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EXISTENCE OF SOLUTIONS FOR NONLINEAR VOLTERRA FREDHOLM INTEGRODIFFERENTIAL EQUATION OF HIGHER ORDER VIA *S*-ITERATION METHOD

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Abstract

In this paper, we study the existence and other properties of the solution of the nonlinear Volterra Fredholm integrodifferential equation of higher order. The tool employed in the analysis is based on the application of the *S*-iteration method. Various properties such as dependence on initial data, closeness of solutions and dependence on parameters and functions involved therein are obtained using the *S*-iteration method. Examples are provided in support of findings.

1. Introduction

Consider the nonlinear integrodifferential equation of the type:

$$x^{(n)}(t) = \mathcal{F}(t, x(t), x'(t), \ldots, x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t)), \tag{1}$$

for $t \in I = [a, b]$, with the given initial conditions

$$x^{(j)}(a) = \alpha_j, \quad j = 0, 1, 2, \ldots, n-1, \tag{2}$$

where

$$(\mathcal{H}x)(t) = \int_a^t \mathcal{K}_1(t, s)\mathcal{M}_1(s, x(s), x'(s), \ldots, x^{(n-1)}(s))ds, \tag{3}$$

$$(\mathcal{L}x)(t) = \int_a^b \mathcal{K}_2(t, s)\mathcal{M}_2(s, x(s), x'(s), \ldots, x^{(n-1)}(s))ds. \tag{4}$$

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. We assume that $\mathcal{F} \in C(I \times \mathbb{R}^{n+2}, \mathbb{R})$; for $i = 1, 2$ and $a \leq s \leq t$, $\mathcal{K}_i \in C(I^2, \mathbb{R})$, $\mathcal{M}_i \in C(I \times \mathbb{R}^n, \mathbb{R})$ are given functions and α_j $(j = 0, 1, 2, \ldots, n-1)$ are given real constants.

Several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and the rate of convergence [1, 3, 4, 5, 7, 9, 10, 18-25, 30-32]. The most of iterations are devoted for both analytical and numerical approaches. Since the S-iteration method is simple and fast, we employ this method in this paper.

The problems of existence, uniqueness and other properties of solutions of special forms of IVP (1)-(2) and its variants have been studied by several researchers under variety of hypotheses by using different techniques [2, 8, 11-17, 26-29, 33-35] and some of references cited therein. Recently, Atalan et al. [6] studied the special version of equation (1) for different qualitative properties of solutions.

The main objective of this paper is to use normal S-iteration method to establish the existence and uniqueness of solution of the initial value problem (1)-(2) and other qualitative properties of solutions. Also, extend the results of Atalan et al. [6].

2. Existence of Solution via *S*-iteration

Let $E = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) be the product space. For continuous functions $x^{(j)} : I \to \mathbb{R}$ $(j = 0, 1, ..., n-1)$, we denote by

$$| x(t) |_E = \sum_{j=0}^{n-1} | x^{(j)}(t) |,$$

for $(x(t), x'(t), ..., x^{(n-1)}(t)) \in E$, $t \in I$. Denote by $B = C^{n-1}(I) = C^{n-1}(I, \mathbb{R})$, the space of those functions x which are $(n-1)$ times continuously differentiable on I endowed with norm

$$\| x \|_B = \max_{t \in I}\{| x(t) |_E\}. \tag{5}$$

It is easy to see that B with norm defined by (5) is a Banach space.

By a solution of equations (1)-(2), we mean a continuous function $x(t)$, $t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies equations (1)-(2). It is easy to observe that the solution $x(t)$ of equations (1)-(2) and its derivatives satisfy the integral equations of the form:

$$x^{(j)}(t) = \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), ..., x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s))ds, \quad (6)$$

for $t \in I$ and $0 \leq j \leq n-1$.

We need the following pair of known results:

Theorem 1 [30, p. 194]. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $m \in [0, 1)$ and fixed point x^*. Let α_k and β_k be two real sequences in $[0, 1]$ such that $\alpha \leq \alpha_k \leq 1$ and $\beta \leq \beta_k < 1$ for all $k \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_k, v_k and w_k in C as follows*:

S-iteration process: $\begin{cases} u_{k+1} = (1-\alpha_k)Tu_k + \alpha_k Ty_k, \\ y_k = (1-\beta_k)u_k + \beta_k Tu_k, \; k \in \mathbb{N}. \end{cases}$

Picard iteration: $v_{k+1} = Tv_k, \; k \in \mathbb{N}$.

Mann iteration process: $w_{k+1} = (1-\beta_k)w_k + \beta_k Tw_k, \; k \in \mathbb{N}$.

Then

(a) $\| u_{k+1} - x^* \| \leq m^k [1-(1-m)\alpha\beta]^k \| u_1 - x^* \|$, *for all* $k \in \mathbb{N}$.

(b) $\| v_{k+1} - x^* \| \leq m^k \| v_1 - x^* \|$, *for all* $k \in \mathbb{N}$.

(c) $\| w_{k+1} - x^* \| \leq [1-(1-m)\beta]^k \| w_1 - x^* \|$, *for all* $k \in \mathbb{N}$.

Moreover, the S-iteration process is faster than the Picard and Mann iteration processes.

In particular, for $\alpha_k = 1, k \in \mathbb{N}$, the *S*-iteration process can be written as:

$$\begin{cases} u_1 \in C, \\ u_{k+1} = Ty_k, \\ y_k = (1-\beta_k)u_k + \beta_k T u_k, \; k \in \mathbb{N}. \end{cases} \tag{7}$$

Lemma 1 [32, p. 4]. *Let* $\{\beta_k\}_{k=0}^{\infty}$ *be a nonnegative sequence for which there exists* $k_0 \in \mathbb{N} \bigcup \{0\}$, *such that for all* $k \geq k_0$,

$$\beta_{k+1} \leq (1-\mu_k)\beta_k + \mu_k \gamma_k, \tag{8}$$

where $\mu_k \in (0, 1)$, *for* $k \in \mathbb{N} \bigcup \{0\}$, $\sum_{k=0}^{\infty} \mu_k = \infty$ *and* $\gamma_k \geq 0$, $\forall k \in \mathbb{N} \bigcup \{0\}$. *Then the following inequality holds*:

$$0 \leq \limsup_{k \to \infty} \beta_k \leq \limsup_{k \to \infty} \gamma_k. \tag{9}$$

We list the following hypotheses for convenience:

(H_1) The function $\mathcal{F}$ in equation (1) satisfies the condition:

$$\begin{aligned} &| \mathcal{F}(t, x(t), x'(t), ..., x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t)) \\ &\quad - \mathcal{F}(t, y(t), y'(t), ..., y^{(n-1)}(t), (\mathcal{H}y)(t), (\mathcal{L}y)(t)) | \\ &\leq p(t)\left[\sum_{i=0}^{n-1} | x^{(i)}(t) - y^{(i)}(t) | + | (\mathcal{H}x)(t) - (\mathcal{H}y)(t) | + | (\mathcal{L}x)(t) - (\mathcal{L}y)(t) |\right], \end{aligned}$$

where $p \in C(I, \mathbb{R}_+)$.

(H_2) The functions $\mathcal{M}_i$ $(i = 1, 2)$ in equations (3) and (4) satisfy:

$$\begin{aligned} &| \mathcal{M}_i(t, x(t), x'(t), ..., x^{(n-1)}(t)) - \mathcal{M}_i(t, y(t), y'(t), ..., y^{(n-1)}(t)) | \\ &\leq q_i(t) \sum_{i=0}^{n-1} | x^{(i)}(t) - y^{(i)}(t) |, \end{aligned}$$

where $q_i \in C(I, \mathbb{R}_+)$.

(H_3) There exist non-negative constants K_i^* $(i = 1, 2)$ such that

$$|\mathcal{K}_i(t, s)||q_i(s)| \leq K_i^*, \text{ for } a \leq s \leq t \leq b.$$

(H_4) $NP(1 + \alpha + \beta)(b - a) < 1,$ where

$$N = \sum_{j=0}^{n-1} \frac{(b-a)^{n-j-1}}{(n-j-1)!}, \; P = \sup\{p(t) : a \leq t \leq b\}, \; K_1^*(b-a) = \alpha, \text{ and}$$

$K_2^*(b - a) = \beta.$

Now, we are able to state and prove the following theorem which deals with the existence of solutions of equations (1)-(2).

