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NUMERICAL SIMULATION OF ROSENAU-KORTEWEG-DE VRIES REGULARIZED LONG WAVE EQUATION WITH FLUX LIMITERS METHOD

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Abstract

In this paper, the flux limiter technique based on the method of lines is designed to simulate the nonlinear Rosenau-Korteweg -de Vries-regularized long wave equation. In order to illustrate the efficiency, accuracy and essentially non-oscillatory property of the present method, the error norms, discrete mass, momentum and energy conservative properties have been calculated. These calculations give good agreement for the exact solutions and numerical solutions of solitary and shock wave.

1. Introduction

Nonlinear evolution equations play an important role for the studies appeared in nonlinear sciences. The numerical solutions of nonlinear wave equations are necessary since most of these types of equations are not solvable analytically. The Korteweg-de Vries (KdV) equation is suitable for small-amplitude long waves such as shallow water waves, longitudinal wave in a channel, for example. The regularized long wave (RLW) equation is used to simulate wave motion in media with nonlinear wave steeping, dispersion and describes also shallow water waves, nonlinear dispersive waves, ion-acoustic plasma waves. The Rosenau equation was proposed for explaining the dynamic of dense discrete systems since the case of wave-wave and wave-wall interactions cannot be explained by the KdV and RLW equations. Monotonicity property preservation like positivity is essential for numerical schemes to approximate non-smooth solutions. Improved discontinuities capture need to increase accuracy of numerical scheme in order to reduce diffusivity. We use a process which is high order, non-oscillating, able to capture the shocks by using total variation diminishing (TVD) schemes, preserve monotonicity and therefore increase accuracy.

However, in the region of the domain where the change of the flux is not fast, we can recover a high order accuracy by using flux limiters method where flux limiter is a function which controls smoothness of the numerical solution and allows limiting gradients to prevent the occurrence of oscillations. We focus on the one-dimensional Rosenau-Korteweg-de Vries - regularized long wave (Rosenau-KdV-RLW) equation, model which is difficult to solve numerically because of the high order mixed derivative and nonlinearity terms. Many numerical schemes have been employed to simulate Rosenau-KdV-RLW equations but there are very few numerical schemes which have been presented for the shock wave of this equation [1-3].

In this paper, we use flux limiters method under method of lines to solve numerically the Rosenau-KdV-RLW equation model. Section 2 is a quick overview of method of lines and flux limiter technique. Section 3 provides numerical example and compares the results with the exact solution of the problem. Error norms and three conservative properties which, respectively, correspond to mass, momentum and energy are calculated to demonstrate the efficiency and accuracy of the present method. Section 4 presents the concluding remarks.

2. Quick Overview of Method of Lines and Flux Limiter Technique

2.1. Method of lines

The method of lines (MOL) is a general way of viewing a partial differential equation (PDE) as a system of ordinary differential equations (ODEs). The space variables are discretized to obtain a system of ODEs in the time variable and then a suitable initial-value problem solver is used to solve this ODE system. The method provides very accurate numerical solution for linear or nonlinear partial differential equations (PDEs). As the number of lines increases, the accuracy of the MOL representation of the original system increases [4]. The method of lines consists of two steps: space discretization and time integration. In a first step, the derivatives with respect to the space variable are approximated by finite difference schemes for example and the second step is numerical integration in time by well-known ode integrator like ode15s of MATLAB [5, 6].

2.2. Flux limiter technique

Flux limiters are widely used in numerical simulations to prevent spurious oscillation in the flow with strong property gradients [7]. In the case of PDEs with a flux term $f(u)_x$, we can approximate the partial derivative by using flux limiter. The main idea is to avoid the spurious oscillations that occur with high-order spatial discretization due to shocks, discontinuities, or steep gradients in the solution domain. Use of flux limiters, together with an appropriate high-resolution scheme, makes the solutions total variation diminishing [8, 9]. Consider the following general form of partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.$$

The flux limiters semi-discrete scheme of the above equation is written as:

$$\frac{du_i}{dt} + \frac{1}{\Delta x_i}\left[f^l_{i+\frac{1}{2}} - f^l_{i-\frac{1}{2}} - \phi(r_i)\left(f^l_{i+\frac{1}{2}} - f^h_{i+\frac{1}{2}} \right) + \phi(r_{i-1})\left(f^l_{i-\frac{1}{2}} - f^h_{i-\frac{1}{2}} \right)\right] = 0,$$

where ϕ is the flux limiter function, r represents the ratio of successive gradients on the solution mesh, $f^l_{i\pm\frac{1}{2}}$, $f^h_{i\pm\frac{1}{2}}$ are low-order flux and high-order flux at the grid points $x_{i\pm\frac{1}{2}}$, respectively. In this paper, flux limiters tested are, respectively, Van-Leer, Van-Albada 1&2, Sweby, Superbee, Ospre, Osher, SMART, Koren, MC (monotonized central difference), minmod and we shall retain best of them for Rosenau-KdV-RLW equation [9].

