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THE INVERSE LAPLACE TRANSFORM OF SOME COMPLEX EXPRESSIONS ARISING IN SOLAR ENERGY MODELS

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Abstract

Recently, many mathematical models are proposed to describe storage of solar energy. Most of these models are governed by boundary value problems (BVPs). The explicit solutions of such BVPs depend in determining the inverse Laplace transform of complex expressions. This paper overcomes some of these difficulties arising on account of

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this. The results can be invested to construct the analytic solutions of solar energy models and also models of other fields in engineering sciences.

1. Introduction

In the past decade, many scientific models have been proposed to extract solar energy by means of nanofluids. These models play a vital role in the field of renewable energy which helps in reducing the harmful effects of climate change. This field attracted considerable interest of many researchers [1-7]. The Laplace transform (LT) is a well-known approach to solve initial/boundary value problems (IVPs/BVPs). The LT is of wide application to exactly solve several scientific problems in various fields of science, biology, and engineering [8-16]. It can be seen in these references that the obtained exact/analytic solutions depend mainly on obtaining the inverse LT of complex expressions. So, the objective of this paper is to provide the inversion of some complex expressions which can be invested in future to explore several scientific models.

2. Analysis

This section is devoted to introduce some useful integrals. These integrals are essential to find the inverse LT of complex expressions.

Definition 1. The *complementary error function* (*erfc*) is defined as [17]:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du. \tag{1}$$

Lemma 1. *The integral* $I_1 = \int_0^t u^{-3/2} e^{-\left(bu + \frac{a^2}{4u}\right)} du$ *is given by*

$$I_1 = \frac{\sqrt{\pi}}{a}\left[e^{-a\sqrt{b}}\, erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) + e^{a\sqrt{b}}\, erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right], \quad a > 0. \tag{2}$$

Proof. Let

$$I_1 = \int_0^t u^{-3/2} e^{-\left(bu+\frac{a^2}{4u}\right)} du. \tag{3}$$

Suppose that

$$z_1 = \sqrt{bu} + \frac{a}{2\sqrt{u}}, \quad z_2 = \sqrt{bu} - \frac{a}{2\sqrt{u}}. \tag{4}$$

Then

$$I_1 = \frac{2}{a}\int_0^t \left[\left(\frac{\sqrt{b}}{2}u^{-1/2} + \frac{a}{4}u^{-3/2}\right)e^{-z_2^2 - a\sqrt{b}} - \left(\frac{\sqrt{b}}{2}u^{-1/2} - \frac{a}{4}u^{-3/2}\right)e^{-z_1^2 + a\sqrt{b}}\right] du, \tag{5}$$

and hence

$$I_1 = \frac{2}{a}\left[\int_{[z_2]_{u=0}}^{[z_2]_{u=t}} e^{-z_2^2 - a\sqrt{b}} dz_2 - \int_{[z_1]_{u=0}}^{[z_1]_{u=t}} e^{-z_1^2 + a\sqrt{b}} dz_1\right], \tag{6}$$

which is

$$I_1 = \frac{2}{a}\left[e^{-a\sqrt{b}}\int_{-\infty}^{\sqrt{bt} - \frac{a}{2\sqrt{t}}} e^{-z_2^2} dz_2 - e^{a\sqrt{b}}\int_{\infty}^{\sqrt{bt} + \frac{a}{2\sqrt{t}}} e^{-z_1^2} dz_1\right], \quad a > 0. \tag{7}$$

So

$$I_1 = \frac{2}{a}\left[-e^{-a\sqrt{b}}\int_{\infty}^{\frac{a}{2\sqrt{t}} - \sqrt{bt}} e^{-z_3^2} dz_3 + e^{a\sqrt{b}}\int_{\frac{a}{2\sqrt{t}} + \sqrt{bt}}^{\infty} e^{-z_1^2} dz_1\right], \quad z_3 = -z_2, \tag{8}$$

i.e.,

$$I_1 = \frac{2}{a}\left[e^{-a\sqrt{b}}\int_{\frac{a}{2\sqrt{t}}-\sqrt{bt}}^{\infty} e^{-z_3^2}dz_3 + e^{a\sqrt{b}}\int_{\frac{a}{2\sqrt{t}}+\sqrt{bt}}^{\infty} e^{-z_1^2}dz_1\right]. \tag{9}$$