Theorem 2. *Assume that hypotheses* (H_1)-(H_4) *hold. Let* $\{\xi_k\}_{k=0}^{\infty}$ *be a real sequence in* $[0, 1]$ *satisfying* $\sum_{k=0}^{\infty} \xi_k = \infty$. *Then equations* (1)-(2) *have a unique solution* $x \in B$ *and normal S-iterative method* (7) $($*with* $u_1 = x_0)$ *converges to* $x \in B$ *with the following estimate*:

$$\| x_{k+1} - x \|_B \leq \frac{[NP(1 + \alpha + \beta)(b - a)]^{k+1}}{e^{[1-NP(\alpha+\beta+\gamma)(b-a)]\sum_{i=0}^{k}\xi_i}} \| x_0 - x \|_B. \tag{10}$$

Proof. For $x(t) \in B,$ define the operator

$$(Tx)(t) = \sum_{i=0}^{n-1} \alpha_i \frac{(t-a)^i}{(i)!} + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s))ds, \tag{11}$$

for $t \in I = [a, b].$

Differentiating both sides of (11) with respect to t, we have

$$(Tx)^{(j)}(t) = \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s))ds, \quad (12)$$

for $t \in I$ and $0 \leq j \leq n-1$.

Let $\{x_k\}_{k=0}^{\infty}$ and $\{x_k^{(j)}\}_{k=0}^{\infty}$ $(j = 1, \ldots, n-1)$ be iterative sequences generated by normal S-iteration method (4) for the operators given in (11) and (12), respectively.

We show that $x_k \to x$ as $k \to \infty$.

From iteration (7), equations (6), (12) and hypotheses, we obtain

$$|x_{k+1}(t) - x(t)|_E$$

$$= \sum_{j=0}^{n-1} |x_{k+1}^{(j)}(t) - x^{(j)}(t)|$$

$$= \sum_{j=0}^{n-1} |(Ty_k)^{(j)}(t) - (Tx)^{(j)}(t)|$$

$$= \sum_{j=0}^{n-1} \Bigg| \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, y_k(s), y_k'(s), \ldots, y_k^{(n-1)}(s), (\mathcal{H}y_k)(s), (\mathcal{L}y_k)(s))ds$$

$$- \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} - \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s))ds \Bigg|$$

$$= \sum_{j=0}^{n-1} \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times | \mathcal{F}(s, y_k(s), y_k'(s), ..., y_k^{(n-1)}(s), (\mathcal{H}y_k)(s), (\mathcal{L}y_k)(s))$$

$$- \mathcal{F}(s, x(s), x'(s), ..., x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s)) | ds$$

$$\leq \sum_{j=0}^{n-1} \frac{(b-a)^{n-j-1}}{(n-j-1)!} \int_a^t p(s) \left[\sum_{i=0}^{n-1} | (y_k)^{(i)}(s) - x^{(i)}(s) | \right.$$

$$\left. + | (\mathcal{H}y_k)(s) - (\mathcal{H}x)(s) | + | (\mathcal{L}y_k)(s) - (\mathcal{L}x)(s) | \right] ds$$

$$\leq NP \int_a^t \left[\sum_{i=0}^{n-1} | (y_k)^{(i)}(s) - x^{(i)}(s) | \right.$$

$$\left. + | (\mathcal{H}y_k)(s) - (\mathcal{H}x)(s) | + | (\mathcal{L}y_k)(s) - (\mathcal{L}x)(s) | \right] ds. \tag{13}$$

From (3) and hypotheses (H_2) - (H_3), we obtain

$$| (\mathcal{H}y_k)(s) - (\mathcal{H}x)(s) |$$

$$= \left| \int_a^s \mathcal{K}_1(s, \tau) \mathcal{M}_1(\tau, y_k(\tau) y_k'(\tau), ..., y_k^{(n-1)}(\tau)) ds \right.$$

$$\left. - \int_a^s \mathcal{K}_1(s, \tau) \mathcal{M}_1(\tau, x(\tau) x'(\tau), ..., x^{(n-1)}(\tau)) d\tau \right|$$

$$\leq \int_a^s | \mathcal{K}_1(s, \tau) || \mathcal{M}_1(\tau, y_k(\tau) y_k'(\tau), ..., y_k^{(n-1)}(\tau))$$

$$- \mathcal{M}_1(\tau, x(\tau) x'(\tau), ..., x^{(n-1)}(\tau)) | d\tau$$

$$\leq \int_a^s |\mathcal{K}_1(s, \tau)| q_1(\tau) \sum_{i=0}^{n-1} |(y_k)^{(i)}(\tau) - x^{(i)}(\tau)| d\tau$$

$$\leq \int_a^s K_1^* \sum_{i=0}^{n-1} |(y_k)^{(i)}(\tau) - x^{(i)}(\tau)| d\tau. \tag{14}$$

Similarly, from (4) and hypotheses (H_2)-(H_3), we have

$$|(\mathcal{L}y_k)(s) - (\mathcal{L}x)(s)| \leq \int_a^b K_2^* \sum_{i=0}^{n-1} |(y_k)^{(i)}(\tau) - x^{(i)}(\tau)| d\tau. \tag{15}$$

Therefore, using (14) and (15) in (13), we have

$$|x_{k+1}(t) - x(t)|_E$$

$$\leq NP \int_a^t \Bigg[\sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - x^{(i)}(s)| + \int_a^s K_1^* \sum_{i=0}^{n-1} |(y_k)^{(i)}(\tau) - x^{(i)}(\tau)| d\tau$$

$$+ \int_a^b K_2^* \sum_{i=0}^{n-1} |(y_k)^{(i)}(\tau) - x^{(i)}(\tau)| d\tau \Bigg] ds$$

$$\leq NP \int_a^t \Bigg[|y_k(s) - x(s)|_E + \int_a^s K_1^* |y_k(\tau) - x(\tau)|_E d\tau$$

$$+ \int_a^b K_2^* |y_k(\tau) - x(\tau)|_E d\tau \Bigg] ds. \tag{16}$$

Now, we estimate

$$|y_k(t) - x(t)|_E$$

$$= \sum_{j=0}^{n-1} |y_k^{(j)}(t) - x^{(j)}(t)|$$

$$= \sum_{j=0}^{n-1} [(1 - \xi_k) |x_k^{(j)}(t) - x^{(j)}(t)| + \xi_k |(Tx_k)^{(j)}(t) - (Tx)^{(j)}(t)|]$$

$$= \left[(1-\xi_k)\sum_{j=0}^{n-1} | x_k^{(j)}(t) - x^{(j)}(t) | + \xi_k \sum_{j=0}^{n-1} | (Tx_k)^{(j)}(t) - (Tx)^{(j)}(t) | \right]$$

$$\leq (1-\xi_k)\sum_{j=0}^{n-1} | x_k^{(j)}(t) - x^{(j)}(t) |$$

$$+ \xi_k NP \int_a^t \left[| x_k(s) - x(s) |_E + \int_a^s K_1^* | x_k(\tau) - x(\tau) |_E d\tau \right.$$

$$\left. + \int_a^b K_2^* | x_k(\tau) - x(\tau) |_E d\tau \right] ds$$

$$\leq (1-\xi_k) | x_k(t) - x(t) |_E$$

$$+ \xi_k NP \int_a^t \left[| x_k(s) - x(s) |_E + \int_a^s K_1^* | x_k(\tau) - x(\tau) |_E d\tau \right.$$

$$\left. + \int_a^b K_2^* | x_k(\tau) - x(\tau) |_E d\tau \right] ds. \tag{17}$$

By taking supremum in the above inequalities, we obtain

$$\| x_{k+1} - x \|_B$$

$$\leq NP \int_a^t \left[\| y_k - x \|_B + \int_a^s K_1^* \| y_k - x \|_B d\tau + \int_a^b K_2^* \| y_k - x \|_B d\tau \right] ds$$

$$\leq NP \int_a^t [1 + K_1^*(b-a) + K_2^*(b-a)] ds \| y_k - x \|_B$$

$$\leq NP[1 + \alpha + \beta](b-a) \| y_k - x \|_B \tag{18}$$

and

$$\| y_k - x \|_B \leq [(1-\xi_k) \| x_k - x \|_B + \xi_k NP[1 + \alpha + \beta](b-a) \| x_k - x \|_B]$$

$$= [(1-\xi_k) + \xi_k NP(1 + \alpha + \beta)(b-a)] \| x_k - x \|_B$$

$$= [1 - \xi_k(1 - NP(1 + \alpha + \beta)(b-a))] \| x_k - x \|_B, \tag{19}$$

respectively.