3. The Numerical Scheme

3.1. Description of Rosenau-KdV-RLW equation

Rosenau-KdV-RLW equation is used for describing the dynamics of shallow water waves that appear along lake shores [10]. Along the beaches and lake shores, the dynamics of dispersive shallow water wave have been treated by Rosenau-KdV-RLW equation. In oceanography, the study of dynamics of shallow water waves is very important. The dynamics of water waves can be modeled by the different differential equations as KdV, Rosenau, Rosenau-KdV, RLW, Rosenau-RLW, and Rosenau-KdV-RLW [11]. Here, we focus on one-dimensional Rosenau-KdV-RLW equation. It should be noted that, the model is difficult to solve numerically because of the great computational cost caused by high order mixed derivative term and the selective wave behavior caused by the power of nonlinearity term [2]. The Rosenau-KdV-RLW equation with the initial and boundary conditions is given by:

$$u_t + \alpha u_x + \beta (u^p)_x - \gamma u_{xxt} + \delta u_{xxx} + \mu u_{xxxxt} = 0, \quad x \in [a; b], \quad t > 0. \qquad (1)$$

The initial condition is:

$$u(x, 0) = u_0(x), \quad x \in [a; b], \qquad (2)$$

and the boundary conditions are:

$$u(a, t) = f_1(t), \quad u(b, t) = f_2(t),$$

$$u_x(a, t) = g_1(t), \quad u_x(b, t) = g_2(t),$$

$$u_{xx}(a, t) = h_1(t), \quad u_{xx}(b, t) = h_2(t), \quad t > 0, \qquad (3)$$

where u is wave profile, $\alpha \geq 0$ is the drifting of the wave coefficient, $\beta \geq 0$ is the coefficient of non-linearity, $\delta \geq 0$ is the coefficient of third order spatial dispersion, while $\gamma \geq 0$ accounts for spatio-temporal dispersion in order to consider equal-width effect. Inclusion of this spatio-temporal dispersion surely makes this model well-posed, p is the power-law nonlinearity parameter and $\mu \geq 0$ is the coefficient of highest derivative [12] and for the existence of soliton, it is necessary that $p \geq 2$. The Rosenau-KdV-RLW equation is a combination of the two forms of dispersive shallow water waves that is analogue to KdV equation. Therefore, this equation models dispersive shallow water waves [13]. When $-a \gg 0$ and $b \gg 0$, the initial boundary value problem (1)-(3) is consistent and the boundary condition (3) is reasonable [14].

3.2. Method of line discretization

We now rewrite Rosenau-Korteweg-de Vries - regularized long wave equation (1) as follows:

$$\frac{\partial}{\partial t}\left(u - \gamma \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^4 u}{\partial x^4}\right) = -\left(\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u^p}{\partial x} + \delta \frac{\partial^3 u}{\partial x^3}\right) = -(F(u))_x. \quad (4)$$

To apply the method of lines for solving the Rosenau-Korteweg-de Vries - regularized long wave equation, firstly we subdivide the solution domain into uniform meshes by the line. We use a uniform mesh of cell size h in space. The uniform mesh is distributed as follows:

$$x_i = ih, \quad i = 0, 1, 2, \ldots, n,$$

$$h = \frac{b - a}{n}$$

and we obtain the semi-discrete form by discretizing (4) at (x_i, t):

$$\left(\frac{\partial}{\partial t}\left(u - \gamma \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^4 u}{\partial x^4}\right)\right)_i = -\alpha\left(\frac{\partial u}{\partial x}\right)_i - \left(\left(\beta \frac{\partial u^p}{\partial x} + \delta \frac{\partial^3 u}{\partial x^3}\right)\right)_i. \quad (5)$$

For the second and fourth derivatives at (x_i, t) in (5), we use three-point central finite difference and five-point central finite difference, respectively, to approximate them, so we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + o(h^2),$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{h^4} + o(h^2).$$

Then equation (5) can be rewritten as following:

$$\frac{d}{dt}\left[u_i - \gamma \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mu \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{h^4}\right] = F_i.$$

Introducing the boundary conditions, this yields a system of ordinary differential equations which depend on t in the following form:

$$A\frac{dU_i}{dt} = F(U_i), \quad i = 1, 2, ..., n, \tag{6}$$

where the $n \times n$ matrix A is:

$$A = \begin{pmatrix} 1+\frac{2\gamma}{h^2}+\frac{6\mu}{h^4} & -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) & \frac{\mu}{h^4} & 0 & 0 & \cdots & 0 \\ -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) & 1+\frac{2\gamma}{h^2}+\frac{6\mu}{h^4} & -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) & \frac{\mu}{h^4} & 0 & \cdots & 0 \\ \frac{\mu}{h^4} & -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) & 1+\frac{2\gamma}{h^2}+\frac{6\mu}{h^4} & -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) & \frac{\mu}{h^4} & \ddots & \vdots \\ 0 & \frac{\mu}{h^4} & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \frac{\mu}{h^4} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) \\ 0 & 0 & \cdots & 0 & \frac{\mu}{h^4} & -\left(\frac{\gamma}{h^2}+\frac{6\mu}{h^4}\right) & 1+\frac{2\gamma}{h^2}+\frac{6\mu}{h^4} \end{pmatrix}.$$

Secondly, this system (6) is solved by using MATLAB solver ode15s. The resulting system of ordinary differential equations is integrated with respect to time.

3.3. Numerical computations and results

In this subsection, we present the numerical results of the proposed method on two test problems. The accuracy of the scheme is measured by using the following error norms:

$$L_2 = \| u_{exact} - u_{num} \|_2 = \sqrt{\sum_{i=0}^{n} | (u_{exact})_i - u_i |^2},$$

$$L_\infty = \| u_{exact} - u_{num} \|_\infty = \max_i | (u_{exact})_i - u_i |.$$

Besides, the error norms used, to demonstrate the efficiency and effectiveness of the proposed method, we can use the fundamental conservation characteristics of the Rosenau-KdV-RLW.

