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NUMERICAL APPROXIMATION OF THE FINAL STATE OF AN INCOMPLETE DATA HEAT PROBLEM

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Abstract

We determine the state at an instant T_0, of a 2D heat problem whose initial condition is partially known on a part of the domain. We use a non-standard method to solve this problem numerically.

0. Introduction

The heat problem can translate the evolution of the temperature in an open domain Ω of $\mathbb{R}^n$, $n = 1, 2, 3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) = f & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \partial\Omega \times]0, T[, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where A is the Laplacian operator, f is a source function, u_0 is the initial condition and $T > 0$.

The problem (1) is said to be *well-posed* (in the sense of Hadamard) if it admits a unique solution depending continuously on the data [9]. In practice, most often we are confronted with the situation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) = f & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \partial\Omega \times]0, T[, \end{cases} \tag{2}$$

where the condition u_0 is partially known.

The evolution of the temperature cannot be studied from the problem (2). We need to know completely u at a time $T_0 \in]0, T[$. We assume to have observations u_{obs} of the state of the system such that

$$u = u_{obs} \text{ in } \omega \times [0, T_0], \tag{3}$$

where $\omega \subseteq \Omega$ is nonempty.

Can we determine $u(T_0)$ in Ω? We provide an answer to this historical question. Classical methods of variational data assimilation can be used to find a solution [3-7]. These methods are based on the resolution of an optimization problem through the adjoint model of the problem. This procedure requires a regularization of the functional by a priori information. These are not accessible in practice [1, 10, 15]. There are other so-called non-standard assimilation methods that circumvent this difficulty. Nudging is a method of data assimilation based on dynamic relaxation, with the aim of fitting the model and constraining it towards the observations [14]. Optimal interpolation (OI) is a system of equations where the fundamental assumption is that for each variable of the model at each grid point, a reduced number of observations is taken into account to perform the analysis [8, 11, 13].

We present a non-standard method to easily generate a numerical approximation of $u(T_0)$. This method has two advantages, that of covering $u(T_0)$ in Ω and that of obtaining a good approximation of the state of the system (2) better than the observations u_{obs}. Indeed, the latter are marred by measurement errors. We were able to implement the method in 1D dimension [2], in this work, we intend to extend it in higher dimension.

1. Problem Formulation

Consider the heat problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x, y, t) & \text{in } Q = \Omega \times]0, T_0[, \\ u = 0 & \text{on } \partial\Omega \times]0, T_0[, \end{cases} \tag{4}$$

where Ω is a regular open set of $\mathbb{R}^2$ and $T_0 > 0$, $f \in L^2(0, T_0; L^2(\Omega))$.

Let $\omega \subseteq \Omega$ be a nonempty open set and u_{obs} an observation function defined in $\omega \times [0, T_0]$. We consider $u(x, y, t)$ solution of the system (4) satisfying the relation

$$u = u_{obs} \text{ in } \omega \times [0, T_0]. \tag{5}$$

We propose to identify $u(T_0)$ in Ω.

Multiplying the heat equation of the system (4) by a sufficiently regular function φ and integrating over Q, we have

$$\int_0^{T_0} \int_\Omega u\left(-\frac{\partial \varphi}{\partial t} - \Delta\varphi\right) + \int_\Omega u(T_0)\varphi(T_0) - \int_\Omega u(0)\varphi(0)$$

$$- \int_0^{T_0} [\nabla u. \vec{n}\, \varphi]_{\partial\Omega} + \int_0^{T_0} [u \nabla \varphi. \vec{n}]_{\partial\Omega}$$

$$= \int_0^{T_0} \int_\Omega f\varphi. \tag{6}$$

Given $\hat{\varphi} \in L^2(\Omega)$, consider φ solution of the controlled adjoint problem

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} - \Delta\varphi = \chi_\omega, v & \text{in } Q = \Omega \times (0, T_0), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T_0), \\ \varphi(T_0) = \hat{\varphi} & \text{in } \Omega, \end{cases} \tag{7}$$

where χ_ω is a characteristic function and $v \in L^2(0, T_0; L^2(\omega))$ represents the control, then equation (6) becomes

$$\int_0^{T_0} \int_\Omega u\chi_\omega \cdot v + \int_\Omega u(T_0)\hat{\varphi} - \int_\Omega u(0)\varphi(0) = \int_0^{T_0} \int_\Omega f\varphi. \tag{8}$$