Therefore, using (19) in (18), we have

$$\| x_{k+1} - x \|_B$$

$$\leq [NP(1+\alpha+\beta)(b-a)][1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\| x_k - x \|_B. \quad (20)$$

Thus, by induction, we get

$$\| x_{k+1} - x \|_B \leq [NP(1+\alpha+\beta)(b-a)]^{k+1}$$

$$\times \prod_{j=0}^{k} [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\| x_0 - x \|_B. \quad (21)$$

Since $\xi_k \in [0, 1]$ for all $k \in \mathbb{N} \cup \{0\}$, the assumption (H_4) yields

$$\xi_k \leq 1 \text{ and } NP(1+\alpha+\beta)(b-a) < 1$$

$$\Rightarrow \xi_k NP(1+\alpha+\beta)(b-a) < \xi_k$$

$$\Rightarrow \xi_k[1-NP(1+\alpha+\beta)(b-a)] < 1, \forall k \in \mathbb{N} \cup \{0\}. \quad (22)$$

From the classical analysis, we know that

$$1 - x \leq e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, \; x \in [0, 1].$$

Hence by utilizing this fact with (22) in (21), we obtain

$$\| x_{k+1} - x \|_B$$

$$\leq (NP(1+\alpha+\beta)(b-a))^{k+1} e^{-(1-NP(1+\alpha+\beta)(b-a))\sum_{j=0}^{k}\xi_j} \| x_0 - x \|_B. \quad (23)$$

This is (10). Since $\sum_{k=0}^{\infty} \xi_k = \infty$,

$$e^{-(1-NP(1+\alpha+\beta)(b-a))\sum_{j=0}^{k}\xi_j} \to 0 \text{ as } k \to \infty, \quad (24)$$

which implies $\lim_{k\to\infty} \| x_{k+1} - x \|_B = 0$. This gives $x_k \to x$ as $k \to \infty$. □

Remark. It is interesting to note that the inequality (23) gives the bounds in terms of known functions, which majorizes the iterations for solutions of equations (1)-(2) as well as its derivatives $x^{(j)}(t)$ $(j = 1, 2, ..., n-1)$ for $t \in I$.

3. Continuous Dependence via *S*-iteration

Suppose that $x(t)$ and $\bar{x}(t)$ are the solutions of (1) with initial data

$$x^{(j)}(a) = \alpha_j, \quad j = 0, 1, 2, ..., n-1 \tag{25}$$

and

$$\bar{x}^{(j)}(a) = \beta_j, \quad j = 0, 1, 2, ..., n-1, \tag{26}$$

respectively, where α_j and β_j are real constants. By a solution of equation (1) along with condition (26), we mean a continuous function $\bar{x}(t),\ t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies equation (1) along with condition (26). It is easy to observe that the solution $\bar{x}(t)$ of equation (1) along with condition (26) and its derivatives satisfy the integral equations of the form

$$\bar{x}^{(j)}(t) = \sum_{i=j}^{n-1} \beta_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, \bar{x}(s), \bar{x}'(s), ..., \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s))ds, \tag{27}$$

for $t \in I$ and $0 \le j \le n-1$.

Then following the steps as in the proof of Theorem 2, we define the operator for equation (1) along with condition (26) as

$$(\overline{T}\,\overline{x})(t) = \sum_{i=0}^{n-1} \beta_i \frac{(t-a)^i}{(i)!} + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!}$$

$$\times \mathcal{F}(s, \overline{x}(s), \overline{x}'(s), ..., \overline{x}^{(n-1)}(s), (\mathcal{H}\overline{x})(s), (\mathcal{L}\overline{x})(s))ds, \quad (28)$$

for $t \in I = [a, b]$.

Differentiating both sides of (28) with respect to t, we obtain

$$(\overline{T}\,\overline{x})^{(j)}(t) = \sum_{i=j}^{n-1} \beta_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, \overline{x}(s), \overline{x}'(s), ..., \overline{x}^{(n-1)}(s), (\mathcal{H}\overline{x})(s), (\mathcal{L}\overline{x})(s))ds, \quad (29)$$

for $t \in I$ and $0 \leq j \leq n-1$.

Now, we deal with the continuous dependence of solutions of equation (1) on initial data.

Theorem 3. *Suppose that hypotheses* (H_1)-(H_4) *hold. Consider the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{\overline{x}_k\}_{k=0}^{\infty}$ *generated by normal S-iterative method associated with operators* T *in* (12) *and* $\overline{T}$ *in* (29), *respectively, with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *If the sequence* $\{\overline{x}_k\}_{k=0}^{\infty}$ *converges to* $\overline{x}$, *then*

$$\| x - \overline{x} \|_B \leq \frac{3M}{[1 - NP(1 + \alpha + \beta)(b-a)]}, \quad (30)$$

where

$$M = \sum_{j=0}^{n-1} \left(\sum_{i=j}^{n-1} | \alpha_i - \beta_i | \frac{(b-a)^{i-j}}{(i-j)!} \right).$$

Proof. Suppose that the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by normal S-iterative method associated with operators T in (12) and $\overline{T}$ in (29), respectively, with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfy $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (7) and equations (6) with (12); (27) with (29) and hypotheses, we obtain

$$
\begin{aligned}
& | x_{k+1}(t) - \bar{x}_{k+1}(t) |_E \\
&= \sum_{j=0}^{n-1} | x_{k+1}^{(j)}(t) - \bar{x}_{k+1}^{(j)}(t) | \\
&= \sum_{j=0}^{n-1} | (Ty_k)^{(j)}(t) - (\bar{T}\bar{y}_k)^{(j)}(t) | \\
&= \sum_{j=0}^{n-1} \Bigg| \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\
&\qquad \times \mathcal{F}(s, y_k(s), y_k'(s), \ldots, y_k^{(n-1)}(s), (\mathcal{H}y_k)(s), (\mathcal{L}y_k)(s)) ds \\
&\qquad - \sum_{i=j}^{n-1} \beta_i \frac{(t-a)^{i-j}}{(i-j)!} - \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\
&\qquad \times \mathcal{F}(s, \bar{y}_k(s), \bar{y}_k'(s), \ldots, \bar{y}_k^{(n-1)}(s), (\mathcal{H}\bar{y}_k)(s), (\mathcal{L}\bar{y}_k)(s)) ds \Bigg| \\
&\leq \sum_{j=0}^{n-1} \left(\sum_{i=j}^{n-1} | \alpha_i - \beta_i | \frac{(t-a)^{i-j}}{(i-j)!} \right) + \sum_{j=0}^{n-1} \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\
&\quad \times | \mathcal{F}(s, y_k(s), y_k'(s), \ldots, y_k^{(n-1)}(s), (\mathcal{H}y_k)(s), (\mathcal{L}y_k)(s)) \\
&\qquad - \mathcal{F}(s, \bar{y}_k(s), \bar{y}_k'(s), \ldots, \bar{y}_k^{(n-1)}(s), (\mathcal{H}\bar{y}_k)(s), (\mathcal{L}\bar{y}_k)(s)) | ds
\end{aligned}
$$

$$\leq M + \sum_{j=0}^{n-1} \frac{(b-a)^{n-j-1}}{(n-j-1)!} \int_a^t p(s) \left[\sum_{i=0}^{n-1} | (y_k)^{(i)}(s) - (\overline{y}_k)^{(i)}(s) | \right.$$

$$\left. + | (\mathcal{H} y_k)(s) - (\mathcal{H} \overline{y}_k)(s) | + | (\mathcal{L} y_k)(s) - (\mathcal{L} \overline{y}_k)(s) | \right] ds$$

$$\leq M + NP \int_a^t \left[\sum_{i=0}^{n-1} | (y_k)^{(i)}(s) - (\overline{y}_k)^{(i)}(s) | \right.$$

$$\left. + | (\mathcal{H} y_k)(s) - (\mathcal{H} \overline{y}_k)(s) | + | (\mathcal{L} y_k)(s) - (\mathcal{L} \overline{y}_k)(s) | \right] ds. \tag{31}$$

Recalling equations (14), (15) and (18), the above inequality becomes

$$\| x_{k+1} - x \|_B \leq M + NP[1 + \alpha + \beta](b-a) \| y_k - \overline{y}_k \|_B. \tag{32}$$

Similarly, we have

$$| y_k(t) - \overline{y}_k(t) |_E$$

$$= \sum_{j=0}^{n-1} | y_k^{(j)}(t) - \overline{y}_k^{(j)}(t) |$$

$$= \sum_{j=0}^{n-1} [(1 - \xi_k) | x_k^{(j)}(t) - \overline{x}_k^{(j)}(t) | + \xi_k | (Tx_k)^{(j)}(t) - (\overline{T} \overline{x}_k)^{(j)}(t) |]$$

$$= \left[(1 - \xi_k) \sum_{j=0}^{n-1} | x_k^{(j)}(t) - \overline{x}_k^{(j)}(t) | + \xi_k \sum_{j=0}^{n-1} | (Tx_k)^{(j)}(t) - (\overline{T} \overline{x}_k)^{(j)}(t) | \right]$$

$$\leq (1 - \xi_k) \sum_{j=0}^{n-1} | x_k^{(j)}(t) - \overline{x}_k^{(j)}(t) |$$

$$+\xi_k M + \xi_k NP\int_a^t \left[\sum_{i=0}^{n-1} |\,(x_k)^{(i)}(s) - (\bar{x}_k)^{(i)}(s)\,| \right.$$

$$\left. + |\,(\mathcal{H}x_k)(s) - (\mathcal{H}\bar{x}_k)(s)\,| + |\,(\mathcal{L}x_k)(s) - (\mathcal{L}\bar{x}_k)(s)\,| \right] ds. \quad (33)$$

