Advances in Differential Equations and Control Processes
© 2023 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324323019
Volume 30, Number 4, 2023, Pages 345-361 **P-ISSN: 0974-3243**

PICARD'S METHOD OF SUCCESSIVE APPROXIMATION FOR FRACTIONAL ORDER INITIAL VALUE PROBLEM

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Abstract

In this paper, we derived Picard's successive approximation technique for fractional differential systems in which the derivative has been taken in the Riemann-Liouville sense. We investigated the existence and uniqueness results of the present method. Two numerical examples are given to show the efficiency of the presented method.

1. Introduction

Fractional calculus serves as an extension of classical calculus, delving into fractional-order derivatives and integrals, along with their associated

Received: August 27, 2023; Accepted: November 9, 2023

2020 Mathematics Subject Classification: 26A33, 34A08, 74S40.

Keywords and phrases: Riemann-Liouville fractional order differential equations, Picard's method of successive approximation, initial value problem, existence.

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How to cite this article: Jag Mohan and Anju Sood, Picard's method of successive approximation for fractional order initial value problem, Advances in Differential Equations and Control Processes 30(4) (2023), 345-361. http://dx.doi.org/10.17654/0974324323019

Published Online: November 30, 2023

properties, as noted in [14]. The genesis of fractional calculus dates back to Leibniz's 1695 letter to l'Hospital [22]. In contemporary times, the significance of fractional calculus has grown due to its capacity to broaden the scope of differential equations from integer to real number orders [9]. Notably, fractional differential equations have captivated the scientific community, offering enhanced precision in describing real-world phenomena, as exemplified by viscoelastic material stress-strain relations that find optimal representation in fractional differential equations. Similarly, while integer-order differential equations might suffice for modeling population growth or decay, complex scenarios such as wars and epidemics necessitate fractional population models [4, 8, 11, 20].

The applications of fractional-order differential equations span diverse domains, including engineering, chemistry, biology, and economics [7, 9, 10, 13, 15, 16, 19, 21]. Consequently, solving fractional-order differential equations remains both extensive and consequential [10]. The arsenal of analytical techniques-comprising the Green function method, Laplace transform method, and power series method-equips researchers with tools to derive exact solutions for fractional-order differential equations [2, 13]. However, it is important to acknowledge that numerous fractional-order differential equations defy exact solutions. In these scenarios, mathematicians have strived to devise approximate solutions by extending existing numerical methods to encompass fractional-order differential equations [5, 6, 24, 25].

Picard's iterative method gives a sequence $y_n(x)$ of approximate solutions to the initial value problem

$$\begin{cases} \dfrac{d}{dx}(y(x)) = f(x, y), \\ y(x_0) = y_0, \end{cases}$$

which converges to the exact solution as $n \to \infty$ [23]. As per the study done by Lyons et al. [13], Picard's iterative method has been extended to solve fractional order initial value problems where the derivative has been taken in the Caputo sense.

Motivated by the above work, we consider the following initial value problem:

$$\begin{cases} {}_{x_0}^{RL}D_x^{\alpha} y(x) = f(x, y), \\ {}_{x_0}^{RL}D_x^{\alpha-1} y(x_0) = y_0, \\ \lim\limits_{x \to x_0} y(x) = y_0, \end{cases} \tag{1}$$

where $0 < \alpha < 1$ and ${}_{x_0}^{RL}D_x^{\alpha} y(x)$ represents the Riemann-Liouville fractional derivative of order α of the function $y(x)$.

2. Preliminaries

This section discusses some basic definitions and important results required to prove our main results.

