Advances in Differential Equations and Control Processes
© 2023 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324323017
Volume 30, Number 4, 2023, Pages 309-321 **P-ISSN: 0974-3243**

NON-CLASSICAL OPTIMAL CONTROL PROBLEM: A CASE STUDY FOR CONTINUOUS APPROXIMATION OF FOUR-STEPWISE FUNCTION

Wan Noor Afifah Wan Ahmad[1], Suliadi Firdaus Sufahani[1,*], Mahmod Abd Hakim Mohamad[2], Mohd Saifullah Rusiman[1], Mohd Zulariffin Md Maarof[3] and Muhamad Ali Imran Kamarudin[4]

[1]Department of Mathematics and Statistics
Faculty of Applied Sciences and Technology
Universiti Tun Hussein Onn Malaysia
Pagoh Higher Educational Hub
84600 Pagoh, Johor, Malaysia
e-mail: nuriz2122@gmail.com
suliadi@uthm.edu.my
saifulah@uthm.edu.my

Received: July 11, 2023; Accepted: August 19, 2023
2020 Mathematics Subject Classification: 65K15, 49J20, 91G80, 49K20, 13P25.
Keywords and phrases: discretization method, optimal control, shooting method.
*Corresponding author

How to cite this article: Wan Noor Afifah Wan Ahmad, Suliadi Firdaus Sufahani, Mahmod Abd Hakim Mohamad, Mohd Saifullah Rusiman, Mohd Zulariffin Md Maarof and Muhamad Ali Imran Kamarudin, Non-classical optimal control problem: a case study for continuous approximation of four-stepwise function, Advances in Differential Equations and Control Processes 30(4) (2023), 309-321. http://dx.doi.org/10.17654/0974324323017

Published Online: September 21, 2023

[2]Department of Mechanical Engineering
Center of Diploma
Universiti Tun Hussein Onn Malaysia
Pagoh Higher Educational Hub
84600 Pagoh, Johor, Malaysia
e-mail: hakim@uthm.edu.my

[3]Department of Sciences and Mathematics
Center of Diploma
Universiti Tun Hussein Onn Malaysia
Pagoh Higher Educational Hub
84600 Pagoh, Johor, Malaysia
e-mail: zulariffin@uthm.edu.my

[4]School of Business Management
College of Business
Universiti Utara Malaysia
06010 Sintok, Kedah, Malaysia
e-mail: aliimran@uum.edu.my

Abstract

The numerical properties of a contemporary optimal control problem (OCP) within the realm of financial aspects deviate from the conventional OCP framework. In our specific scenario, the final state condition is unknown, while the integrand exhibits a piecewise capacity that aligns with the unknown terminal state value. Since this is not a classical OCP, it cannot be solved using Pontryagin's maximum approach under the expected end limit conditions. A free final state in the non-classical issue results in a critical limit condition of the final shadow value not being equal to zero. The new fundamental condition must be comparable to a particular necessary condition because the integrand is a part of the unidentified final state value. By employing the hyperbolic tangent (tanh) function, we showcase a continuous approximation of the piecewise constant integrand function. Furthermore, we tackle a specific scenario utilizing the shooting method in C++ programming language. This is by combining the Newton and Golden Section Search methods in the shooting technique to calculate the limiting free final state value in an external circle emphasis. Discretization methods such as Euler and Runge-Kutta approximations were used in the validation procedure. The program was constructed in AMPL programming language with MINOS solver.

1. Introduction

The calculus of variation (CoV) offers a numerical framework to address extreme practical scenarios where a given function possesses a stationary value that can be either a minimum or a maximum [13]. CoV is enhanced by optimal control (OC), which is a method for figuring out the best OC strategies. By utilizing control factor strategies, OC can increase cost utilitarianism while still satisfying standard differential criteria. The notion of CoV and OC is intriguingly used in relation to financial issues [7].

Several well-known examples that illustrate the application of OC include the optimal production process, drug enforcement strategies, discrete mechanics, strategic action plans, and the challenge of optimal resource allocation [5, 7, 9, 12]. Additionally, as noted in earlier optimization research, OC is a component of optimization strategies [12].

In our financial problems, a company struggling with little demand for its products will purposefully raise the price by reducing its production of the products. Imposing a flat-rate payment on sales leads to an increase in marginal cost and a decrease in yield. However, introducing a nonlinear royalty structure creates an unexpected effect, giving rise to an unusual CoV problem that was not originally anticipated. This research will present a few solutions to the problem without going into the specifics of financial issues.