Thus

$$I_1 = \frac{\sqrt{\pi}}{a}\left[e^{-a\sqrt{b}}erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) + e^{a\sqrt{b}}erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right], \tag{10}$$

which completes the proof. □

Lemma 2. *For* $c > 0$,

$$\int_0^t e^{-cz}erfc\left(\frac{d}{2\sqrt{z}}\right)dz = -\frac{e^{-ct}}{c}erfc\left(\frac{d}{2\sqrt{t}}\right) + \frac{1}{2c}\left[e^{-d\sqrt{c}}erfc\left(\frac{d}{2\sqrt{t}} - \sqrt{ct}\right)\right.$$

$$\left. + e^{d\sqrt{c}}erfc\left(\frac{d}{2\sqrt{t}} + \sqrt{ct}\right)\right]. \tag{11}$$

Proof. The given integral can be decomposed as

$$\int_0^t e^{-cz}erfc\left(\frac{d}{2\sqrt{z}}\right)dz$$

$$= -\frac{1}{c}\left[e^{-ct}erfc\left(\frac{d}{2\sqrt{t}}\right) - \frac{d}{2\sqrt{\pi}}\int_0^t z^{-3/2}e^{-\left(cz+\frac{d^2}{4z}\right)}dz\right]. \tag{12}$$

The integral in the right hand side, in equation (12), can be given from Lemma 1 as

$$\int_0^t z^{-3/2}e^{-\left(cz+\frac{d^2}{4z}\right)}dz$$

$$= \frac{\sqrt{\pi}}{d}\left[e^{-d\sqrt{c}}erfc\left(\frac{d}{2\sqrt{t}} - \sqrt{ct}\right) + e^{d\sqrt{c}}erfc\left(\frac{d}{2\sqrt{t}} + \sqrt{ct}\right)\right]. \tag{13}$$

Employing (13) into (12), we have

$$\int_0^t e^{-cz} erfc\left(\frac{d}{2\sqrt{z}}\right) dz = -\frac{e^{-ct}}{c} erfc\left(\frac{d}{2\sqrt{t}}\right) + \frac{1}{2c}\left[e^{-d\sqrt{c}} erfc\left(\frac{d}{2\sqrt{t}} - \sqrt{ct}\right)\right.$$

$$\left. + e^{d\sqrt{c}} erfc\left(\frac{d}{2\sqrt{t}} + \sqrt{ct}\right)\right], \tag{14}$$

which is the desired result. □

Lemma 3. *The integral* $I_2 = \int_0^t u^{-1/2} e^{-\left(bu + \frac{a^2}{4u}\right)} du$ *is given by*

$$\frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) - e^{-a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right]. \tag{15}$$

Proof. The given integral can be put in the form:

$$I_2 = \frac{1}{\sqrt{b}} \int_0^t u^{-1/2}\left[\left(\frac{\sqrt{b}}{2} u^{-1/2} - \frac{a}{4} u^{-3/2}\right) e^{-z_1^2 - a\sqrt{b}}\right.$$

$$\left. + \left(\frac{\sqrt{b}}{2} \sigma^{-1/2} + \frac{a}{4} u^{-3/2}\right) e^{-z_2^2 + a\sqrt{b}}\right] du, \tag{16}$$

where z_1 and z_2 are defined by equations (4). Thus

$$I_2 = \frac{1}{\sqrt{b}}\left[\int_{[z_1]_{u=0}}^{[z_1]_{u=t}} e^{-z_1^2 - a\sqrt{b}} dz_1 + \int_{[z_2]_{u=0}}^{[z_2]_{u=t}} e^{-z_2^2 + a\sqrt{b}} dz_2\right], \tag{17}$$