Hence from equations (14), (15) and (19), the above inequality takes the form

$$\| y_k - \bar{y}_k \|_B$$

$$\leq (1-\xi_k)\| x_k - \bar{x}_k \|_B + \xi_k M + \xi_k NP(1+\alpha+\beta)(b-a)\| x_k - \bar{x}_k \|_B$$

$$\leq \xi_k M + [(1-\xi_k) + \xi_k NP(1+\alpha+\beta)(b-a)]\| x_k - \bar{x}_k \|_B$$

$$\leq \xi_k M + [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\| x_k - \bar{x}_k \|_B. \quad (34)$$

Therefore, using (34) in (32) and hypothesis (H_4), and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\| x_{k+1} - \bar{x}_{k+1} \|_B$$

$$\leq [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\| x_k - \bar{x}_k \|_B + \xi_k M + 2\xi_k M$$

$$\leq [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\| x_k - \bar{x}_k \|_B$$

$$+ \xi_k(1-NP(1+\alpha+\beta)(b-a))\frac{3M}{(1-NP(1+\alpha+\beta)(b-a))}. \quad (35)$$

We denote

$$\beta_k = \| x_k - \bar{x}_k \|_B \geq 0,$$

$$\mu_k = \xi_k(1-NP(1+\alpha+\beta)(b-a)) \in (0, 1),$$

$$\gamma_k = \frac{3M}{(1-NP(1+\alpha+\beta)(b-a))} \geq 0.$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (35) satisfies all the conditions of Lemma 1 and hence we have

$$0 \le \limsup_{k \to \infty} \beta_k \le \limsup_{k \to \infty} \gamma_k$$

$$\Rightarrow 0 \le \limsup_{k \to \infty} \| x_k - \bar{x}_k \|_B \le \limsup_{k \to \infty} \frac{3M}{(1 - NP(1 + \alpha + \beta)(b - a))}$$

$$\Rightarrow 0 \le \limsup_{k \to \infty} \| x_k - \bar{x}_k \|_B \le \frac{3M}{(1 - NP(1 + \alpha + \beta)(b - a))}. \tag{36}$$

Using the assumptions $\lim_{k \to \infty} x_k = x$ and $\lim_{k \to \infty} \bar{x}_k = \bar{x}$, we get from (36) that

$$\| x - \bar{x} \|_B \le \frac{3M}{[1 - NP(1 + \alpha + \beta)(b - a)]}, \tag{37}$$

which shows the dependency of solutions of IVP (1)-(2) and IVP (1) with (26) on given initial data. □

4. Closeness of Solution via *S*-iteration

In this section, we study the continuous dependence of solutions of (1)-(2) on the given initial data, and function involved therein.

Now, we consider the initial value problem (1)-(2) and the corresponding problem

$$\bar{x}^{(n)}(t) = \mathcal{G}(t, \bar{x}(t), \bar{x}'(t), \dots, \bar{x}^{(n-1)}(t), (\mathcal{H}\bar{x})(t), (\mathcal{L}\bar{x})(t)), \tag{38}$$

for $t \in I = [a, b]$, with the given initial conditions

$$\bar{x}^{(j)}(a) = \beta_j, \quad j = 0, 1, 2, \dots, n - 1, \tag{39}$$

where $\mathcal{G} \in C(I \times \mathbb{R}^{n+2}, \mathbb{R})$; $(\mathcal{H}\bar{x})(t)$, $(\mathcal{L}\bar{x})(t)$ are as in (3), (4) and β_j are given real constants.

By a solution of equations (38)-(39), we mean a continuous function $\bar{x}(t)$, $t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies the IVP (38)-(39). It is easy to observe that the solution $\bar{x}(t)$ of equations (38)-(39) and its derivatives satisfy the integral equations:

$$\bar{x}^{(j)}(t) = \sum_{i=j}^{n-1} \beta_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!} \times \mathcal{G}(s, \bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s))ds, \quad (40)$$

for $t \in I$ and $0 \leq j \leq n-1$.

Let $\bar{x}(t) \in B$. Following steps from the proof of Theorem 2, define the operator for equations (38)-(39):

$$(\bar{T}\bar{x})(t) = \sum_{i=0}^{n-1} \beta_i \frac{(t-a)^{i}}{(i)!} + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} \times \mathcal{G}(s, \bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s))ds, \quad (41)$$

for $t \in I = [a, b]$.

Differentiating both sides of (41) with respect to t, we have

$$(\bar{T}\bar{x})^{(j)}(t) = \sum_{i=j}^{n-1} \beta_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!} \times \mathcal{G}(s, \bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s))ds, \quad (42)$$

for $t \in I$ and $0 \leq j \leq n-1$.

The next theorem deals with the closeness of solutions of the problems (1)-(2) and (38)-(39).

Theorem 4. *Consider the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{\bar{x}_k\}_{k=0}^{\infty}$ *generated by normal S-iterative method associated with operators T in* (12) *and* $\overline{T}$ *in* (42), *respectively, with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *Assume that*

(i) *all conditions of Theorem* 2, *and* $x(t)$ *hold and* $\bar{x}(t)$ *are solutions of* (1)-(2) *and* (38)-(39), *respectively*,

(ii) *there exists a non-negative constant* ε *such that*

$$\begin{aligned} &|\mathcal{F}(t, x(t), x'(t), \ldots, x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t)) \\ &\quad - \mathcal{G}(t, x(t), x'(t), \ldots, x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t))| \\ &\leq \bar{p}(t)\varepsilon, \forall t \in I, \end{aligned} \tag{43}$$

where $\bar{p}(t) \in \bar{p}(I, \mathbb{R}_+)$.

If the sequence $\{\bar{x}_k\}_{k=0}^{\infty}$ *converges to* $\bar{x}$, *then*

$$\| x - \bar{x} \|_B \leq \frac{3[M + N\overline{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}, \tag{44}$$

where $\overline{P} = \max\{\bar{p}(t) : a \leq t \leq b\}$.

Proof. Suppose that the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\bar{x}_k\}_{k=0}^{\infty}$ generated by normal S-iterative method associated with operators T in (12) and $\overline{T}$ in (42), respectively, with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfy $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (7) and equations (6) with (12); (40) with (42) and hypotheses, we obtain

$$\begin{aligned} &| x_{k+1}(t) - \bar{x}_{k+1}(t) |_E \\ &= \sum_{j=0}^{n-1} | x_{k+1}^{(j)}(t) - \bar{x}_{k+1}^{(j)}(t) | \end{aligned}$$

$$= \sum_{j=0}^{n-1} | (Ty_k)^{(j)}(t) - (\overline{T}\, \overline{y}_k)^{(j)}(t) |$$

$$= \sum_{j=0}^{n-1} \Bigg| \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s,\ y_k(s),\ y_k'(s),\ \ldots,\ y_k^{(n-1)}(s),\ (\mathcal{H} y_k)(s),\ (\mathcal{L} y_k)(s))ds$$

$$- \sum_{i=j}^{n-1} \beta_i \frac{(t-a)^{i-j}}{(i-j)!} - \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{G}(s,\ \overline{y}_k(s),\ \overline{y}_k'(s),\ \ldots,\ \overline{y}_k^{(n-1)}(s),\ (\mathcal{H} \overline{y}_k)(s),\ (\mathcal{L} \overline{y}_k)(s))ds \Bigg|$$

$$\leq \sum_{j=0}^{n-1} \left(\sum_{i=j}^{n-1} | \alpha_i - \beta_i | \frac{(t-a)^{i-j}}{(i-j)!} \right) + \sum_{j=0}^{n-1} \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times | \mathcal{F}(s,\ y_k(s),\ y_k'(s),\ \ldots,\ y_k^{(n-1)}(s),\ (\mathcal{H} y_k)(s),\ (\mathcal{L} y_k)(s))$$

$$- \mathcal{F}(s,\ \overline{y}_k(s),\ \overline{y}_k'(s),\ \ldots,\ \overline{y}_k^{(n-1)}(s),\ (\mathcal{H} \overline{y}_k)(s),\ (\mathcal{L} \overline{y}_k)(s)) | ds$$