Since $u(0)$ is not known, we can look for solution φ of the controlled problem (7) such that

$$\varphi(0) = 0 \text{ in } \Omega. \tag{9}$$

Taking this condition, we obtain the following integral equation:

$$\int_\Omega u(T_0)\hat{\varphi} = \int_0^{T_0} \int_\Omega f\varphi - \int_0^{T_0} \int_\omega u_{obs} v. \tag{10}$$

Solving the problem (7) for zero controllability amounts solving the minimization problem

$$J_\beta(v_\beta) = \min_{v \in L^2(0,\, T_0;\, L^2(\omega))} J_\beta(v), \quad \beta > 0 \tag{11}$$

under the constraint (7), where

$$J_\beta(v) = \frac{1}{2}\int_0^{T_0}\int_\omega |\, v\,|^2 + \frac{1}{2\beta}\int_\Omega \varphi_v^2(0). \tag{12}$$

Thus φ_v is the solution of the problem (7) for the control v and the final state $\hat{\varphi}$ is fixed.

Characterization

For all $\beta > 0$, there is a unique solution $v_\beta \in L^2(0, T; L^2(\omega))$ of the problem (11) which is characterized by the following optimality problem: There exists $p_\beta \in C^1(0, T; H_0^1(\Omega))$ such that

$$-\frac{\partial \varphi_\beta}{\partial t} - \Delta\varphi_\beta = \chi_\omega \cdot v_\beta \text{ in } Q = \Omega \times (0, T_0), \tag{13a}$$

$$\varphi_\beta = 0 \text{ on } \partial\Omega \times (0, T_0), \tag{13b}$$

$$\varphi_\beta(T_0) = \hat{\varphi} \text{ in } \Omega, \tag{13c}$$

$$\frac{\partial p_\beta}{\partial t} - \Delta p_\beta = 0 \text{ in } \Omega \times (0, T_0), \tag{13d}$$

$$p_\beta = 0 \text{ on } \partial\Omega \times (0, T_0), \tag{13e}$$

$$p_\beta(0) = -\frac{1}{\beta}\varphi_\beta(0) \text{ in } \Omega, \tag{13f}$$

$$p_\beta - v_\beta = 0 \text{ in } \omega \times (0, T_0). \tag{13g}$$

Indeed, we have the following result:

Theorem 1.1 [12]**.** *The sequences* (v_β) *and* (φ_β) *converge*, *respectively*, *in* $L^2(0, T_0; L^2(\omega))$ *and in* $L^2(0, T_0; H_0^1(\Omega))$ *when* $\beta \to 0$. *More precisely*,

$$v_\beta \to \overline{v} \text{ in } L^2(0, T_0; L^2(\omega))$$

and

$$\varphi_\beta \to \overline{\varphi} \text{ in } L^2(0, T; H_0^1(\Omega)).$$

In addition, *we have*

$$\begin{cases} -\dfrac{\partial\overline{\varphi}}{\partial t} - \Delta\overline{\varphi} = \chi_\omega \cdot \overline{v} & \text{in } Q = \Omega \times (0, T_0), \\ \overline{\varphi} = 0 & \text{on } \partial\Omega \times (0, T_0), \\ \overline{\varphi}(T_0) = \hat{\varphi} & \text{in } \Omega \end{cases} \tag{14}$$

and

$$\overline{\varphi}(0) = 0 \text{ in } \Omega. \tag{15}$$

2. Numerical Experiments

The discretization in space is based on the pseudo spectral method with Chebyshev collocation points.