i.e.,

$$I_2 = \frac{1}{\sqrt{b}}\left[e^{-a\sqrt{b}} \int_{\infty}^{\sqrt{bt} + \frac{a}{2\sqrt{t}}} e^{-z_1^2} dz_1 + e^{a\sqrt{b}} \int_{-\infty}^{\sqrt{bt} - \frac{a}{2\sqrt{t}}} e^{-z_2^2} dz_2\right], \; a > 0. \tag{18}$$

Consequently,

$$I_2 = \frac{1}{\sqrt{b}}\left[-e^{-a\sqrt{b}} \int_{\sqrt{bt} + \frac{a}{2\sqrt{t}}}^{\infty} e^{-z_1^2} dz_1 - e^{a\sqrt{b}} \int_{\sqrt{bt} - \frac{a}{2\sqrt{t}}}^{-\infty} e^{-z_2^2} dz_2\right] \tag{19}$$

or

$$I_2 = \frac{1}{\sqrt{b}}\left[e^{a\sqrt{b}}\int_{\frac{a}{2\sqrt{t}}-\sqrt{bt}}^{\infty} e^{-z_3^2}dz_3 - e^{-a\sqrt{b}}\int_{\frac{a}{2\sqrt{t}}+\sqrt{bt}}^{\infty} e^{-z_1^2}dz_1\right], \quad z_3 = -z_2, \tag{20}$$

and this gives

$$\begin{aligned} I_2 &= \int_0^{\tau} u^{-1/2} e^{-\left(bu+\frac{a^2}{4u}\right)}du \\ &= \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{a\sqrt{b}}erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) - e^{-a\sqrt{b}}erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right]. \end{aligned} \tag{21}$$

□

3. Inversion of Complex Expressions

In this section, we obtain inversion of some complex expressions. The current results are of great importance to determine the exact solutions for various IVPs/IVPs that arise in solar energy models.

Lemma 4. *For* $a \in \mathbb{R}$ (*see* [17]),

$$\mathcal{L}^{-1}\{e^{-a\sqrt{s+b}}\} = \frac{at^{-3/2}}{2\sqrt{\pi}}\, e^{-\left(bt+\frac{a^2}{4t}\right)}. \tag{22}$$

Theorem 1. *For* $a > 0$,

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s}\right\} = \frac{1}{2}\left[e^{-a\sqrt{b}}erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) + e^{a\sqrt{b}}erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right]. \tag{23}$$

Proof. We have

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\{e^{-a\sqrt{s+b}}\} = \frac{a}{2\sqrt{\pi}}\int_0^t u^{-3/2} e^{-\left(bu+\frac{a^2}{4u}\right)}du, \tag{24}$$

where $(*)$ stands for convolution operation defined by

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du. \tag{25}$$

The integral in equation (24) is already obtained by Lemma 1, hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s}\right\} = \frac{1}{2}\left[e^{-a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) + e^{a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right], \tag{26}$$

and this completes the proof. As a special case, when $b = 0,$ the inversion (26) becomes

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = erfc\left(\frac{a}{2\sqrt{t}}\right), \tag{27}$$

which coincides with [17]. □

Lemma 5. *The inversion of* $\dfrac{e^{-d\sqrt{s}}}{s(s-c)}$ *is*

$$\mathcal{L}^{-1}\left\{\frac{e^{-d\sqrt{s}}}{s(s-c)}\right\} = -\frac{1}{c} erfc\left(\frac{d}{2\sqrt{t}}\right) + \frac{1}{2c}\left[e^{ct-d\sqrt{c}} erfc\left(\frac{d}{2\sqrt{t}} - \sqrt{ct}\right) + e^{ct+d\sqrt{c}} erfc\left(\frac{d}{2\sqrt{t}} + \sqrt{ct}\right)\right]. \tag{28}$$