Let us examine a simple problem of finding the control function denoted as $u(t)$ that maximizes the integral function over the interval $[t_i, t_f]$:

$$J[(\cdot)] = \int_{t_i}^{t_f} h(t,\ y(t),\ u(t),\ y(t_f))\,dt. \tag{1}$$

Furthermore, we are familiar with the standard differential condition denoted as $\dot{y}(t) = u(t)$, where the initial state value $y(t_i)$ is known, but the final state value denoted as $y(t_f)$ is unknown. As the integral relies on the unknown $y(t_f)$, this problem does not fall into the category of a classical OCP.

In a classical OCP, the typical approach involves considering $y(t_f)$ as a known endpoint and applying a limit condition where the shadow value equals zero [13]. When employing Pontryagin's Maximum Principle (PMP), the Hamiltonian function of OC theory becomes apparent [14]. An OCP is employed to determine the most suitable control strategy for transitioning a dynamic system from one state to another while considering state or input control constraints [13].

In this investigation, we have put forth a method to approximate the piecewise constant integrand function continuously by utilizing a tanh approach instead of a discrete step function. The primary objective of this paper is to showcase how such problems can be effectively addressed. In the subsequent section, we will delve into the theoretical framework we have adopted, drawing inspiration from the work of Malinowska and Torres [6] in establishing the limit condition for CoV [2]. To further illustrate the effectiveness of our proposed method, we will present a numerical example and employ C++ programming language for its solution. The results will be thoroughly examined and validated, culminating in concluding remarks summarizing our findings.

2. Non-classical Optimal Control Problem

Equation (1) within the traditional framework of OCP is independent of the free value $y(t_f)$. However, in our specific case, the function h is contingent upon both $y(t_f)$ and $p(t_f)$, which can be likened to a certain integrand. Numerous investigations have offered understanding into the limit condition for CoV on time scales, as observed in works such as [2, 6]. The challenge we face can be associated with the corresponding theory [7].

Theorem 1 [7]. *Let* t_i *and* t_f *be two real numbers satisfying the condition* $t_i < t_f$. *Suppose* $y(\cdot)$ *represents the solution of*

$$J[(\cdot)] = \int_{t_i}^{t_f} h(t, y(t), \dot{y}(t), y(t_f))dt,$$

$$y(t_i) \text{ is known}, \; y(t_f) \text{ is unknown}, \; y(\cdot) \in C^1 \tag{2}$$

subject to

$$\frac{d}{dt} h_{\dot{y}}(t, y(t), \dot{y}(t), y(t_f)) = h_y(t, y(t), \dot{y}(t), y(t_f)), \tag{3}$$

where $t \in [t_i, t_f]$. *Furthermore,*

$$h_{\dot{y}}(t, y(t), \dot{y}(t), y(t_f)) = -\int_{t_i}^{t_f} h_z(t, y(t), \dot{y}(t), y(t_f))dt. \tag{4}$$

From the OC perspective, $p(t_f) = h_{\dot{y}}(t, y(t), \dot{y}(t), y(t_f))$. *Hence,*

$$p(t_f) = -\int_{t_i}^{t_f} h_z(t, y(t), \dot{y}(t), y(t_f))dt. \tag{5}$$

In this particular scenario, $p(t)$ *is denoted as the Hamiltonian multiplier.*

Theorem 1 reveals that the fundamental optimality condition $p(t_f)$ is non-zero. Given this observation, it is worth considering a piecewise function of ρ. The selection of the independent variable plays a significant role in the progression of the piecewise function. For instance, let us consider the following objective function:

$$J[(\cdot)] = \int_{t_i}^{t_f} h(t,\ y,\ u,\ z)\,dt \tag{6}$$

subject to $z = y(t_f)$ while

$$h(t,\ y,\ u,\ z) = \frac{1}{2}u^3 + z\rho y\cos(t) + \frac{5}{7}z, \tag{7}$$

where

$$\rho = \begin{cases} A & \text{for } y \le \frac{1}{2}z \\ B & \text{for } y > \frac{1}{2}z. \end{cases} \tag{8}$$

Suppose that A and B are real numbers in this scenario, which suggests the presence of an additional condition to address the main problem. Typically, a step function is implemented using an if-else condition in programming. However, this approach often leads to inaccuracies in the approximation. A tanh approach was presented to overcome this issue. The function represents a continuous approach for the stepwise function. It is connected to equation (8) and substituted into equation (7).