Theorem 2.1 (The mean value theorem for derivatives) [1]**.** *If* $f(x)$ *is continuous function on the interval* $[a, b]$ *and differentiable on the open interval* (a, b), *then there exists in* (a, b) *at least one number c such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Definition 2.2 [17]**.** The Gamma function is the generalization of factorial *n*! and it allows *n* to take any real or even complex value. The *Gamma function* is defined by the formula

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{(x-1)} dt, \quad x \in \mathbb{R}^+.$$

Definition 2.3 [17]**.** The *Beta function* is defined by the definite integral

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad x, y \in \mathbb{R}^+.$$

Definition 2.4 [18]**.** Let $f(x)$ be a piecewise continuous function on $(0, \infty)$, integrable on any finite sub interval of $[0, \infty]$ and α be a non negative real number. Then for $t > 0$, *Riemann-Liouville fractional integral* of $f(x)$ of order α is defined by the formula

$$ {}_{x_0}^{RL}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0. $$

We used the following symbol ${}_{x_0}^{RL}I_x^{\alpha} f(x)$ for Riemann-Liouville integration of $f(x)$ that is

$$ {}_{x_0}^{RL}I_x^{\alpha} f(x) = {}_{x_0}^{RL}D_x^{-\alpha} f(x). $$

A special case of Riemann-Liouville fractional integral is when $x_0 = 0$. Then the Riemann-Liouville operator becomes ${}_{0}^{RL}D_x^{-\alpha} y(x)$, and the formula becomes

$$ {}_{0}^{RL}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0. $$

From above equation we can easily see that

$$ {}_{x_0}^{RL}I_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} x^{(\mu+\alpha)}, \quad \alpha > 0,\ \mu > -1,\ x > 0. $$

Definition 2.5 [26]**.** Let $f(t)$ be continuous on $[0, \infty]$ and $\mu, \nu > 0$. Then for all *t*,

$$ D^{-\mu}[D^{-\nu} f(t)] = D^{-(\mu+\nu)} f(t) = D^{-\nu}[D^{-\mu} f(t)]. $$

Definition 2.6 (Riemann-Liouville fractional derivatives [3])**.** The *fractional derivative* of order α is defined by using the fractional integration as

$$ {}_{x_0}^{RL}D_x^{\alpha} f(x) = D^n[{}_{x_0}^{RL}D_x^{-(n-\alpha)} f(x)], \quad \alpha \in \mathbb{R}^+,\ n-1 < \alpha < n. $$

Definition 2.7 [3]**.** The *Caputo derivative* ${}_{x_0}^{C}D_x^{\alpha} f(x)$ of order $\alpha > 0$ for the real valued function $f(x)$ is defined as

$$ {}_{x_0}^{C}D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{x_0}^{x} f^{n}(t) \frac{1}{(x-t)^{\alpha+1-n}} dt, \quad n-1 < \alpha < n. $$

3. Results

In the present section, we have given Picard's method of successive approximation for the fractional order differential equations in which we have taken Riemann-Liouville fractional order derivative. After applying the Riemann-Liouville fractional derivative of order $1-\alpha$ in (1) we have

$$ y' = {}_{x_0}^{RL}D_x^{1-\alpha} f(x, y). \tag{2} $$

Suppose that ${}_{x_0}^{RL}D_x^{1-\alpha} f(x, y)$ is continuous function in some neighbourhood of (x_0, y_0). Therefore, from (2) we can write

$$ y(x) = y(x_0) + \int_{x_0}^{x} {}_{x_0}^{RL}D_x^{1-\alpha} f(t, y(t)) dt. \tag{3} $$

In order to avoid confusion, we have used the dummy variable t instead of x. By using the index law of fractional integral and fractional differential operator [12] in the above equation, we have

$$ y(x) = y(x_0) + {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t)). \tag{4} $$

For finding the solution of (1), we will find the solution of (4) by an approximate method and improve it by using a repeatable process and try to take it near to an exact solution as much as we want.

Clearly, $y(x_0) = y_0$ is a constant function and is a solution of (4). In order to find the first approximation we have used the $y(x_0) = y_0$ in the right hand side of (4). The first approximation y_1 is given by the equation

$$ y_1 = y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y_0). $$

We will use y_1 in the right hand side of (4) to get the next approximation y_2,

$$y_2 = y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y_1).$$

By continuing in this way we get the *n*th approximation y_n,

$$y_n = y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y_{n-1}).$$

In next theorem, we have shown that sequence y_n converges to the exact solution of (4).

Theorem 3.1. *Let* $f(x, y)$, ${}_{x_0}^{RL}D_x^{1-\alpha} f(x, y)$ *and* $\frac{\partial f}{\partial y}$ *be the continuous functions on some closed rectangle R and whose sides are parallel to the axes. Let* (x_0, y_0) *be an interior point of R. Then there exists a real number* $h > 0$ *such that in the interval* $|x - x_0| \leq h$ *initial value problem represented by* (1) *has a unique solution* $y = y(x)$, *where* $0 < \alpha < 1$.