The Rosenau-KdV-RLW equation possesses three conservative properties corresponding to mass, momentum and energy, respectively, [14-16].

Mass, momentum and energy are defined as follow, respectively:

$$I_1(t) = \int_a^b u(x, t) dx = I_1(0),$$

$$I_2(t) = \int_a^b \left(u^2(x, t) + \gamma \left(\frac{\partial u}{\partial x} \right)^2 + \mu \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \right) dx = I_2(0),$$

$$I_3(t) = \int_a^b \left(\frac{\beta}{p+1} u^{p+1}(x, t) + \frac{\alpha}{2} u^2(x, t) - \frac{\delta}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right) dx = I_3(0).$$

The initial boundary value problem (1)-(3) includes the conservative quantities I_1, I_2 and I_3. Those quantities are applied to measure the conservation properties and are calculated by using Simpson's rule. The implementation of the weighted sum for Simpson's rule applied to the function $u(x, t)$ is defined as follows:

$$\int_a^b u(x, t) dx \approx \frac{h}{3} \left[u_1 + \sum_{i=2}^{n-2} (4u_i + 2u_{i+1}) + u_n \right].$$

In the simulation of solitary wave motion, the invariants I_1, I_2 and I_3 are observed to check the conversation of the numerical algorithm [17].

We compare the results obtained by using flux limiters methods to those obtained by MATMOL finite difference operators dss002 (differentiation in space subroutines) [18].

3.3.1. Example

Let us consider solitary wave solution of Rosenau-Korteweg-de Vries-regularized long wave (1) with parameters $\alpha = \delta = \gamma = \mu = 1$, $\beta = 0.5$, $p = 2$, $a = -40$ and $b = 100$.

The equation is [19, 20]:

$$u_t + u_x + 0.5(u^2)_x - u_{xxt} + u_{xxx} + u_{xxxxt} = 0.$$

The boundary conditions are

$$u(-40, t) = u(100, t) = 0,$$

$$u_x(-40, t) = u_x(100, t) = 0,$$

$$u_{xx}(-40, t) = u_{xx}(100, t) = 0$$

and the exact solitary wave solution is:

$$u(x, t) = A\operatorname{sech}^4(B(x - vt)),$$

where the values of the amplitude A, wave width B and wave speed v are, respectively:

$$A = \frac{-125 + 65\sqrt{457}}{456},$$

$$B = \sqrt{\frac{-13 + \sqrt{457}}{288}},$$

$$v = \frac{241 + 13\sqrt{457}}{266}.$$

The numerical simulation of equations (1)-(3) using MATLAB subroutine dss002 gives the following figures and tables. The profiles of the solitary wave at different times $t = 0, 5, 10, 15, 20, 25, 30$ are given in Figures 1, 2 and 3.

Figures 1, 2 and 3 show the motion of propagation of the solitary wave which travels to the right at an invariable speed and nearly conserves its amplitude and shape with the increasing time. We can also observe a good agreement between the exact and numerical solutions. The values of invariants and error norms for single solitary wave are tabulated in Table 1.

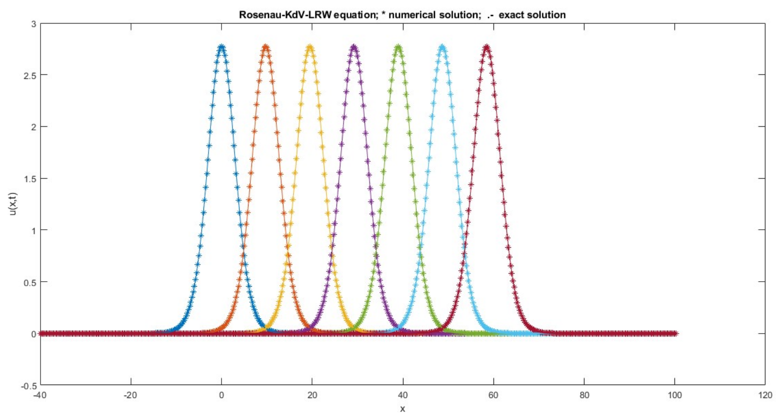

Figure 1. Numerical solution with the exact solution.

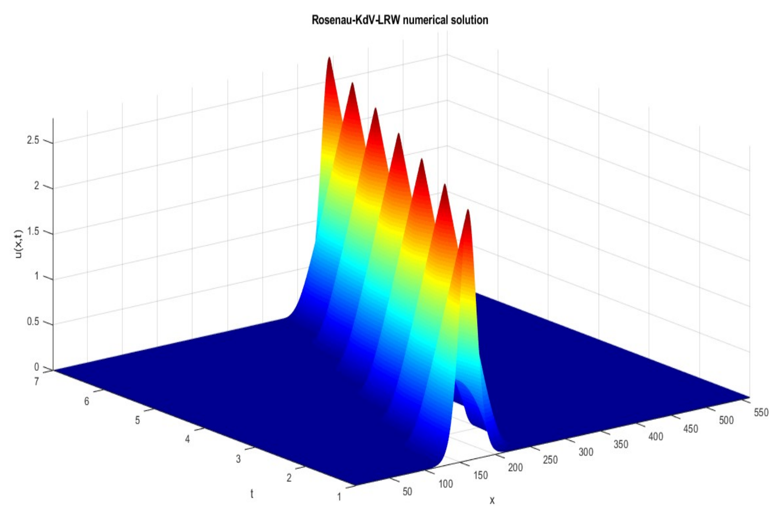

Figure 2. 3D plot of the numerical solution.

