### 2.2. Main scheme

Consider the nonautonomous multi-dimensional case with scalar additive noise $d = 1, 2, \ldots$ and $m = 1$. There is a family of implicit two-step order 2 strong schemes for which $i$th component is

$$\begin{aligned}\hat{x}_i^{j+1} &= (1-\gamma_i) + \gamma_i x_i^{j-1} \\ &\quad + \frac{1}{2}\{\underline{a}((j+1), \hat{x}^{j+1}) + (1+\gamma_i)\underline{a} + \gamma_i \underline{a}_i((j-1), \hat{x}^{j-1})\}\Delta \\ &\quad - \frac{1}{2}(1-\gamma_i)\underline{L}^1 \underline{a}_i((j-1), \hat{x}^{j-1})\Delta W^{j-1}\Delta \\ &\quad - \frac{1}{4}(1-\gamma_i)\underline{L}^1 \underline{L}^1 \underline{a}_i (\Delta W_i)^2 \Delta \\ &\quad + V_i^j + \gamma_i V_i^{j-1} \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} V_i^j &= b^j \Delta W_i + \frac{\partial b_i}{\partial t}\{\Delta W_i - \Delta Z_i\} \\ &\quad + \underline{L}^1 \underline{a}_i \left\{\Delta Z_i - \frac{1}{2}\Delta W_i \Delta\right\} \\ &\quad + \underline{L}^1 \underline{L}^1 \underline{a}_i \left\{J_{(1,1,0)_{jh,(j+1)h}} - \frac{1}{4}(\Delta W_i)^2 \Delta\right\}, \end{aligned} \tag{2.2}$$

where $\underline{L}^1 = b\frac{\partial}{\partial \hat{x}}$ and $\gamma_i \in [0, 1]$ for $i = 1, \ldots, d$.

### 2.3. Experimental example

The following system of Stratonovich SDEs had been simulated by using (2.1) for number of steps $N = 400$:

$$\begin{cases} dx_1 = x_1 dt + (\sin^2(x_1) + 1) \circ dW_t - (\cos^2(x_2)) \circ dV_t, \\ dx_2 = \dfrac{t+2}{1+x_2^2} dt + \cos^2(x_1) \circ dW_t + (\sin^2(x_2) + 1) \circ dV_t, \\ x_1(0) = 1, \\ x_2(0) = 2, \\ 0 \le t \le 1. \end{cases} \tag{2.3}$$

Figure 1 displays a piecewise linear curve representing approximate solutions within the interval $[0, 1]$ at time steps of $h = \frac{1}{N}$, where $N$ is equal to 200. These approximations are obtained through the utilization of equation (2.1).

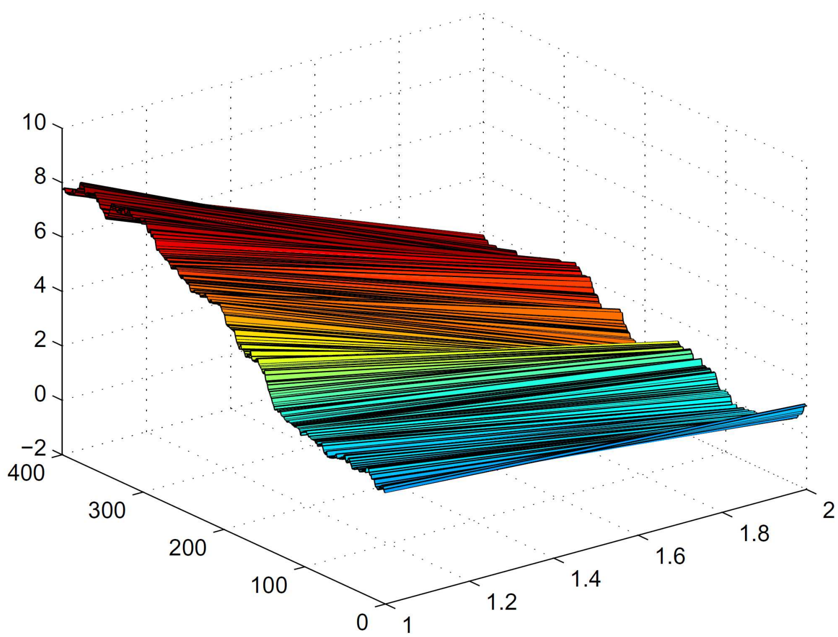


**Figure 1.** Approximate solutions of (2.3).

## 3. Conclusion

The sponsored method has importance across various fields due to the absence of an efficient approach for computing higher-order numerical solutions within a two-step framework for stochastic differential equations (SDEs). Its applicability extends to the computation of strong approximations for SDEs of varying dimensions without incurring a substantial computational burden.

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