The solution is well known on the boundary of Ω. However, we are interested in the interior points. Let $\Omega = (-1, 1) \times (-1, 1)$. Consider $(x_i, y_i)_{1 \leq i \leq (N-1)^2}$ the interior points of the grid of domain Ω and let $\hat{\varphi} \in L^2(\Omega)$. Let $\mu_i = \hat{\varphi}(x_i, y_i)$, and for $k = 1, ..., (N-1)^2$, we introduce the following functions:

$$\hat{\varphi}^{(k)} : [-1, 1] \times [-1, 1] \to \mathbb{R}$$

such that

$$\hat{\varphi}^{(k)}(x_i, y_i) = \begin{cases} \mu_k & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

For every k, there is a control v^k such that the corresponding solution $\varphi^{(k)}$ of the problem (7), where v is replaced by $v^{(k)}$ and $\hat{\varphi}$ by $\hat{\varphi}^{(k)}$, satisfies

$$\varphi^{(k)}(0) = 0.$$

Thus, for all $k = 1, ..., (N-1)^2$, the relation (10) becomes

$$\int_{\Omega} u(T_0)\hat{\varphi}^{(k)} = \int_0^{T_0}\int_{\Omega} f\varphi^{(k)} - \int_0^{T_0}\int_{\omega} u_{obs}v^{(k)}. \tag{17}$$

Consider $(t_j)_{j=0,...,M}$ a regular subdivision of $(0, T_0)$ such that $\Delta t = \frac{T_0}{M}$, $M \in \mathbb{N}^*$. Performing a numerical integration, for $k = 1, ..., (N-1)^2$, equation (17) implies

$$\begin{aligned}&\Delta x_k \Delta y_k u(T_0)(x_k, y_k)\mu_k \\ &= \Delta t \sum_{j=0}^{M}\sum_{i=1}^{N-1} \Delta x_i \Delta y_i f(x_i, y_i, t_j)\varphi^{(k)}(x_i, y_i, t_j) \\ &- \Delta t \sum_{j=0}^{M}\sum_{i=1}^{N-1} \Delta x_i \Delta y_i u_{obs}(x_i, y_i, t_j)v^{(k)}(x_i, y_i, t_j). \end{aligned} \tag{18}$$

Taking $\hat{\varphi}$ such that $\hat{\varphi}^{(k)}(x_k, y_k) \neq 0$ for all $k = 1, ..., (N-1)^2$, the system of equation (18) gives an approximation of the final state $u(T_0)$ at the collocation points $(x_i, y_i)_{1 \leq i \leq (N-1)^2}$. However, determining $u(T_0)$ requires computing $v^{(k)}$ and $\varphi^{(k)}$ such that $\varphi^{(k)}(0) = 0$.

Let

$$\psi^{(k), j} = (\varphi^{(k)}(x_1, y_1, t_j), \varphi^{(k)}(x_2, y_2, t_j), ..., \varphi^{(k)}(x_{(N-1)^2}, y_{(N-1)^2}, t_j))^T$$

be the solution vector of the numerical scheme of the system (7).

$$\begin{cases} -\dfrac{\psi^{(k),\,j+1} - \psi^{(k),\,j}}{\Delta t} - D\psi^{(k),\,j} = V^{(k),\,j}, \quad j = M-1, \ldots, 0, \\ \psi^{(k),\,M} = \hat{\psi}^{(k)}, \end{cases} \tag{19}$$

where D is the pseudo-spectral differentiation matrix associated with the Laplacian operator Δ.

$$V^{(k),\,j} = (v^{(k)}(x_1, y_1, t_j), v^{(k)}(x_2, y_2, t_j), \ldots, v^{(k)}(x_{(N-1)^2}, y_{(N-1)^2}, t_j))^T \tag{20}$$

is the control vector at time t_j, $j = 0, 1, \ldots, M$, where $v^{(k)}(x_m, y_m, t_j) = 0$ if $(x_m, y_m) \in \Omega \backslash \omega$.

The system (19) can be rewritten in the form

$$\begin{cases} \psi^{(k),\,M} = \hat{\varphi}^{(k)}, \\ G\psi^{(k),\,j} = \psi^{(k),\,j+1} + \Delta t V^{(k),\,j}, \; j = M-1, \ldots, 0, \end{cases} \tag{21}$$

where $G = I_{(N-1)^2} - \Delta t D$.

Solving (21) requires knowledge of $V^{(k),\,j}$.

Determine an approximate solution of the problem (13d)-(13f) by solving the numerical scheme

$$\begin{cases} P_\beta^{(k),\,0} = -\dfrac{1}{\beta}\varphi_\beta^{(k)}(0), \\ GP_\beta^{(k),\,j+1} = P_\beta^{(k),\,j}, \quad j = 0, \ldots, M-1, \end{cases} \tag{22}$$

where $\varphi\beta^{(k)}(0)$ is chosen such that its norm is close to zero and β is as close to zero as possible.