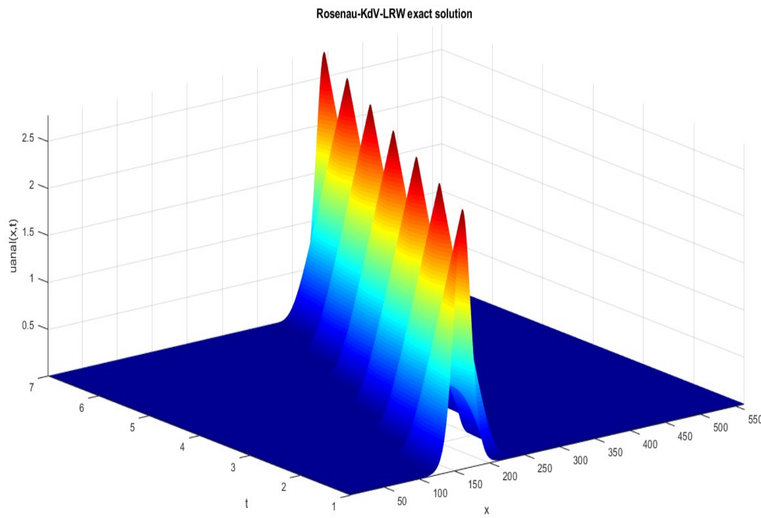

Figure 3. 3D plot of the exact solution.

Table 1. The error norms and invariants for solitary wave solution with $\alpha = \delta = \gamma = \mu = 1, \beta = 0.5, p = 2, n = 560$ and $h = 0.25$

Time (t)	L_2	L_∞	I_1	I_2	I_3
0	0	0	21.6793	43.7171	50.3397
10	2.26e-02	1.30e-02	21.6793	43.7169	50.3373
20	4.29e-02	2.33e-02	21.6793	43.7164	50.3355
30	6.28e-02	3.62e-02	21.6793	43.7164	50.3355

Table 1 shows that the invariants are practically constant over time and the error norms are acceptable.

We now use the flux limiters techniques for numerical simulation and then compare the results to those obtained by classical MATLAB finite difference subroutine dss002 method. The profiles of the solitary wave at different time levels are shown in Figures 4, 5 and 6.

Figures 4, 5 and 6 display that flux limiters method performs better. The motion of propagation of the solitary wave is very satisfactory. The solitary wave moves to the right at nearly unchanged speed and conserves its amplitude and shape with increasing time. The simulation is run from $t = 0$ to $t = 30$, and values of the error norms and invariant quantities with limiters functions are listed in Table 2 and Table 3.

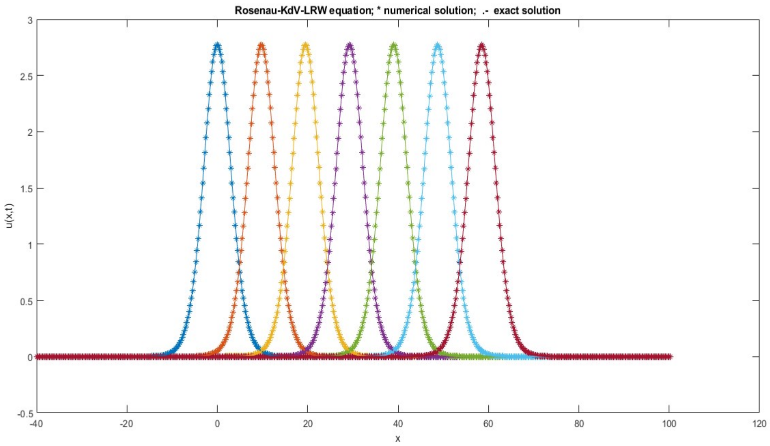

Figure 4. Numerical solution and exact solution with SMART flux limiter.

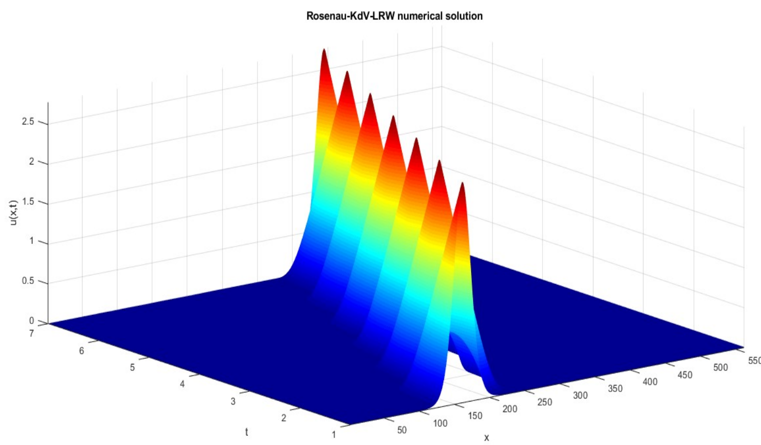

Figure 5. 3D plot of the numerical solution with SMART flux limiter.