Proof. It is clear that any solution of the initial value problem represented by (1) is a continuous solution of the integral equation (4). We have used it to prove that the initial value problem given by (1) has a unique solution in the interval $x - x_0 \leq h$ and consequently in the interval $|x - x_0| \leq h$. For this, we have proved that the sequence given by

$$\begin{aligned} y_0(x) &= y_0, \\ y_1(x) &= y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y_0), \\ y_2(x) &= y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y_1), \\ &\vdots \\ y_n(x) &= y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y_{n-1}), \qquad (5) \\ &\vdots \end{aligned}$$

converges to an exact solution of (4). It is easy to observe that sequence $y_n(x)$ is the sequence of *n*th partial sum of the infinite series given by

$$y_0(x) + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)]$$
$$= y_0(x) + [y_1(x) - y_0(x)]$$
$$+ [y_2(x) - y_1(x)] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots. \quad (6)$$

Therefore convergence of (6) will ensure the convergence of (5). For the completion of the proof we will find $h > 0$ that gives the interval $|x - x_0| \le h$ and on this interval we will show that

Step 1. The series (6) converges to a function $y(x)$.

In the hypothesis of the theorem we have taken $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on closed rectangle *R*. Therefore, $f(x, y)$ and $\frac{\partial f}{\partial y}$ are bounded on *R*. Therefore, there exist numbers $M > 0$ and $K > 0$ such that

$$|f(x, y)| \le M, \quad (7)$$

$$\left|\frac{\partial f}{\partial y}\right| \le K, \quad (8)$$

for all points in rectangle *R*.

Let (x, y_1) and (x, y_2) be two distinct points in rectangle *R*. Then by mean value theorem for derivative, we have

$$|f(x, y_1) - f(x, y_2)| \le \left|\frac{\partial}{\partial y} f(x, y^*)\right| |y_1 - y_2|, \quad (9)$$

where y^* lies between y_1 and y_2. From (8) and (9), we have

$$|f(x, y_1) - f(x, y_2)| \le K |y_1 - y_2|, \quad (10)$$

where (x, y_1) and (x, y_2) lie on the same vertical line in rectangle *R*. Now we choose $h > 0$ such that

$$kh^{\alpha} < 1, \quad (11)$$

and the rectangle formed by $|x - x_0| \leq h$ and $|y - y_0| \leq \max\left\{\frac{M}{\Gamma(\alpha+1)} h^{\alpha}\right\}$ lies in rectangle R. Let us denote this rectangle by R_1. It is easy to observe that such h must exist because (x_0, y_0) is an interior point of rectangle R.

For the proof of Step 1, it is enough to show that series

$$|y_0(x)| + \sum_{n=1}^{\infty} |[y_n(x) - y_{n-1}(x)]|$$

$$= |y_0(x)| + |[y_1(x) - y_0(x)]| \tag{12}$$

$$+ |[y_2(x) - y_1(x)]| + \cdots + |[y_n(x) - y_{n-1}(x)]| + \cdots. \tag{13}$$

In order to prove the convergence of (12) we will find $|y_n(x) - y_{n-1}(x)|$. Firstly we will show that graph of $y_n(x)$ lies in R_1 for all n and consequently in R. It is true for $y_0(x) = y_0$. Therefore, points $(x, y_0(x))$ are lies in R_1 and hence from (7) we have $|f(x, y_0(x))| \leq M$,

$$\begin{aligned} |y_1(x) - y_0| &= |{}_{x_0}^{RL}I_x^{\alpha} f(t, y_0(t))| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t, y_0(t)) dt \right| \\ &\leq \frac{M}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} dt \\ &\leq \frac{M}{\Gamma(\alpha+1)} h^{\alpha}. \end{aligned}$$

Therefore,

$$|y_1(x) - y_0| \leq \frac{M}{\Gamma(\alpha+1)} h^{\alpha}.$$

Hence, graph of $y_1(x)$ lies in R_1.