3. Hyperbolic Tangent Function as a Continuous Approach

Previous attempts to address the issue involved using the exact piecewise function without any approximation. However, these efforts yielded inconsistent and non-differentiable results across the domain. Alternative approximations, such as Fourier series approximation, were also explored but proved to be inadequate, requiring a significant number of terms. In order to ensure differentiability throughout, we propose a continuous approximation method. By transforming the discontinuous piecewise constant integrand function, commonly referred to as the step function, into a continuous integrand function, we aim to achieve a smooth and differentiable function. To accomplish this, we present a tanh method for the continuous approximation of the piecewise constant integrand function, resulting in an overall differentiable and smoothly continuous function.

4. Optimization of the Objective Function

Suppose that the ordinary differential equation (ODE) system follows

$$\dot{y}(t) = u(t) \tag{9}$$

as mentioned in Section 1, let us consider the following objective function:

$$J[u(t)] = \int_{t_i}^{t_f} g(t, y(t), u(t))dt \tag{10}$$

such that

$$g = \int_{t_i}^{t_f} (a(t)u^{1-\alpha(t)}(t) - (\rho(y(t)) + m_0(t) + c_0(t)e^{-\lambda(t)t})u(t))e^{-r(t)t}dt. \tag{11}$$

In the given context, we have various variables and functions. Let $a(t)$ represent the demand, which follows an exponential growth pattern defined as $e^{\frac{1}{40}t}$. The price elasticity of demand is denoted by $\alpha(t)$. The royalty payment, $\rho(y(t))$, is a four-stage piecewise function. The learning curve is characterized by $m_0(t)$, an asymptote equal to one, and the component of unit cost that is subject to learning is denoted as $c_0(t)$, also equal to one. The parameter $\lambda(t)$ determines the speed of learning, set at 12%. The discount factor is indicated as $r(t)$ and is equal to 10%. The control variable is represented by $u(t)$.

The function g depends on $y(t)$ and ρ, where ρ takes the form of a piecewise constant function, with $y(t_f)$ determining its value. We set the initial time as $t_i = 0$ and the final time as $t_f = 10$. The known initial state is $y(t_i) = 0$, while $y(t_f)$ is unknown. Generally, the value of ρ can be at any stage. In this case, the four-stage piecewise function is the value of ρ. This setting was implemented in order to maximize the objective function in equation (10). Further,

$$\rho(y(t)) = \begin{cases} \frac{1}{10} & \text{for } 0 \le y(t) \le \frac{1}{4}z, \\ \frac{6}{5} & \text{for } \frac{1}{4}z < y(t) \le \frac{1}{2}z, \\ \frac{6}{25} & \text{for } \frac{1}{2}z < y(t) \le \frac{3}{4}z, \\ \frac{3}{25} & \text{for } \frac{3}{4}z < y(t) \le z. \end{cases} \tag{12}$$

The continuous $\rho(y(t))$ was approximated by applying the tanh function:

$$\rho(y(t)) = \frac{11}{100} + \frac{11}{20}\tanh\left(k\left(y - \frac{1}{4}z\right)\right) - \frac{12}{25}\tanh\left(k\left(y - \frac{1}{2}z\right)\right) - \frac{3}{50}\tanh\left(k\left(y - \frac{3}{4}z\right)\right). \tag{13}$$

The smoothing value k was set equal to 250. The smoothness of the ρ will increase with the number of k. The Hamiltonian is denoted as H where the function is a summation of the integrand g with the product of shadow value $p(t)$ and control variable $u(t)$. The Hamiltonian behaviour can be expressed in the following expressions:

$$\dot{y}(t) = H_p,$$
$$\dot{p}(t) = -H_y,$$
$$H_u = 0. \tag{14}$$

Therefore, the state value satisfies

$$\dot{y}(t) = \left(e^{\frac{1}{40}t}u^{\frac{1}{2}} - \left(\rho + 1 + e^{-\frac{3}{25}y}\right)u\right)e^{-\frac{1}{10}t} + u. \tag{15}$$