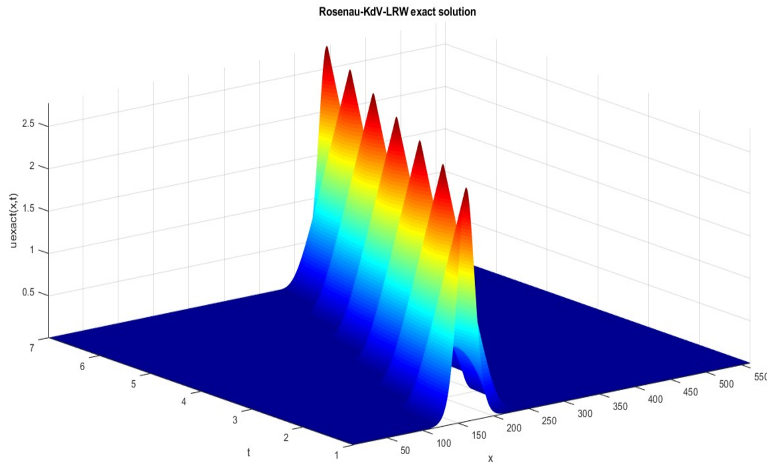

Figure 6. 3D plot of the exact solution with SMART flux limiter.

Table 2. The error norms for solitary wave solution with $\alpha = \delta = \gamma = \mu = 1$, $\beta = 0.5$, $p = 2$, $n = 560$, $h = 0.25$ at times $t = 0, 10, 20$ and 30 with flux limiters

	L_2				L_∞			
Time (t)	0	10	20	30	0	10	20	30
Van leer	0	2.01e-02	2.04e-02	2.25e-02	0	1.25e-02	1.3e-02	1.38e-02
Van albada1	0	1.91e-02	2.08e-02	4.5e-02	0	1.3e-02	1.17e-02	3.19e-02
SMART	0	5.3e-03	8.3e-03	1.8e-02	0	2.4e-03	5.8e-03	1.13e-02
Osher	0	1.47e-02	3.32e-02	5.68e-02	0	9.7e-03	1.91e-02	3.19e-02
MC	0	2.19e-02	2.76e-02	2.54e-02	0	1.35e-02	1.86e-02	1.86e-02
Ospre	0	1.97e-02	1.96e-02	2.95e-02	0	1.27e-02	1.21e-02	2.02e-02

Table 3. The invariants for solitary wave solution with $\alpha = \delta = \gamma = \mu = 1$, $\beta = 0.5$, $p = 2$, $n = 560$, $h = 0.25$ at times $t = 0, 10, 20$ and 30 with flux limiters

	I_1				I_2				I_3			
Time (t)	0	10	20	30	0	10	20	30	0	10	20	30
Van leer	21.6793	21.6755	21.6717	21.6680	43.7171	43.5823	43.4482	43.3149	50.3397	50.1447	49.9421	49.7400
V.alb1	21.6793	21.6596	21.6400	21.6206	43.7171	43.5247	43.3333	43.1436	50.3397	50.0559	49.7666	49.4793
SMART	21.6772	21.6751	21.6709	21.6668	43.6966	43.6762	43.6354	43.5947	50.3130	50.2829	50.2210	50.1592

Osher	21.6793	21.6755	21.6717	21.6680	43.7171	43.6715	43.6261	43.5807	50.3397	50.2709	50.2011	50.1321
MC	21.6793	21.6717	21.6679	21.6566	43.7171	43.6408	43.6026	43.4884	50.2348	50.1776	50.0045	50.2348
Ospre	21.6793	21.6635	21.6399	21.6321	43.7171	43.6408	43.6026	43.4884	50.3397	50.1125	49.7615	49.6451

The table clearly shows that the error norms obtained by the proposed method are less than the others. The results obtained show that the fundamental conservation properties of the Rosenau Korteweg-de Vries - regularized long wave equation are preserved with flux limiters numerical schemes. The figures and tables obtained show that the method is better.

3.3.2. Example

Let us consider shock wave solution of Rosenau-Korteweg-de Vries-regularized long wave equation (1) with parameters $\alpha = 1$, $\delta = 0.025$, $\gamma = 0.04$, $\mu = 1$, $\beta = 0.5$, $p = 3$.

The exact solution is [20]:

$$u(x, t) = A \tanh^{\frac{4}{p-1}}[B(x - vt)]$$

with

$$B = \sqrt{\frac{10\alpha\mu - \sqrt{100\alpha^2\mu^2 + 46\delta\mu(\delta - \alpha\gamma)}}{92\gamma\mu}},$$

$$v = \frac{\alpha - 8\delta B^2}{136\mu B^4 - 8\gamma B^2 + 1},$$

$$A = 2B^2\sqrt{\frac{30\mu v}{\beta}}.$$

Numerical simulation of Rosenau Korteweg-de Vries-regularized long wave equation (1) with above parameters and classical MATLAB finite difference subroutine dss002 gives the following figures and tables.

The profiles of the single soliton at times $t = 0, 10, 20, 30, 40$ are given in Figure 7, 8 and 9.