Similarly,

$$|\, y_2(x) - y_0 \,| \le \frac{M}{\Gamma(\alpha + 1)} h^{\alpha}.$$

By continuing in this way we can say that graph of $y_n(x)$, lies in R_1 for all n. Since every continuous function defined on a closed interval attains its maximum value. Here $y_1(x)$ is continuous, therefore there exists some positive number L such that

$$|\, y_1(x) - y_0 \,| \le L.$$

Since the points $(x,\ y_1(x))$ and $(x,\ y_0(x))$ lie in R_1, from (10), we have

$$|\, f(x,\ y_1(x)) - f(x,\ y_0(x)) \,| \le K |\, y_1(x) - y_0(x) \,| \le KL.$$

Therefore,

$$|\, y_2(x) - y_1(x) \,| = |\, {}_{x_0}^{RL}I_x^{\alpha} f(t,\ y_1(t)) - f(t,\ y_0(t)) \,|$$

$$\le {}_{x_0}^{RL}I_x^{\alpha} KL$$

$$\le KL \frac{h^{\alpha}}{\Gamma(\alpha + 1)}.$$

Similarly,

$$|\, f(x,\ y_2(x)) - f(x,\ y_1(x)) \,| \le K |\, y_2(x) - y_1(x) \,|$$

$$\le KLK \frac{h^{\alpha}}{\Gamma(\alpha + 1)}$$

$$= LK^2 \frac{h^{\alpha}}{\Gamma(\alpha + 1)}.$$

Therefore,

$$| y_3(x) - y_2(x) | = | {}_{x_0}^{RL}I_x^{\alpha} f(t,\ y_2(t)) - f(t,\ y_1(t))|$$

$$\leq {}_{x_0}^{RL}I_x^{\alpha} LK^2 \frac{h^{\alpha}}{\Gamma(\alpha+1)}$$

$$= L\frac{(Kh^{\alpha})^2}{(\Gamma(\alpha+1))^2}.$$

If we continue in this way, we get

$$| y_n(x) - y_{n-1}(x) | \leq L\frac{(Kh^{\alpha})^{n-1}}{(\Gamma(\alpha+1))^{n-1}} \leq L(Kh^{\alpha})^{n-1}.$$

Now from (11) and (12), we have

$$y_0(x) + \sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)]$$

$$\leq | y_0(x) | + \sum_{n=1}^{\infty} | [y_n(x) - y_{n-1}(x)] |$$

$$\leq | y_0 | + L[(Kh^{\alpha}) + (Kh^{\alpha})^2 + (Kh^{\alpha})^3 + \cdots].$$

From equation (11), it is clear that series

$$| y_0 | + L[(Kh^{\alpha}) + (Kh^{\alpha})^2 + (Kh^{\alpha})^3 + \cdots]$$

is convergent. Therefore, by M_n-test, (6) is uniformly convergent. If the series (6) converges to $y(x)$, then sequence $y_n(x)$ converges to $y(x)$.

Step 2. $y(x)$ is a continuous solution.

Since $y_n(x)$ converges uniformly to $y(x)$ and $y_n(x)$ are continuous functions, therefore, $y(x)$ is a continuous function of (4).

Step 3. $y(x)$ is the unique solution of (4).

In this part, we have proved that $y(x)$ is the unique solution of (4). In order to show that $y(x)$ is the solution of (4), we show that

$$y(x) - y_0 - {}_{x_0}^{RL}I_x^{\alpha} f[t, y(t)] = 0. \tag{14}$$

But we have

$$y_n(x) - y_0 - {}_{x_0}^{RL}I_x^{\alpha} f[t, y_{n-1}(t)] = 0, \tag{15}$$

$$| y(x) - y_0 - {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t))|$$

$$= | y(x) - y_n(x) + y_n(x) - {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t))$$

$$+ {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t)) - y_0 - {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t))|$$

$$\leq | y(x) - y_n(x) | + | {}_{x_0}^{RL}I_x^{\alpha}(t, y_{n-1}(t)) - f(t, y(t))|.$$

Now, graph of $y(x)$ lies in rectangle R therefore from (10), we can write

$$| y(x) - y_0 - {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t))|$$

$$\leq | y(x) - y_n(x) | + \frac{Kh^{\alpha}}{\Gamma(\alpha + 1)} \max| y_{n-1}(x) - y(x) |$$

$$\leq | y(x) - y_n(x) | + Kh^{\alpha} \max| y_{n-1}(x) - y(x) |.$$

Since $y_n(x)$ converges uniformly to $y(x)$ therefore right hand side of the above inequality can be made as small as we like. Therefore,

$$y(x) - y_0 - {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t)) = 0.$$

Hence, $y(x)$ is the solution of (4).