Proof. The convolution theorem gives

$$\mathcal{L}^{-1}\left\{\frac{e^{-d\sqrt{s}}}{s(s-c)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-c}\right\} * \mathcal{L}^{-1}\left\{\frac{e^{-d\sqrt{s}}}{s}\right\} \tag{29}$$

$$= e^{ct} * erfc\left(\frac{d}{2\sqrt{t}}\right) = \int_0^t e^{c(t-u)} erfc\left(\frac{d}{2\sqrt{u}}\right)du \tag{30}$$

$$= e^{ct}\int_0^t e^{-cu} erfc\left(\frac{d}{2\sqrt{u}}\right)du. \tag{31}$$

Using the result in Lemma 2, we have

$$\int_0^\tau e^{-cu} erfc\left(\frac{d}{2\sqrt{u}}\right) du = -\frac{e^{-ct}}{c} erfc\left(\frac{d}{2\sqrt{t}}\right) + \frac{1}{2c}\left[e^{-d\sqrt{c}} erfc\left(\frac{d}{2\sqrt{t}} - \sqrt{ct}\right) + e^{d\sqrt{c}} erfc\left(\frac{d}{2\sqrt{\tau}} + \sqrt{ct}\right)\right]. \tag{32}$$

Employing (32) in (31) leads to the result of this lemma. □

Lemma 6.

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s^2}\right\} = \left(\frac{t}{2} e^{-a\sqrt{b}} - \frac{a}{4\sqrt{b}} e^{a\sqrt{b}}\right) erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) + \left(\frac{\tau}{2} e^{a\sqrt{b}} - \frac{a}{4\sqrt{b}} e^{-a\sqrt{b}}\right) erfc\left(\frac{a}{2\sqrt{\tau}} + \sqrt{b\tau}\right). \tag{33}$$

Proof. Applying the convolution theorem, we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\{e^{-a\sqrt{s+b}}\}, \tag{34}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s^2}\right\} = \frac{a}{2\sqrt{\pi}}\left(t * t^{-3/2} e^{-\left(bt + \frac{a^2}{4t}\right)}\right)$$

$$= \frac{a}{2\sqrt{\pi}} \int_0^t (t-u) u^{-3/2} e^{-\left(bu + \frac{a^2}{4u}\right)} du$$

$$= \frac{a}{2\sqrt{\pi}} (tI_1 - I_2), \tag{35}$$

where I_1 and I_2 were already obtained by Lemma 1 and Lemma 3, respectively, i.e.,

$$I_1 = \int_0^t u^{-3/2} e^{-\left(bu + \frac{a^2}{4u}\right)} du$$

$$= \frac{\sqrt{\pi}}{a}\left[e^{-a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{u}} - \sqrt{bt}\right) + e^{a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right], \qquad (36)$$

while

$$I_2 = \int_0^t u^{-1/2} e^{-\left(bu + \frac{a^2}{4u}\right)} du$$

$$= \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) - e^{-a\sqrt{b}} erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right)\right]. \qquad (37)$$

Employing I_1 and I_2 into equation (35) and simplifying, we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+b}}}{s^2}\right\} = \left(\frac{t}{2} e^{-a\sqrt{b}} - \frac{a}{4\sqrt{b}} e^{a\sqrt{b}}\right) erfc\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right)$$

$$+ \left(\frac{t}{2} e^{a\sqrt{b}} - \frac{a}{4\sqrt{b}} e^{-a\sqrt{b}}\right) erfc\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right). \qquad (38)$$

□

4. Conclusions

The explicit solutions of many mathematical models describing storage of solar energy depend on evaluating the inverse Laplace transform of complex expressions. In this paper, some of the difficulties arising due to it are solved. Accordingly, the obtained results are useful in establishing explicit solutions of several scientific models including solar energy and also other areas of engineering sciences.

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