Let Λ be a diagonal matrix such that

$$\Lambda_{ii} = \begin{cases} 1 & \text{if } (x_i, y_i) \in \omega, \\ 0 & \text{else if}. \end{cases} \tag{23}$$

Thus, the control vector is generated by

$$V^{(k),\,j} = \Lambda P^{(k),\,j}, \quad j = 1, \ldots, M. \tag{24}$$

For $\psi^{(k),0} = 0,$ it suffices that

$$\psi^{(k),1} + \Delta t V^{(k),0} = 0. \tag{25}$$

By fixing the parameters *N* and *M*, we can generate an approximation of the state $u(T_0)$ at the collocation points of Ω by the following algorithm:

---

(1) *that is* β *quite small*

(2) *For* $k = 1, \ldots, (N-1)^2$

- *Give* μ_k *nonzero real*
- *give vector* $\varphi_\beta^{(k)}$ *such as* $\frac{\| \varphi_\beta^{(k)} \|}{\beta}$ *is bounded by a constant*
- *Calculate* $p_\beta^{(k)}$ *system fix* (22)
- *Generate* $V^{(k),\,j} = \Lambda P^{(k),\,j},\ j = 1, \ldots, M$ (*formula* (24))
- *Calculate* $\psi^{(k),\,j},\ j = M, \ldots, 1$ (*formula* (21))
- *Determine* $V^{(k),0} = -\frac{1}{\Delta t}\psi^{(k),1}$ *from equation* (25)

(3) *Calculate* $u(T_0)(x_k,\ y_k)$ *from* (18).

End

For the validation of the numerical scheme, we consider the problem (4) with $f(x,\ y,\ t) = \lambda(\pi^2 - \alpha)e^{-\alpha t}\sin(\pi x)\sin(\pi y),\quad \Omega = (-1,\ 1)\times(-1,\ 1)$ and $T = \frac{1}{10},\ \mu_k = 10 - e^{-k}$. The observation function u_{obs} is obtained by the twin experiment method. It is generated from a random perturbation of the values of the exact solution.

When $\omega = \Omega$ and $T_0 = T$, the method makes it possible to best approach the exact solution instead of being satisfied with observation measurements which are generally marred by errors.

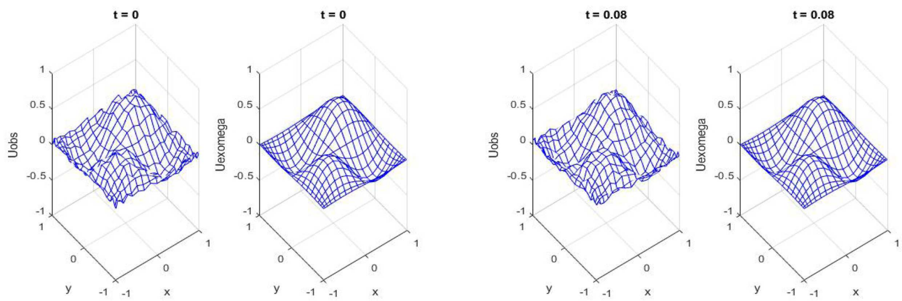

Figure 1. Observations Uobs and exact state Uexomega on Ω at time $t = 0$ (on the left), at time $t = 0.08$ (on the right).

We evaluate the error between the exact state and the observations as a function of time $Er = \| U_{ex}(t_j) - U_{obs}(t_j) \|_\infty$, $j = 0, \ldots, M$. The behavior of this curve (2) reflects the randomness used to generate the observations.

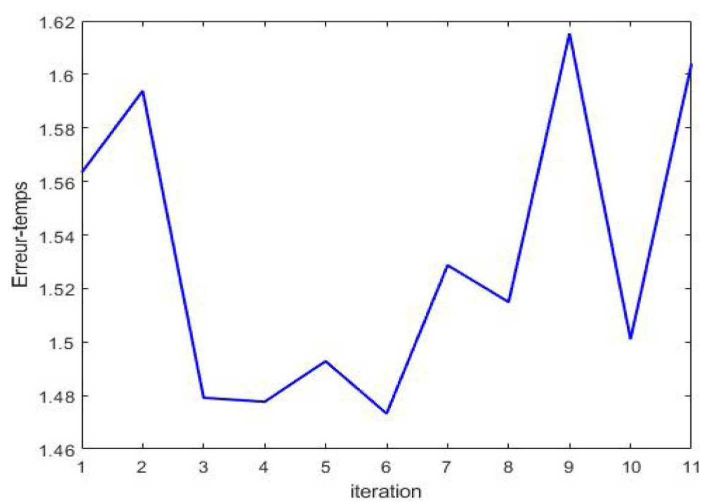

Figure 2. Maximum deviation between system state and observations.