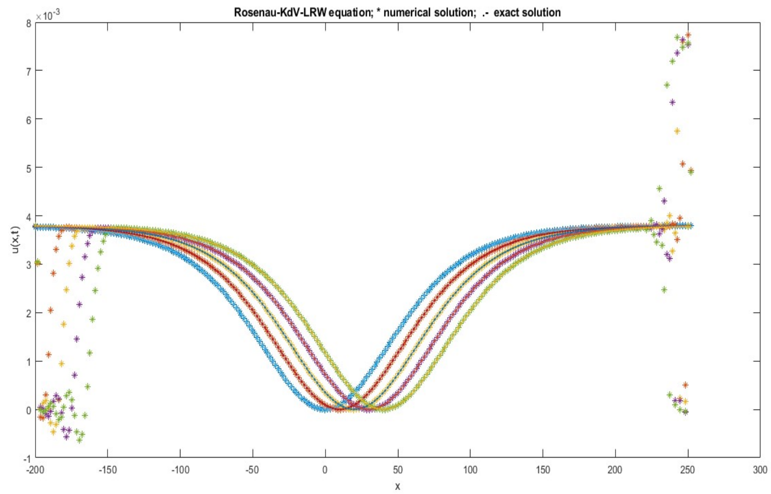

Figure 7. Numerical solution with the exact solution.

We observe from Figures 7, 8 and 9 that solitary wave travels to the right at a constant speed and keeps its amplitude with some oscillations. The different figures show that there are spurious oscillations at the boundaries. As it is seen, the maximum errors happen at the boundary position of the solitary wave. In Table 4, we give the error norms and invariant values at various time steps.

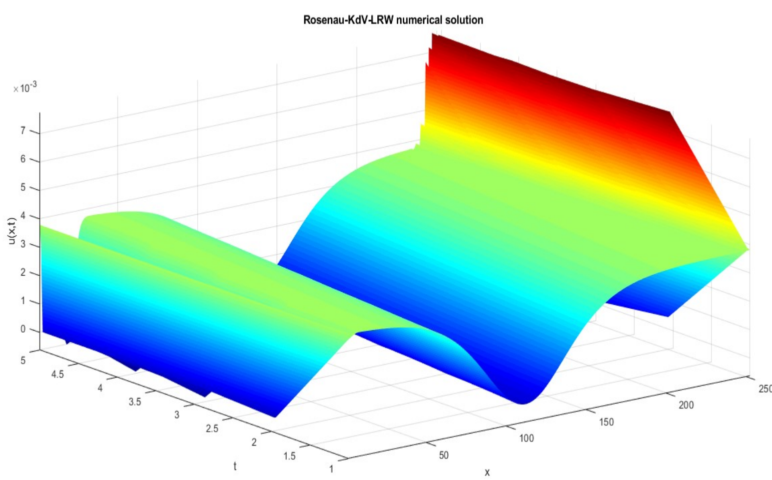

Figure 8. 3D plot of the numerical solution.

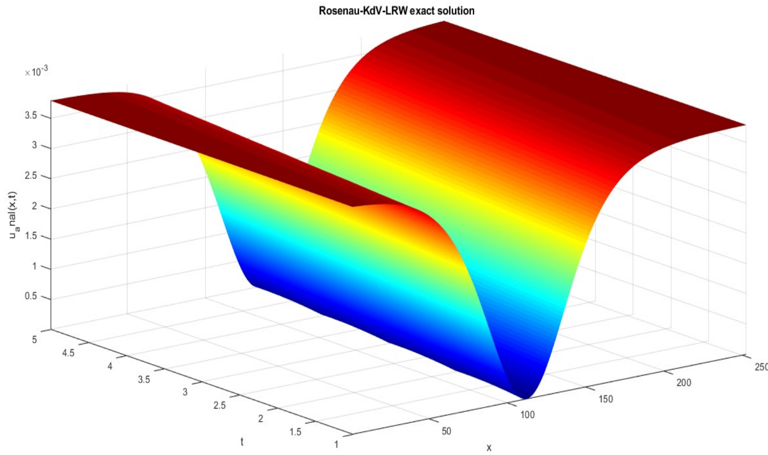

Figure 9. 3D plot of the exact solution.

Table 4. The error norms for shock wave solution with $\alpha = 1$, $\delta = 0.025$, $\gamma = 0.04$, $\mu = 1$, $\beta = 0.5$, $p = 3$, $n = 250$ at times $t = 0, 20, 30, 40$

Time (t)	$L_2 \times 10^{-3}$	$L_\infty \times 10^{-3}$	I_1	$I_2 \times 10^{-3}$	$I_3 \times 10^{-3}$
0	0	0	1.2240	4.000	2.000
20	1.100	1.100	1.1670	4.400	2.000
30	1.200	1.100	1.1339	4.600	2.000
40	1.700	1.300	1.1009	4.600	1.900

Table 4 indicates that the error norms are almost no negligible and invariants are relatively constant.

Now, we turn our attention to the flux limiter method. To this end, we compute the numerical solutions using the flux-limiter scheme with limiter functions mentioned in Subsection 2.2 for four values of the time and the best are retained. The profiles of the single soliton at times $t = 0, 10, 20, 30$ and 40 are given in Figures 10, 11 and 12.