$$+ \sum_{j=0}^{n-1} \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times | \mathcal{F}(s,\ \overline{y}_k(s),\ \overline{y}_k'(s),\ \ldots,\ \overline{y}_k^{(n-1)}(s),\ (\mathcal{H} \overline{y}_k)(s),\ (\mathcal{L} \overline{y}_k)(s))$$

$$- \mathcal{G}(s,\ \overline{y}_k(s),\ \overline{y}_k'(s),\ \ldots,\ \overline{y}_k^{(n-1)}(s),\ (\mathcal{H} \overline{y}_k)(s),\ (\mathcal{L} \overline{y}_k)(s)) | ds$$

$$\leq M + \sum_{j=0}^{n-1} \frac{(b-a)^{n-j-1}}{(n-j-1)!} \int_a^t p(s) \left[\sum_{i=0}^{n-1} | (y_k)^{(i)}(s) - (\overline{y}_k)^{(i)}(s) | \right.$$

$$\left. + | (\mathcal{H} y_k)(s) - (\mathcal{H} \overline{y}_k)(s) | + | (\mathcal{L} y_k)(s) - (\mathcal{L} \overline{y}_k)(s) | \right] ds$$

$$+\sum_{j=0}^{n-1}\frac{(b-a)^{n-j-1}}{(n-j-1)!}\int_a^t \overline{p}(s)\varepsilon ds$$

$$\leq M + NP\int_a^t\left[\sum_{i=0}^{n-1}|(y_k)^{(i)}(s)-(\overline{y}_k)^{(i)}(s)|\right.$$

$$\left.+|(\mathcal{H}y_k)(s)-(\mathcal{H}\overline{y}_k)(s)|+|(\mathcal{L}y_k)(s)-(\mathcal{L}\overline{y}_k)(s)|\right]ds$$

$$+N\overline{P}\varepsilon(b-a). \tag{45}$$

Recalling equations (14), (15) and (18), the above inequality becomes

$$\| x_{k+1}-\overline{x}_{k+1}\|_B \leq M + N\overline{P}\varepsilon(b-a)+NP[1+\alpha+\beta](b-a)\| y_k-\overline{y}_k\|_B. \tag{46}$$

Similarly, we have

$$|y_k(t)-\overline{y}_k(t)|_E$$

$$=\sum_{j=0}^{n-1}|y_k^{(j)}(t)-\overline{y}_k^{(j)}(t)|$$

$$=\sum_{j=0}^{n-1}[(1-\xi_k)|x_k^{(j)}(t)-\overline{x}_k^{(j)}(t)|+\xi_k|(Tx_k)^{(j)}(t)-(\overline{T}\,\overline{x}_k)^{(j)}(t)|]$$

$$=\left[(1-\xi_k)\sum_{j=0}^{n-1}|x_k^{(j)}(t)-\overline{x}_k^{(j)}(t)|+\xi_k\sum_{j=0}^{n-1}|(Tx_k)^{(j)}(t)-(\overline{T}\,\overline{x}_k)^{(j)}(t)|\right]$$

$$\leq (1-\xi_k)\sum_{j=0}^{n-1}|x_k^{(j)}(t)-\overline{x}_k^{(j)}(t)|+\xi_k[M+N\overline{P}\varepsilon(b-a)]$$

$$+\xi_k NP\int_a^t\left[\sum_{i=0}^{n-1}|(y_k)^{(i)}(s)-(\overline{y}_k)^{(i)}(s)|\right.$$

$$\left.+|(\mathcal{H}y_k)(s)-(\mathcal{H}\overline{y}_k)(s)|+|(\mathcal{L}y_k)(s)-(\mathcal{L}\overline{y}_k)(s)|\right]ds. \tag{47}$$

Hence from equations (14), (15) and (19), the above inequality takes the form

$$\| y_k - \bar{y}_k \|_B$$

$$\leq (1 - \xi_k)\| x_k - \bar{x}_k \|_B + \xi_k [M + N\bar{P}\varepsilon(b-a)]$$

$$+ \xi_k NP(1 + \alpha + \beta)(b-a)\| x_k - \bar{x}_k \|_B$$

$$\leq \xi_k [M + N\bar{P}\varepsilon(b-a)] + [(1 - \xi_k) + \xi_k NP(1 + \alpha + \beta)(b-a)]\| x_k - \bar{x}_k \|_B$$

$$\leq \xi_k [M + N\bar{P}\varepsilon(b-a)] + [1 - \xi_k (1 - NP(1 + \alpha + \beta)(b-a))]\| x_k - \bar{x}_k \|_B. \tag{48}$$

Therefore, using (46) in (48) and hypothesis (H_4), and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\| x_{k+1} - \bar{x}_{k+1} \|_B$$

$$\leq [1 - \xi_k (1 - NP(1 + \alpha + \beta)(b-a))]\| x_k - \bar{x}_k \|_B$$

$$+ \xi_k [M + N\bar{P}\varepsilon(b-a)] + 2\xi_k [M + N\bar{P}\varepsilon(b-a)]$$

$$\leq [1 - \xi_k (1 - NP(1 + \alpha + \beta)(b-a))]\| x_k - \bar{x}_k \|_B$$

$$+ \xi_k [1 - NP(1 + \alpha + \beta)(b-a)] \frac{3[M + N\bar{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}. \tag{49}$$

We denote

$$\beta_k = \| x_k - \bar{x}_k \|_B \geq 0,$$

$$\mu_k = \xi_k [1 - NP(1 + \alpha + \beta)(b-a)] \in (0, 1),$$

$$\gamma_k = \frac{3[M + N\bar{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]} \geq 0.$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (49) satisfies all the conditions of Lemma 1 and hence we have

$$0 \leq \limsup_{k \to \infty} \beta_k \leq \limsup_{k \to \infty} \gamma_k$$

$$\Rightarrow 0 \leq \limsup_{k \to \infty} \| x_k - \bar{x}_k \|_B \leq \limsup_{k \to \infty} \frac{3[M + N\bar{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}$$

$$\Rightarrow 0 \leq \limsup_{k \to \infty} \| x_k - \bar{x}_k \|_B \leq \frac{3[M + N\bar{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}. \tag{50}$$

Using this fact and the assumptions $\lim_{k \to \infty} x_k = x$, $\lim_{k \to \infty} \bar{x}_k = \bar{x}$, we get from (50) that

$$\| x - \bar{x} \|_B \leq \frac{3[M + N\bar{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}. \tag{51}$$

□

Remark. The inequality (51) relates the solutions of the problems (1)-(2) and (38)-(39) in the sense that if $\mathcal{F}$ and $\mathcal{G}$ are close, then not only the solutions of the problems (1)-(2) and (38)-(39) are close to each other (i.e., $\| x - \bar{x} \|_B \to 0$), but also depend continuously on the functions involved therein and initial data.

5. Parameter Dependence via *S*-iteration

We next consider the following initial value problems containing certain parameters:

$$x^{(n)}(t) = \mathcal{F}(t, x(t), x'(t), \ldots, x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t)\, \mu_1), \tag{52}$$

$$\bar{x}^{(n)}(t) = \mathcal{F}(t, \bar{x}(t), \bar{x}'(t), \ldots, \bar{x}^{(n-1)}(t), (\mathcal{H}\bar{x})(t), (\mathcal{L}\bar{x})(t), \mu_2), \tag{53}$$

for $t \in I = [a, b]$, with the given initial conditions (2), where $\mathcal{F} \in C(I \times \mathbb{R}^{n+3}, \mathbb{R})$; $\mathcal{H}, \mathcal{L}$ as defined in (3), (4) are given functions and μ_1, μ_2 are real parameters.

By a solution of equations (52) with condition (2), we mean a continuous function $x(t)$, $t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies equations (52) with condition (2). It is easy to observe that the solution $x(t)$ of equations (52) with condition (2) and its derivatives satisfy the integral equations of the form

$$x^{(j)}(t) = \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s), \mu_1)ds, \quad (54)$$

for $t \in I$ and $0 \leq j \leq n-1$.

Let $x(t) \in B$. Following steps from the proof of Theorem 2, define the operator for equation (52) as

$$(Tx)(t) = \sum_{i=0}^{n-1} \alpha_i \frac{(t-a)^{i}}{(i)!} + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s), \mu_1)ds, \quad (55)$$

for $t \in I = [a, b]$.

Differentiating both sides of (55) with respect to t, we have

$$(Tx)^{(j)}(t) = \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s), \mu_1)ds, \quad (56)$$

for $t \in I$ and $0 \leq j \leq n-1$.