Before carrying out the determination of an approximation of $u(T_0)$ on a domain ω fixed, we study the variation of the errors according to ω.

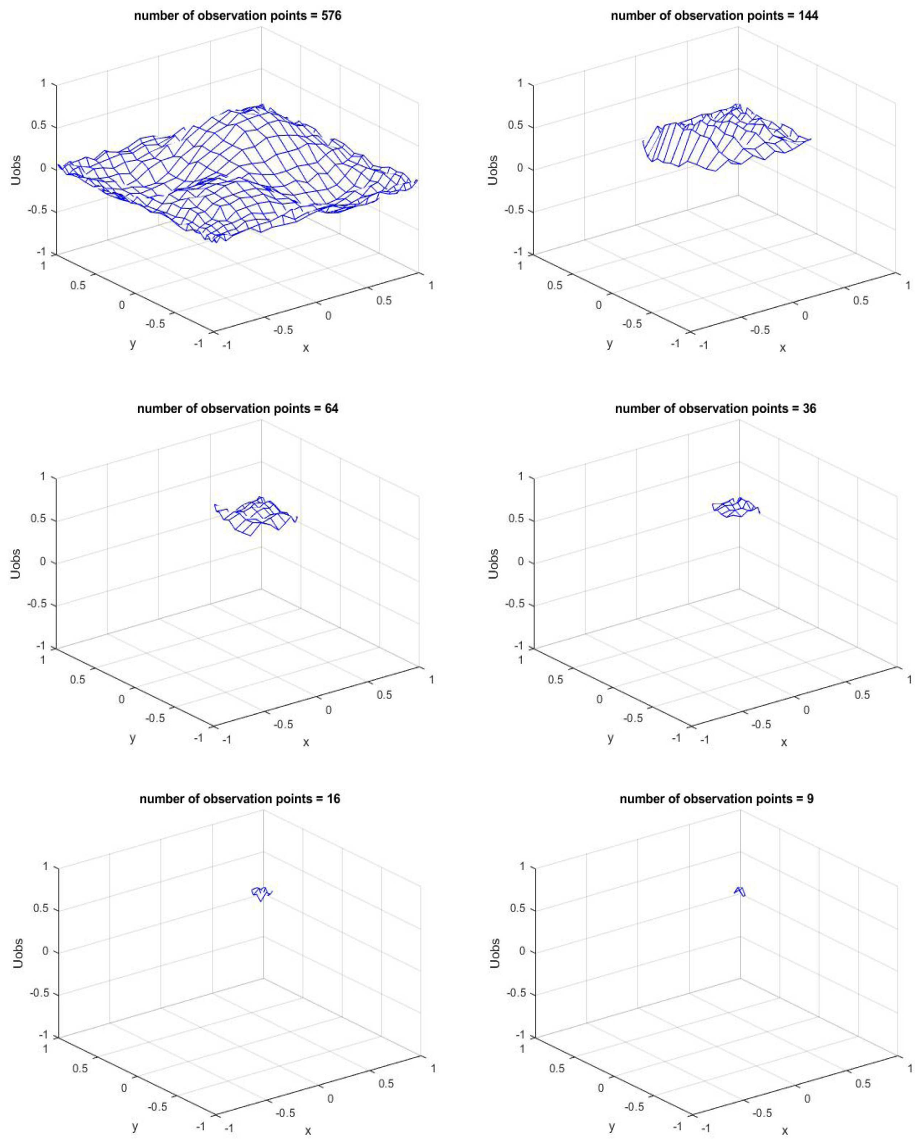

Figure 3. A variation of the domain of observations ω.