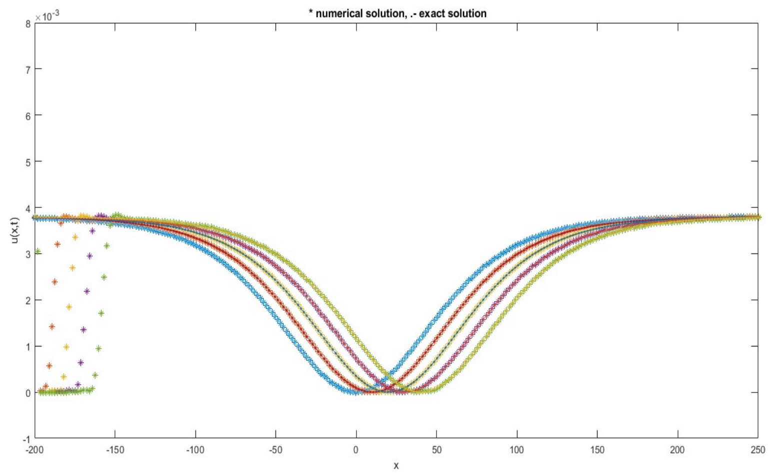

Figure 10. Numerical solution and exact solution with MC flux limiter.

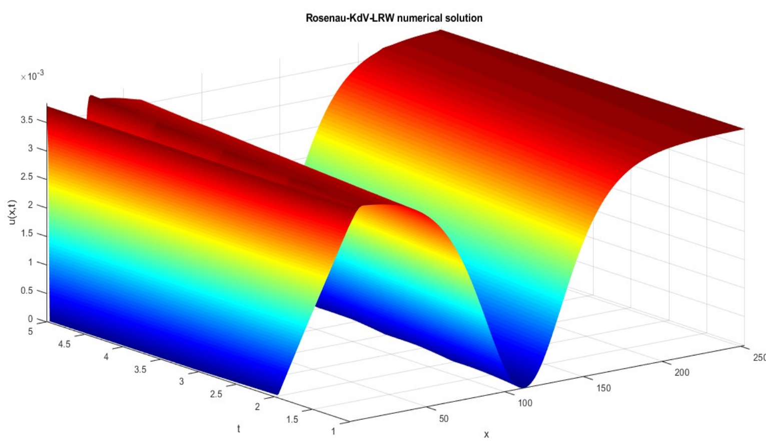

Figure 11. 3D plot of the numerical solution with MC flux limiter.

The overall motion of propagation of the single solitary wave for this example is preserved with no spurious oscillations appearing for results obtained by using the flux-limiter scheme. The computed results satisfy the stability and the shock capturing properties of the proposed flux-limiter scheme. The result obtained by using the flux-limiter scheme is very good. The flux-limiter scheme performs well since it does not diffuse the moving fronts and no spurious oscillations have been observed at the end.

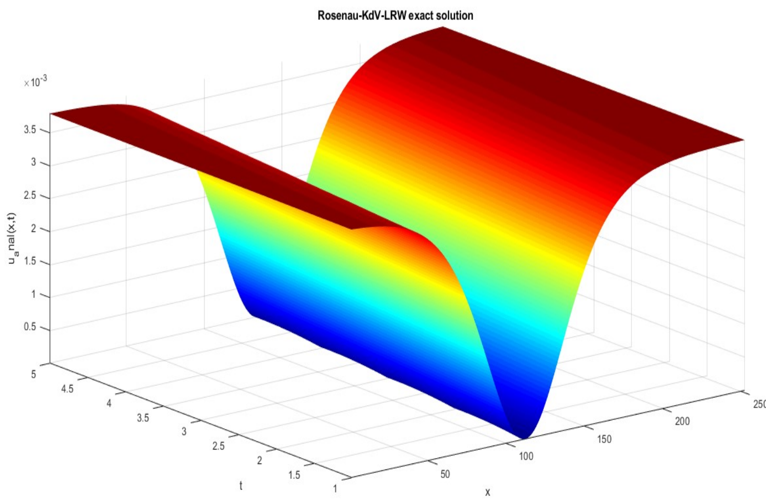

Figure 12. 3D plot of the exact solution with MC flux limiter.

Table 5. The invariants for shock wave solution with $\alpha = 1$, $\delta = 0.025$, $\gamma = 0.04$, $\mu = 1$, $\beta = 0.5$, $p = 3$, $n = 250$ at $t = 0, 20, 30, 40$ with flux limiters

	I_1				$I_2 \times 10^{-3}$				$I_3 \times 10^{-3}$			
Time (t)	0	20	30	40	0	20	30	40	0	20	30	40
Van leer	1.2240	1.1505	1.1127	1.0748	4.000	3.700	3.600	3.500	2.000	1.900	1.800	1.700
V.alb1	1.2240	1.1502	1.1123	1.0745	4.000	3.700	3.600	3.400	2.000	1.900	1.800	1.700
V.alb2	1.2240	1.1503	1.1125	1.0746	4.000	3.700	3.600	3.400	2.000	1.900	1.800	1.700
SMART	1.2240	1.1509	1.1130	1.0752	4.000	3.700	3.600	3.500	2.000	1.900	1.800	1.700
Superbee	1.2240	1.1513	1.1134	1.0756	4.000	3.800	3.600	3.500	2.000	1.900	1.800	1.700
Sweby	1.2240	1.1508	1.1129	1.0751	4.000	3.700	3.600	3.500	2.000	1.900	1.800	1.700
Osher	1.2240	1.1503	1.1124	1.0746	4.000	3.700	3.600	3.400	2.000	1.900	1.800	1.700
MC	1.2240	1.1508	1.1130	1.0752	4.000	3.700	3.600	3.500	2.000	1.900	1.800	1.700

It is noticeably seen from Table 5 that the error norms obtained by our method are found much better than the others. Table 5 also shows that invariants are almost constant for all the limiters used. We can observe from Table 5 that the results from the present study are in good agreement with the exact solutions. We can see that the results obtained with the flux limiters are all good, fundamental preservation quantities remain almost constant and are preserved during simulation time.