Similarly, we define for equation (53):

$$\bar{x}^{(j)}(t) = \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, \bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s), \mu_2) ds, \quad (57)$$

for $t \in I$ and $0 \leq j \leq n-1$.

Let $\bar{x}(t) \in B$. Following steps from the proof of Theorem 2, define the operator for equation (53) as

$$(\bar{T}\bar{x})(t) = \sum_{i=0}^{n-1} \alpha_i \frac{(t-a)^{i}}{(i)!} + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!}$$

$$\times \mathcal{F}(s, \bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s), \mu_2) ds, \quad (58)$$

for $t \in I = [a, b]$.

Differentiating both sides of (58) with respect to t, we have

$$(\bar{T}\bar{x})^{(j)}(t) = \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, \bar{x}(s), \bar{x}'(s), \ldots, \bar{x}^{(n-1)}(s), (\mathcal{H}\bar{x})(s), (\mathcal{L}\bar{x})(s), \mu_2) ds, \quad (59)$$

for $t \in I$ and $0 \leq j \leq n-1$.

The following theorem states the continuous dependency of solutions on parameters.

Theorem 5. *Consider the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{\bar{x}_k\}_{k=0}^{\infty}$ *generated by normal S-iterative method associated with operators* T *in* (56) *and* $\bar{T}$ *in* (59), *respectively, with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* [0, 1] *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *Assume that*

(i) *The hypotheses* (H_2)-(H_4) *hold and* $x(t)$ *and* $\bar{x}(t)$ *are solutions of equation* (52) *with condition* (2) *and equation* (53) *with condition* (2), *respectively*, *and*

(ii) *The function* $\mathcal{F}$ *in equations* (52) *and* (53) *satisfy the conditions*:

$$|\,\mathcal{F}(t, x(t), x'(t), ..., x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t), \mu_1)$$

$$- \mathcal{F}(t, y(t), y'(t), ..., y^{(n-1)}(t), (\mathcal{H}y)(t), (\mathcal{L}y)(t), \mu_1)\,|$$

$$\leq p(t)\left[\sum_{i=0}^{n-1}|\,x^{(i)}(t) - y^{(i)}(t)\,| + |\,(\mathcal{H}x)(t) - (\mathcal{H}y)(t)\,| + |\,(\mathcal{L}x)(t) - (\mathcal{L}y)(t)\,|\right],$$

and

$$|\,\mathcal{F}(t, x(t), x'(t), ..., x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t), \mu_1)$$

$$- \mathcal{F}(t, x(t), x'(t), ..., x^{(n-1)}(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t), \mu_2)\,|$$

$$\leq r(t)|\,\mu_1 - \mu_2\,|,$$

where $p, r \in C(I, \mathbb{R}_+)$.

If the sequence $\{\bar{x}_k\}_{k=0}^{\infty}$ *converges to* $\bar{x}$, *then*

$$\|\,x - \bar{x}\,\|_B \leq \frac{3[N\bar{R}\,|\,\mu_1 - \mu_2\,|\,(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}, \tag{60}$$

where $\bar{R} = \max\{r(t) : a \leq t \leq b\}$.

Proof. Suppose that the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\bar{x}_k\}_{k=0}^{\infty}$ generated by normal S-iterative method associated with operators T in (56) and $\bar{T}$ in (59), respectively, with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfy $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (7) and equations (54) with (56), (57) with (59) and hypotheses, we obtain

$$|x_{k+1}(t) - \bar{x}_{k+1}(t)|_E$$

$$= \sum_{j=0}^{n-1} |x_{k+1}^{(j)}(t) - \bar{x}_{k+1}^{(j)}(t)|$$

$$= \sum_{j=0}^{n-1} |(Ty_k)^{(j)}(t) - (\bar{T}\bar{y}_k)^{(j)}(t)|$$

$$= \sum_{j=0}^{n-1} \Bigg| \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} + \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, y_k(s), y_k'(s), \ldots, y_k^{(n-1)}(s), (\mathcal{H}y_k)(s), (\mathcal{L}y_k)(s), \mu_1)ds$$

$$- \sum_{i=j}^{n-1} \alpha_i \frac{(t-a)^{i-j}}{(i-j)!} - \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times \mathcal{F}(s, \bar{y}_k(s), \bar{y}_k'(s), \ldots, \bar{y}_k^{(n-1)}(s), (\mathcal{H}\bar{y}_k)(s), (\mathcal{L}\bar{y}_k)(s), \mu_2)ds \Bigg|$$

$$\leq \sum_{j=0}^{n-1} \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times |\mathcal{F}(s, y_k(s), y_k'(s), \ldots, y_k^{(n-1)}(s), (\mathcal{H}y_k)(s), (\mathcal{L}y_k)(s), \mu_1)$$

$$- \mathcal{F}(s, \bar{y}_k(s), \bar{y}_k'(s), \ldots, \bar{y}_k^{(n-1)}(s), (\mathcal{H}\bar{y}_k)(s), (\mathcal{L}\bar{y}_k)(s), \mu_1)|ds$$

$$+ \sum_{j=0}^{n-1} \int_a^t \frac{(t-s)^{n-j-1}}{(n-j-1)!}$$

$$\times |\mathcal{F}(s, \bar{y}_k(s), \bar{y}_k'(s), \ldots, \bar{y}_k^{(n-1)}(s), (\mathcal{H}\bar{y}_k)(s), (\mathcal{L}\bar{y}_k)(s), \mu_1)$$

$$- \mathcal{F}(s, \bar{y}_k(s), \bar{y}_k'(s), \ldots, \bar{y}_k^{(n-1)}(s), (\mathcal{H}\bar{y}_k)(s), (\mathcal{L}\bar{y}_k)(s), \mu_2)|ds$$

$$\leq \sum_{j=0}^{n-1} \frac{(b-a)^{n-j-1}}{(n-j-1)!} \int_a^t p(s)\Bigg[\sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - (\bar{y}_k)^{(i)}(s)|$$

$$+ |(\mathcal{H} y_k)(s) - (\mathcal{H}\bar{y}_k)(s)| + |(\mathcal{L} y_k)(s) - (\mathcal{L}\bar{y}_k)(s)| \Bigg] ds$$

$$+ \sum_{j=0}^{n-1} \frac{(b-a)^{n-j-1}}{(n-j-1)!} \int_a^t r(s) |\mu_1 - \mu_2| ds$$

$$\leq NP \int_a^t \Bigg[\sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - (\bar{y}_k)^{(i)}(s)|$$

$$+ |(\mathcal{H} y_k)(s) - (\mathcal{H}\bar{y}_k)(s)| + |(\mathcal{L} y_k)(s) - (\mathcal{L}\bar{y}_k)(s)| \Bigg] ds$$

$$+ N\bar{R} |\mu_1 - \mu_2| (b-a). \tag{61}$$

Recalling equations (14), (15) and (18), the above inequality becomes

$$\| x_{k+1} - \bar{x}_{k+1} \|_B$$

$$\leq N\bar{R} |\mu_1 - \mu_2| (b-a) + NP(1+\alpha+\beta)(b-a) \| y_k - \bar{y}_k \|_B. \tag{62}$$

Similarly, we have

$$| y_k(t) - \bar{y}_k(t) |_E$$

$$= \sum_{j=0}^{n-1} | y_k^{(j)}(t) - \bar{y}_k^{(j)}(t) |$$

$$= \sum_{j=0}^{n-1} [(1-\xi_k) | x_k^{(j)}(t) - \bar{x}_k^{(j)}(t) | + \xi_k | (Tx_k)^{(j)}(t) - (\bar{T}\bar{x}_k)^{(j)}(t) |]$$

$$= \Bigg[(1-\xi_k) \sum_{j=0}^{n-1} | x_k^{(j)}(t) - \bar{x}_k^{(j)}(t) | + \xi_k \sum_{j=0}^{n-1} | (Tx_k)^{(j)}(t) - (\bar{T}\bar{x}_k)^{(j)}(t) | \Bigg]$$

$$\leq (1-\xi_k)\sum_{j=0}^{n-1}|\, x_k^{(j)}(t)-\bar{x}_k^{(j)}(t)\,| + \xi_k[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)]$$

$$+\ \xi_k NP\int_a^t\left[\sum_{i=0}^{n-1}|\,(y_k)^{(i)}(s)-(\bar{y}_k)^{(i)}(s)\,|\right.$$

$$\left. +\ |\,(\mathcal{H}y_k)(s)-(\mathcal{H}\bar{y}_k)(s)\,| + |\,(\mathcal{L}y_k)(s)-(\mathcal{L}\bar{y}_k)(s)\,|\right]ds. \quad (63)$$