Thus, the shadow value satisfies

$$\dot{p}(t) = \begin{pmatrix} \frac{109}{100}k - \frac{11}{20}k\tanh\left(k\left(y - \frac{1}{4}z\right)\right)^2 \\ + \frac{12}{25}k\tanh\left(k\left(y - \frac{1}{2}z\right)\right)^2 \\ + \frac{3}{50}k\tanh\left(k\left(y - \frac{3}{4}z\right)\right)^2 - \frac{3}{25}e^{-\frac{3}{25}y} \end{pmatrix} ue^{-\frac{1}{10}t}. \tag{16}$$

At the same time, the control value fulfils

$$u(t) = \frac{\frac{1}{4}\left(e^{\frac{1}{40}t}\right)^2\left(e^{-\frac{1}{10}t}\right)^2}{\left(\rho e^{-\frac{1}{10}t} + e^{-\frac{3}{25}y}e^{-\frac{1}{10}t} + e^{-\frac{1}{10}t} - p\right)^2}. \tag{17}$$

Hence, the final shadow value satisfies

$$p(t_f) = \int_0^{10} \begin{pmatrix} \frac{59}{400}k + \frac{11}{80}k\tanh\left(k\left(y - \frac{1}{4}z\right)\right)^2 \\ - \frac{6}{25}k\tanh\left(k\left(y - \frac{1}{2}z\right)\right)^2 \\ - \frac{9}{200}k\tanh\left(k\left(y - \frac{3}{4}z\right)\right)^2 \end{pmatrix} ue^{-0.1t}dt. \tag{18}$$

In order to compare approaches, we solved the identical issue using a different method, such as discrete-time nonlinear programming (NLP) [1, 7]. The discretization method, such as the Euler and Runge-Kutta approximation, was used to find the unknown $y(t_f)$ and optimal objective function. The problem was solved using the AMPL programming language with MINOS solver, where the step size was set as $S = 45$ [3].

Table 1 showcases the results obtained from the nonlinear shooting method, as well as the optimal solutions achieved using the Euler and Runge-Kutta methods. Notably, the nonlinear shooting method yielded a highly accurate optimal solution.

Table 1. Optimal final state value, objective function and initial shadow value generated from shooting and discretization methods

Method	Final state value	Objective function	Initial shadow value
Shooting method			
Newton and Golden Section Search	0.585314	0.820619	–0.277648
Discretization method			
Euler	0.543349	0.814081	–0.082824
Runge-Kutta	0.588223	0.826710	–0.246044

According to the findings, the objective function values for the shooting and Runge-Kutta methods are comparable up to one decimal point. The final state values for all techniques are similar to one decimal place. The initial shadow values yield an optimal solution identical to one decimal place for the shooting and Runge-Kutta methods.

The obtained results indicate that the plots of the state variables and shadow values for the shooting, Euler, and Runge-Kutta methods exhibit similarities. Figure 1 depicts the optimal curves of the state variables, shadow values, control variables, and the objective function. Upon comparing the Euler and Runge-Kutta methods with the shooting method, the graphs of the control variables show distinct variations between the approaches. As previously stated, the calculation likely contains errors since S increases inaccuracy. This has an impact on the control plots.