4. Concluding Remarks

In this article, to solve the solitary wave and shock wave problem of Rosenau-Korteweg-de Vries-regularized long wave equation, we have used flux limiters schemes instead of finite difference approximation of advection terms. The performance of the method has been examined on two problems having exact solutions. The efficiency and accuracy have been shown by calculating the error norms and the discrete invariants conservative properties. Numerical simulations show that the method is very efficient with the advantages of being non-oscillant. Results obtained in this paper demonstrate that the flux limiter method based on the method of lines is a remarkably successful numerical technique for solving the Rosenau-Korteweg-de Vries-regularized long wave equation and can be efficiently applied to a broad class of nonlinear PDEs with power law nonlinearity.

References

[1] N. M. Yagmurlu, B. Karaagac and S. Kutluay, Numerical solutions of Rosenau-RLW equation using Galerkin cubic B-Spline finite element method, American Journal of Computational and Applied Mathematics 7(1) (2017), 1-10.

[2] Muyassar Ahmat and Jianxian Qiu, SSP IMEX Runge-Kutta WENO scheme for generalized Rosenau-KdV-RLW equation, Journal of Mathematical Study (JMS) 55(1) (2022), 1-21.

[3] P. Razborova, A. Bouthina and Anjan Biswas, Solitons, shock waves and conservation laws of Rosenau-KdV-RLW equation with power law nonlinearity, Appl. Math. Inf. Sci. 8(2) (2014), 485-491.

[4] S. Pamuk and I. Atac, The method of lines for the numerical solution of a mathematical model in the initiation of angiogenesis, TWMS Journal of Applied and Engineering Mathematics 3(2) (2013), 182-197.

[5] H. O. Bakodah and M. A. Banaja, The method of lines solution of the regularized long-wave equation using Runge-Kutta time discretization method, Math. Probl. Eng. (2013), Art. ID 804317, 8 pp.

[6] J. P. Kauthen, The method of lines for parabolic partial integro-differential equations, Journal of Integral Equations and Applications 4(1) (1992), 69-72.

[7] Adek Tasri, Comparison of errors caused by flux limiters on the numerical solution of advection-diffusion problem, Nigerian Journal of Technological Development 19(4) (2022), 309-312.

[8] F. S. Gökhan, G. W. Griffiths and W. E. Schiesser, Method of lines solution to the transient SBS equations for nanosecond Stokes pulses, Journal of the European Optical Society-Rapid Publications 8 (2013), 13049-3-13049-6.

[9] Ouédraogo Mamadou, Somé Longin and Lamien Kassiénou, Using some flux limiters methods to solve three test problems, Far East J. Appl. Math. 93(2) (2015), 83-108.

[10] Polina Razborova, Bouthina Ahmed and Anjan Biswas, Solitons, shock waves and conservation laws of Rosenau-KdV-RLW equation with power law nonlinearity, Appl. Math. Inform. Sci. 8(2) (2014), 485-487.

[11] Ankur, Ram Jiwari and Naresh Kumar, Analysis and simulation of Korteweg-de Vries-Rosenau-regularised long-wave model via Galerkin finite element method, Computers and Mathematics with Applications 135 (2023), 134-148.

[12] P. Razborova, L. Moraru and A. Biswas, Perturbation of dispersive shallow water waves with Rosenau-KdV-RLW equation and power law nonlinearity, Romanian Journal of Physics 59(7-8) (2014), 658-676.

[13] Ahlem Ghiloufi and Khaled Omrani, New conservative difference schemes with fourth-order accuracy for some model equation for nonlinear dispersive waves, Numer. Methods Partial Differential Equations 34(2) (2018), 451-500.

[14] Sibel Özer, An effective numerical technique for the Rosenau-KdV-RLW equation, Journal of Balikesir University Institute of Science and Technology 20(3) (2018), 1-14.

[15] Jiaxiang Cai, Hua Liang and Chun Zhang, Efficient high-order structure-preserving methods for the generalized Rosenau-type equation with power law nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 59 (2018), 122-131.

[16] Sibel Özer, Two efficient numerical methods for solving Rosenau-KdV-RLW equation, The Kuwait Journal of Science 48(1) (2021), 14-24.

[17] S. Battal Gazi Karakoca and Turgut Akb, Numerical simulation of dispersive shallow water waves with Rosenau-KdV equation, International Journal of Advances in Applied Mathematics and Mechanics 3(3) (2016), 32-40.

[18] Graham W. Griffiths and William E. Schiesser, A compendium of partial differential equations, Method of Lines Analysis with Matlab, Cambridge University Press, 2009.

[19] Sibel Özer, Numerical solution of the Rosenau-KdV-RLW equation by operator splitting techniques based on B-spline collocation method, Numer. Methods Partial Differential Equation 35(5) (2019), 1928-1943.

[20] Mohammadreza Foroutan and Ali Ebadian, Chebyshev rational approximations for the Rosenau-KdV-RLW equation on the whole line, International Journal of Analysis and Applications 16(1) (2018), 1-15.

[21] S. B. G. Karakoc, Fuzheng Gao and S. K. Bhowmik and B. Seydi, Solitons and shock waves solutions for the Rosenau-KdV-RLW equation, Journal of Science and Arts 4(45) (2018), 1073-1088.