Finally, we show that solution $y(x)$ is the unique solution of (4). Let $\bar{y}(x)$ be another solution of (4) on the interval $| x - x_0 | \leq h.$ Since $\bar{y}(x)$ is a continuous solution of (4), we have

$$\bar{y}(x) = y_0 + {}_{x_0}^{RL}I_x^{\alpha} f(t, y(t)).$$

Let $T = \max| \bar{y}(x) - y_0 |$. Then in the interval $x_0 \le x \le x_0 + h$ we have

$$| \bar{y}(x) - y_1(x) | = | {}_{x_0}^{RL}I_x^{\alpha} f(t, \bar{y}(t)) - f(t, y_0(t)) |$$

$$\le K \, {}_{x_0}^{RL}I_x^{\alpha} | \bar{y}(t) - y_0 |$$

$$\le KT \frac{h^{\alpha}}{\Gamma(\alpha + 1)}.$$

Similarly,

$$| \bar{y}(x) - y_2(x) | = | {}_{x_0}^{RL}I_x^{\alpha} f(t, \bar{y}(t)) - f(t, y_1(t)) |$$

$$\le K {}_{x_0}^{RL}I_x^{\alpha} | \bar{y}(t) - y_1(t) |$$

$$\le K^2 T \frac{h^{\alpha}}{\Gamma(\alpha + 1)} \frac{h^{\alpha}}{\Gamma(\alpha + 1)}$$

$$\le \frac{T(Kh^{\alpha})^2}{(\Gamma(\alpha + 1))^2}.$$

In general, we can write that

$$| \bar{y}(x) - y_n(x) | \le \frac{T(Kh^{\alpha})^n}{(\Gamma(\alpha + 1))^n}.$$

In the same way, similar result holds in $(x_0 - h) \le x \le x_0$. Now by (11) $Kh^{\alpha} < 1$, therefore the limit of the right hand side of the above equation tends to zero as n tends to ∞. Hence $\bar{y}(x) = y(x)$. Therefore, solution $y(x)$ of (4) is a unique. Hence, initial value problem given by (1) has unique solution in the interval $| x - x_0 | \le h$.

4. Numerical Simulation

Example 4.1. In Figure 1 with the help of Matlab software, we have plotted 20 iterations of Picard's iterative method with the step size $h = 0.1$ by taking the interval $[0, 3]$ for the fractional order system

$$ {}_{x_0}^{RL}D_x^{\frac{1}{2}} y(x) = y, $$

$$ y(0) = 1. $$

It is clear from Figure 1 that after the 11th iteration all the iterations are superimposing on each other, therefore approximate solutions are converging towards a function. According to our result, this function is the required solution to the fractional order system given above. Hence, the convergence of our method is confirmed.