Hence from equations (14), (15) and (19), the above inequality takes the form

$$\begin{aligned}\|\, y_k-\bar{y}_k\,\|_B &\leq (1-\xi_k)\|\, x_k-\bar{x}_k\,\|_B + \xi_k[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)] \\ &\quad + \xi_k NP(1+\alpha+\beta)(b-a)\|\, x_k-\bar{x}_k\,\|_B \\ &\leq \xi_k[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)] \\ &\quad + [(1-\xi_k)+\xi_k NP(1+\alpha+\beta)(b-a)]\|\, x_k-\bar{x}_k\,\|_B \\ &\leq \xi_k[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)] \\ &\quad + [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\|\, x_k-\bar{x}_k\,\|_B. \end{aligned} \quad (64)$$

Therefore, using (64) in (62) and hypothesis (H_4), and $\frac{1}{2}\leq \xi_k$ for all $k\in \mathbb{N}\cup\{0\}$, the resulting inequality becomes

$$\begin{aligned}&\|\, x_{k+1}-\bar{x}_{k+1}\,\|_B \\ &\leq [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\|\, x_k-\bar{x}_k\,\|_B \\ &\quad + \xi_k[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)] + 2\xi_k[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)] \\ &\leq [1-\xi_k(1-NP(1+\alpha+\beta)(b-a))]\|\, x_k-\bar{x}_k\,\|_B \\ &\quad + \xi_k[1-NP(1+\alpha+\beta)(b-a)]\frac{3[N\bar{R}|\,\mu_1-\mu_2\,|(b-a)]}{[1-NP(1+\alpha+\beta)(b-a)]}. \end{aligned} \quad (65)$$

We denote

$$\beta_k = \| x_k - \bar{x}_k \|_B \geq 0,$$

$$\mu_k = \xi_k [1 - NP(1 + \alpha + \beta)(b - a)] \in (0, 1),$$

$$\gamma_k = \frac{3[N\bar{R} | \mu_1 - \mu_2 | (b - a)]}{[1 - NP(1 + \alpha + \beta)(b - a)]} \geq 0.$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily see that (65) satisfies all the conditions of Lemma 1 and hence we have

$$0 \leq \limsup_{k \to \infty} \beta_k \leq \limsup_{k \to \infty} \gamma_k$$

$$\Rightarrow 0 \leq \limsup_{k \to \infty} \| x_k - \bar{x}_k \|_B \leq \limsup_{k \to \infty} \frac{3[N\bar{R} | \mu_1 - \mu_2 | (b - a)]}{[1 - NP(1 + \alpha + \beta)(b - a)]}$$

$$\Rightarrow 0 \leq \limsup_{k \to \infty} \| x_k - \bar{x}_k \|_B \leq \frac{3[N\bar{R} | \mu_1 - \mu_2 | (b - a)]}{[1 - NP(1 + \alpha + \beta)(b - a)]}. \tag{66}$$

Using this fact and the assumptions $\lim_{k \to \infty} x_k = x$ and $\lim_{k \to \infty} \bar{x}_k = \bar{x}$, we get from (66) that

$$\| x - \bar{x} \|_B \leq \frac{3[N\bar{R} | \mu_1 - \mu_2 | (b - a)]}{[1 - NP(1 + \alpha + \beta)(b - a)]}. \tag{67}$$

□

Remark. The result dealing with this property of a solution is called “dependence of solutions on parameters”. Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters is an important aspect in various physical problems.

6. Examples

We consider the following example for $n = 2$:

$$x'' = \frac{t^3}{5}[\sin x(t) \cos x(t)] - \frac{t^3}{7}\sin x'(t)$$

$$+ \frac{t^3}{9}\int_0^t \left(\frac{t^2}{4 + e^s}\right) x(s) ds + \frac{t^3}{7}\int_0^1 t(4s^2 + 1)\frac{x(s)}{5 + x(s)} ds, \tag{68}$$

where $t \in I = [0, 1]$, with initial conditions

$$x(0) = 0, \quad x'(0) = \frac{1}{2}. \tag{69}$$

Comparing this equation with proposed equation (1), we get $\mathcal{F} \in C(I \times \mathbb{R}^4, \mathbb{R})$, with

$$\mathcal{F}(t, x(t), x'(t), (\mathcal{H}x)(t), (\mathcal{L}x)(t))$$

$$= \frac{t^3}{5}[\sin x(t) \cos x(t)] - \frac{t^3}{7}\sin x'(t) + \frac{t^3}{9}(\mathcal{H}x)(t) + \frac{t^3}{7}(\mathcal{L}x)(t),$$

where

$$(\mathcal{H}x)(t) = \int_0^t \left(\frac{t^2}{4 + e^s}\right) x(s) ds \quad \text{and} \quad (\mathcal{L}x)(t) = \int_0^1 t(4s^2 + 1)\frac{x(s)}{5 + x(s)} ds.$$

Also, we get $\mathcal{K}_1(s, t) = \dfrac{t^2}{4 + e^s}$, $\mathcal{K}_2(s, t) = t(4s^2 + 1)$ and $\mathcal{M}_1(t, x, x') = x$, $\mathcal{M}_2(t, x, x') = \dfrac{x}{5 + x}$.

Now, we have

$$|\mathcal{F}(t, x, x', \mathcal{H}x, \mathcal{L}x) - \mathcal{F}(t, y, y', \mathcal{H}y, \mathcal{L}y)|$$

$$\leq \frac{t^3}{5}\left|\frac{\sin 2x(t)}{2} - \frac{\sin 2y(t)}{2}\right| + \frac{t^3}{7}|\sin x'(t) - \sin y'(t)|$$

$$+\frac{t^3}{9}|\mathcal{H}x-\mathcal{H}y|+\frac{t^3}{7}|\mathcal{L}x-\mathcal{L}y|$$

$$\leq \frac{t^3}{5}|x-y|+\frac{t^3}{7}|x'-y'|+\frac{t^3}{9}|\mathcal{H}x-\mathcal{H}y|+\frac{t^3}{7}|\mathcal{L}x-\mathcal{L}y|$$

$$\leq \frac{t^3}{5}[|x-y|+|x'-y'|+|\mathcal{H}x-\mathcal{H}y|+|\mathcal{L}x-\mathcal{L}y|]$$

$$\leq p(t)[|x-y|+|x'-y'|+|\mathcal{H}x-\mathcal{H}y|+|\mathcal{L}x-\mathcal{L}y|],$$

where $p(t)=\frac{t^3}{5}$.

Also, we get

$$|\mathcal{M}_1(t, x, x')-\mathcal{M}_1(t, y, y')|\leq |x-y|$$

and

$$|\mathcal{M}_2(t, x, x')-\mathcal{M}_2(t, y, y')|\leq \left|\frac{x}{5+x}-\frac{y}{5+y}\right|\leq \frac{1}{5}|x-y|.$$

This gives

$$q_1(x)=1 \quad \text{and} \quad q_2(x)=\frac{1}{5}.$$

Therefore, we have

$$\mathcal{K}_1(t, s)||q_1|=\left|\frac{t^2}{4+e^s}\right||1|\leq \frac{1}{5}=\mathcal{K}_1^*$$

and

$$\mathcal{K}_2(t, s)||q_2|=|t(4s^2+1)|\left|\frac{1}{5}\right|\leq \frac{5}{5}=1=\mathcal{K}_2^*.$$

Further, we have

$$P=\sup\{p(t): 0\leq t\leq 0\}=\sup\left\{\frac{t^3}{5}: 0\leq t\leq 0\right\}=\frac{1}{5},$$

$$\alpha = \mathcal{K}_1^*(b - a) = \frac{1}{5}(1 - 0) = \frac{1}{5},$$

$$\beta = \mathcal{K}_2^*(b - a) = 1(1 - 0) = 1,$$

$$N = \sum_{j=0}^{1} \frac{(b-a)^{n-j-1}}{(n-j-1)!} = \sum_{j=0}^{1} \frac{1}{(2-j-1)!} = \frac{1}{(2-0-1)!} + \frac{1}{(2-1-1)!} = 2.$$

Therefore,

$$NP(1 + \alpha + \beta)(b - a) = 2\left(\frac{1}{5}\right)\left[1 + \frac{1}{5} + 1\right](1 - 0) = \frac{22}{25} = 0.88 < 1.$$

6.1. Existence and uniqueness of solutions

Now, we define the operator $T : B \to B$ by

$$(Tx)(t) = \frac{t}{2} + \int_0^t (t - s)\mathcal{F}(s, x(s), x'(s), (\mathcal{H}x)(s), (\mathcal{L}x)(s))ds. \tag{70}$$

From the above discussion, it follows that the operator T satisfies all the conditions of Theorem 2. Hence, the sequence $\{x_k\}$ associated with the normal S-iterative method (7) for the operator T in (70) converges to a unique solution x of IVP (68)-(69) in B.