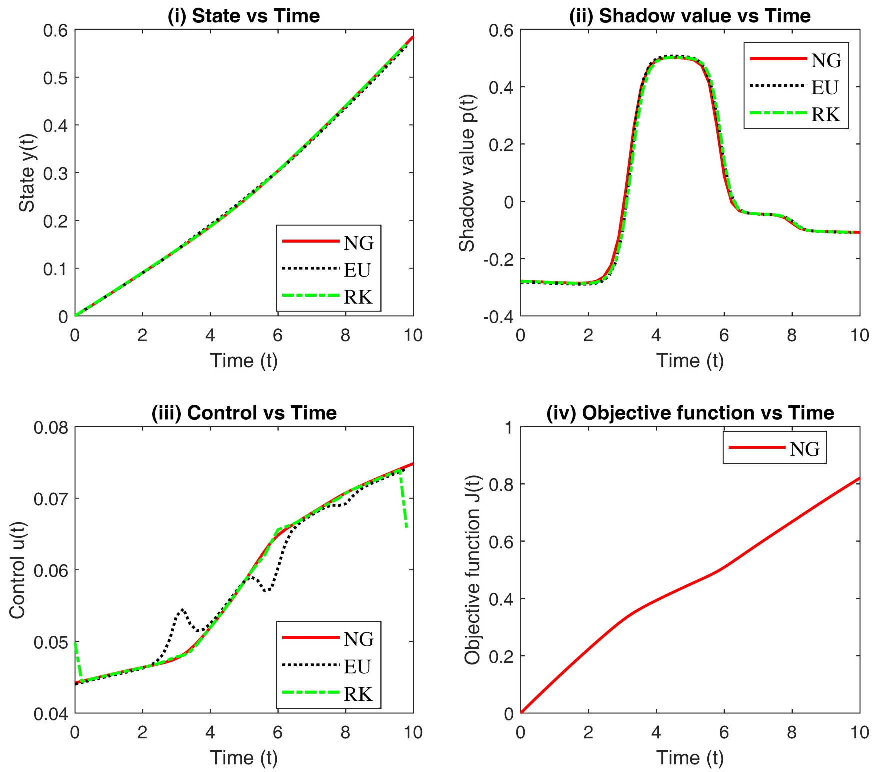

Figure 1. Optimal plot for the state, shadow value, control variable and objective function. (NG = Newton and Golden Section Search, EU = Euler, and RK = Runge-Kutta).

5. Conclusion

This article shows how to use a continuous approximation stepwise function over a tanh to solve an unconventional optimum control issue. We have shown how to apply the necessary parameters and numerical techniques to find the best answer. The approach applies to the real climatic problem where the Lagrangian is piecewise continuous. The numerical method was used to approve a simple illustration, which was then resolved using sophisticated PC programming. It was possible to acquire a highly accurate solution and compare it to other NLP techniques.

Acknowledgements

The Ministry of Higher Education (MOHE) supported this research through the Fundamental Research Grant Scheme (FRGS/1/2021/STG06/UTHM/03/3). We would like to express our gratitude to the Research Management Center (RMC) of Universiti Tun Hussein Onn Malaysia (UTHM) for overseeing the publication and research processes.

References

[1] J. T. Betts, Practical methods for optimal control using nonlinear programming, Advances in Design and Control, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

[2] R. A. C. Ferreira and D. F. M. Torres, Higher-order calculus of variations on time scales, Mathematical Control Theory and Finance, Springer, Berlin, 2008, pp. 149-159.

[3] R. Fourer, D. M. Gay and B. W. Kernighan, AMPL: A Modelling Language for Mathematical Programming, Duxbury Press/Brooks/Cole Publishing Company, 2002.

[4] G. Stefani, Hamiltonian approach to optimality conditions in control theory and nonsmooth analysis, Nonsmooth Analysis Control Theory and Differential Equations, INDAM, Roma, 2009. http://events.math.unipd.it/nactde09/sites/default/files/stefani.pdf.

[5] D. Leonard and N. V. Long, Optimal Control Theory and Static Optimization in Economics, Cambridge University Press, Cambridge, 1992.

[6] A. B. Malinowska and D. F. M. Torres, Natural boundary conditions in the calculus of variations, Mathematical Methods in the Applied Science 33(14) (2010), 1712-1722. doi:10.1002/mma.1289.

[7] A. F. C. Pedro, D. F. M. Torres and A. S. I. Zinober, A non-classical class of variational problems with application in economics, International Journal of Mathematical Modeling and Numerical Optimization 1(3) (2010), 227-236. doi:10.1504/IJMMNO.2010.031750.

[8] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes: The Art of Scientific Computing, 3rd ed., Cambridge University Press, Cambridge, 2007.

[9] S. P. Sethi and G. L. Thompson, Optimal Control Theory: Applications on Management Science and Economics, 2nd ed., Kluwer Academic Publishers, Boston, MA, 2000.

[10] R. Vinter, System and Control: Foundation and Application Optimal Control, Birkha Boston, Springer-Verlag, New York, 2000.

[11] A. S. I. Zinober and K. Kaivanto, Optimal production subject to piecewise continuous royalty payment obligations, University of Sheffield, 2008.

[12] A. S. I. Zinober and S. F. Sufahani, A non-standard optimal control problem arising in an economics application, Pesquisa Operacional 33(1) (2013), 63-71.

[13] D. E. Kirk, Optimal Control Theory: An Introduction, Prentice-Hall Network Series, New Jersey, 1970.

[14] E. R. Pinch, Optimal Control and the Calculus of Variations, Oxford University Press, Oxford, 1993.