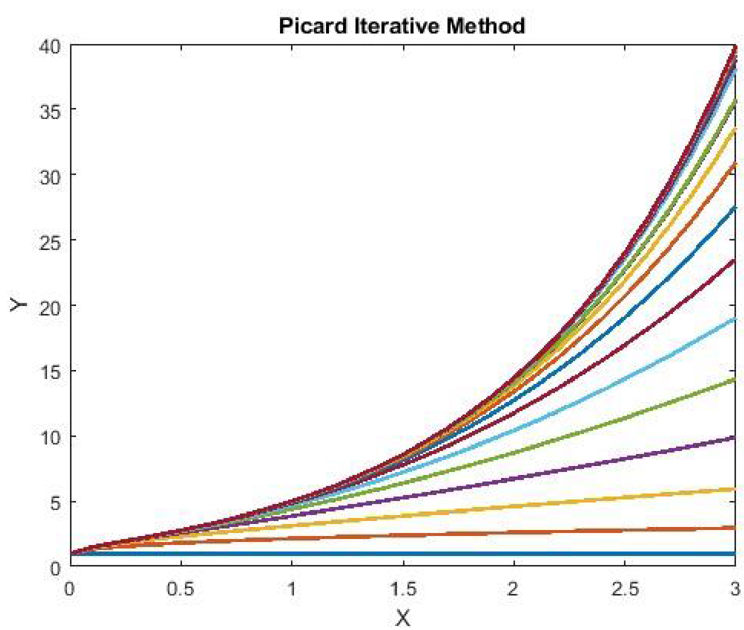

Figure 1. Graphs of ${}_{x_0}^{RL}D_x^{\frac{1}{2}} y(x) = y,\ y(0) = 1$ in $[0, 3]$.

Example 4.2. In Figure 2 with the help of Matlab software, we have plotted 20 iterations of Picard's iterative method in the interval $[0, 4]$ with the step size $h = 0.1$ of the initial value problem

$$ {}_{x_0}^{RL}D_x^{\frac{1}{2}} y(x) = -y + x^2 + \frac{8}{3\sqrt{\pi}x^{\frac{3}{2}}}, $$

$$ y(0) = 0. $$

In Figure 2, curve with red circles is the curve of exact solution and exact solution of above initial value problem is $y = x^2$. From Figure 2, it is clear that as the number of iterations are increasing the approximate solutions are converging towards exact solution.

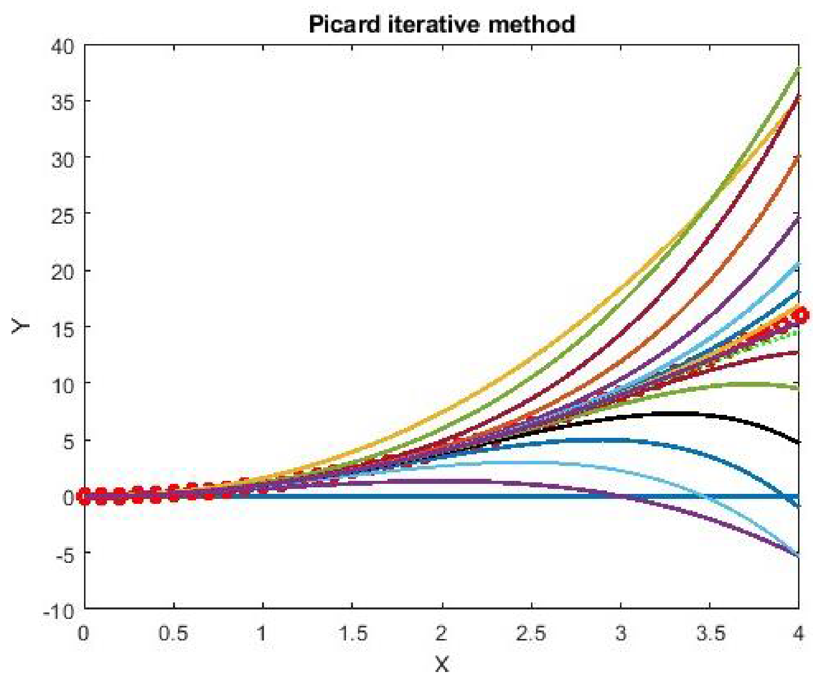

Figure 2. Graphs of ${}_{x_0}^{RL}D_x^{\frac{1}{2}} y(x) = -y + x^2 + \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}}$, $y(0) = 0$ in $[0, 4]$.

5. Conclusion

Given the established understanding that analytic approaches cannot yield solutions for all fractional-order differential equations, the reliance on numerical methods becomes paramount. In this context, numerical techniques offer a viable avenue for approximating solutions to fractional-order differential equations. In our current endeavor, we have expanded upon Picard's iterative method, tailoring it for fractional-order differential equations that entail derivatives in the Riemann-Liouville sense. To substantiate our findings, we have furnished two illustrative examples. In both cases, we observed that after a limited number of iterations, the approximated solutions progressively converge toward the precise solutions. This underscores the effectiveness and reliability of the extended method presented in our work.

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