Error in infinite norm, at time T_0, on a variation of ω, is given by

$$Er_{ExactObs} = \| U_{ex}(T_0) - U_{obs}(T_0) \|_{\infty},$$

$$Er_{ExactAppGo} = \left\| U_{ex}(T_0) - U_{app}(T_0) \right\|_{\infty}^{\Omega},$$

$$Er_{ExactAppPo} = \left\| U_{ex}(T_0) - U_{app}(T_0) \right\|_{\infty}^{\omega}.$$

Figure 4 represents the error curves by varying the domain of observations ω with a constant discretization of the domain Ω. This figure shows us a clear difference between the approximate state and the observations compared to the exact state of the system on ω. Indeed, the observation error curve increases according to the number of discretization points on ω contrary to that of error relating to the approximated state on the same domain. The error curve of the approximate state on Ω is globally decreasing and tends towards zero. This representation can guide our choice of the number of observation points necessary to have an excellent approximation of the state of the system on Ω.

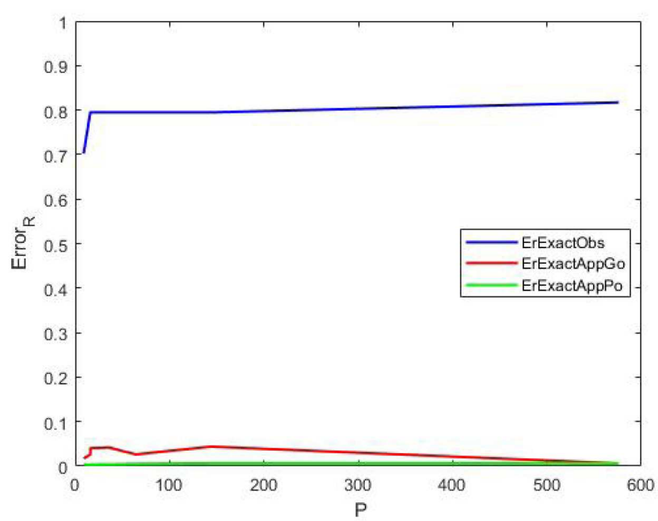

Figure 4. Error in infinite norm, at time T_0, on a variation of ω.

To determine an approximation of the state of the system at time T_0 on Ω, we set the following parameters:

$$N = 25, \quad M = 10, \quad \lambda = 1/\pi, \quad \alpha = .01, \quad \beta = 10^{-24}.$$

Choice of observation domain ω: On this domain, there are 9 data collection points.

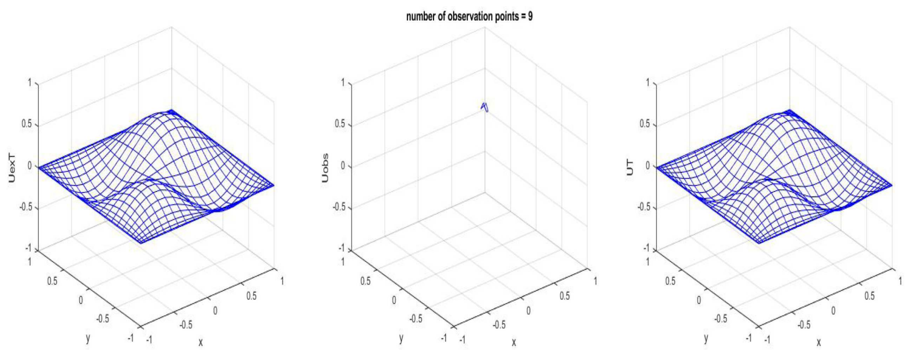

Figure 5. At time T_0: the exact state (left), observation (middle), and approximation of the state (right).

3. Conclusion and Perspectives

We are interested in this article in the approximation of the state of a heat problem with incomplete data in time T_0. Among several methods that exist, we have opted the non-standard methods for our resolution.

The discretization in space is based on the Chebyshev pseudo-spectral method and the discretization in time being taken like regular. The numerical tests carried out show a good stability of our numerical scheme. This method allowed us to obtain an excellent approximation of $u(T_0)$ without going through an optimization process that requires a priori information. We can improve the numerical scheme by trying to use the best tools in terms of numerical scheme of PDEs and numerical integration. We can also extend the method to nonlinear PDEs or to Ω domains with complex geometry.

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