6.2. Error estimate

Further for any $x_0 \in B$, we have

$$\begin{aligned} \| x_{k+1} - x \|_B &\le \frac{[NP(1 + \alpha + \beta)(b - a)]^{k+1}}{e^{[1 - NP(1+\alpha+\beta)(b-a)]\sum_{i=0}^{k} \xi_i}} \| x_0 - x \|_B \\ &\le \frac{[0.88]^{k+1}}{e^{(0.12)\sum_{i=0}^{k} \xi_i}} \| x_0 - x \|_B, \end{aligned} \tag{71}$$

where, we have chosen $\xi_i = \frac{1}{1+i} \in [0, 1]$. The estimate in (71) is called the *bound for the error* (due to truncation of computation at the kth iteration).

6.3. Continuous dependence

Indeed, for $\alpha_0 = 0$, $\alpha_1 = \frac{1}{2}$ and $\beta_0 = \frac{1}{2}$, $\beta_1 = 1$, we have

$$\| x - \bar{x} \|_B \leq \frac{3M}{[1 - NP(1 + \alpha + \beta)(b - a)]},$$

where

$$\begin{aligned} M &= \sum_{j=0}^{n-1}\left(\sum_{i=j}^{n-1} | \alpha_i - \beta_i | \frac{(b-a)^{i-j}}{(i-j)!}\right) \\ &= | \alpha_0 - \beta_0 | \left(\frac{1}{0!}\right) + | \alpha_1 - \beta_1 | \left(\frac{1}{1!}\right) + | \alpha_1 - \beta_1 | \left(\frac{1}{0!}\right) \\ &= \left| 0 - \frac{1}{2} \right| + \left| \frac{1}{2} - 1 \right| + \left| \frac{1}{2} - 1 \right| \\ &= \frac{3}{2}. \end{aligned}$$

Therefore,

$$\| x - \bar{x} \|_B \leq \frac{3\left(\frac{3}{2}\right)}{1 - \frac{22}{25}} = \frac{4.5}{0.12} = 37.5.$$

6.4. Closeness of solutions

Next, we consider the perturbed equation:

$$\begin{aligned} \bar{x}''(t) &= \frac{t^3}{5}[\sin \bar{x}(t) \cos \bar{x}(t)] - \frac{t^3}{7} \sin \bar{x}'(t) + \frac{t^3}{9} \int_0^t \left(\frac{t^2}{4 + e^s}\right) \bar{x}(s) ds \\ &\quad + \frac{t^3}{7} \int_0^1 t(4s^2 + 1) \frac{\bar{x}(s)}{5 + \bar{x}(s)} ds - \frac{t^3}{10} \end{aligned} \tag{72}$$

with the given initial conditions

$$\bar{x}(0) = \frac{1}{2}, \quad \bar{x}'(0) = 1. \tag{73}$$

Similarly, comparing with equation (40), we have

$$\mathcal{G}(s,\ \bar{x}(s),\ \bar{x}'(s),\ (\mathcal{H}\bar{x})(s),\ (\mathcal{L}\bar{x})(s))$$

$$= \frac{t^3}{5}[\sin \bar{x}(t) \cos \bar{x}(t)] - \frac{t^3}{7}\sin \bar{x}'(t) + \frac{t^3}{9}\int_0^t \left(\frac{t^2}{4+e^s}\right)\bar{x}(s)ds$$

$$+ \frac{t^3}{7}\int_0^1 t(4s^2+1)\frac{\bar{x}(s)}{5+\bar{x}(s)}ds - \frac{t^3}{10}. \tag{74}$$

Now, we define the operator $\overline{T} : B \to B$ by

$$(\overline{T}\,\bar{x})(t) = t + \frac{1}{2} + \int_0^t (t-s)\mathcal{G}(s,\ \bar{x}(s),\ \bar{x}'(s),\ (\mathcal{H}\bar{x})(s),\ (\mathcal{L}\bar{x})(s))ds. \tag{75}$$

It is easy to see that the perturbed equation (72) satisfies all the conditions of Theorem 2. Hence, the sequence $\{\bar{x}_k\}$ associated with the iterative method (7) for the operator (75) converges to a unique solution $\bar{x} \in B.$ Now, we have the following estimate:

$$|\ \mathcal{F}(t,\ x(t),\ x'(t),\ (\mathcal{H}x)(t),\ (\mathcal{L}x)(t)) - \mathcal{G}(t,\ x(t),\ x'(t),\ (\mathcal{H}x)(t),\ (\mathcal{L}x)(t))\ |$$

$$\leq \frac{t^3}{5}\left|\frac{1}{2}\right| = \frac{t^3}{5}\frac{1}{2}. \tag{76}$$

Therefore, $\varepsilon = \frac{1}{2}.$

Consider the sequences $\{x_k\}_{k=0}^{\infty}$ with $x_k \to x$ as $k \to \infty$, and $\{\bar{x}_k\}_{k=0}^{\infty}$ with $\bar{x}_k \to \bar{x}$ as $k \to \infty$ generated by iterative method (7) associated to operators T in (70) and $\overline{T}$ in (75), respectively, with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}.$ Then we have from Theorem 4 that for $\beta_0 = \frac{1}{2},\ \beta_1 = 1,\ \varepsilon = 0.5$ with $\bar{p}(t) = p(t) = \frac{t^3}{5}.$

Therefore, $\overline{P} = \max\{\bar{p}(t) : 0 \leq t \leq 1\} = \frac{1}{5}.$

Thus,

$$\| x - \bar{x} \|_B \leq \frac{3[M + N\bar{P}\varepsilon(b-a)]}{[1 - NP(1 + \alpha + \beta)(b-a)]}$$

$$= \frac{3\left[\frac{3}{2} + 2\left(\frac{1}{5}\right)\left(\frac{1}{2}\right)(1-0)\right]}{[1 - 0.88]} = \frac{5.1}{0.12} = 42.5. \tag{77}$$

This shows that the closeness of solutions depends on involved functions.

6.5. Dependence on parameters

Finally, we prove the dependency of solutions on real parameters. Consider the following integrodifferential equations:

$$x''(t) = \frac{t^3}{5}[\sin x(t) \cos x(t)] - \frac{t^3}{7} \sin x'(t) + \frac{t^3}{9} \int_0^t \left(\frac{t^2}{4 + e^s} \right) x(s) ds$$

$$+ \frac{t^3}{7} \int_0^1 t(4s^2 + 1) \frac{x(s)}{5 + x(s)} ds + \mu_1, \tag{78}$$

where $t \in I = [0, 1]$, with initial conditions

$$x(0) = 0, \quad x'(0) = \frac{1}{2} \tag{79}$$

and

$$\bar{x}''(t) = \frac{t^3}{5}[\sin \bar{x}(t) \cos \bar{x}(t)] - \frac{t^3}{7} \sin \bar{x}'(t) + \frac{t^3}{9} \int_0^t \left(\frac{t^2}{4 + e^s} \right) \bar{x}(s) ds$$

$$+ \frac{t^3}{7} \int_0^1 t(4s^2 + 1) \frac{\bar{x}(s)}{5 + \bar{x}(s)} ds + \mu_2, \tag{80}$$

where $t \in I = [0, 1]$, with initial conditions

$$\bar{x}(0) = \frac{1}{2}, \quad \bar{x}'(0) = 1. \tag{81}$$

Following the above discussion, we have $p(t) = \bar{p}(t) = r(t) = \frac{t^3}{5}$. Therefore, $\bar{R} = \max\{\bar{p}(t) : 0 \le t \le 1\} = \frac{1}{5}$. Hence, by making similar arguments and from Theorem 5,

$$\| x - \bar{x} \|_B \le \frac{3[N\bar{R} \,|\, \mu_1 - \mu_2 \,|\, (b - a)]}{[1 - NP(1 + \alpha + \beta)(b - a)]} = \frac{3\left[2\left(\frac{1}{5}\right) |\, \mu_1 - \mu_2 \,|\, (1 - 0)\right]}{(1 - 0.88)}.$$

In particular, for $\mu_1 = 1, \mu_2 = \frac{1}{2}$, the above inequality becomes

$$\| x - \bar{x} \|_B \le \frac{3\left[2\left(\frac{1}{5}\right)\left|1 - \frac{1}{2}\right|\right]}{0.12} = 5.$$

This proves the dependence on both initial data and real parameters.

7. Conclusion

We established the existence and uniqueness of the solution to the IVP (1)-(2) by the *S*-iteration method. Further, we discussed various properties of solutions such as continuous dependence on the initial data, closeness of solutions, and dependence on parameters and functions involved therein. Finally, we provided examples in support of our results.

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