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# ON REALIZATION OF THE SUPERPOSITION PRINCIPLE FOR A FINITE BUNDLE OF INTEGRAL CURVES OF A SECOND-ORDER BILINEAR DIFFERENTIAL SYSTEM

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## Abstract

We investigate the solvability of the problem of realization of operator-functions of invariant linear regulator (*IL*-regulator) of a second order nonstationary differential system (*D*-system), which allows for a finite bundle of integral curves of "*trajectory*, *control*"

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type, induced in this *D*-system by different bilinear regulators, to reduce this bundle to a subfamily of admissible solutions of this *D*-system through action of *IL*-regulator. The problem under consideration belongs to the type of nonstationary coefficient-operator inverse problems for evolution equations (including the hyperbolic) in separable Hilbert space. The problem is solved on the basis of a qualitative study of the continuity and semiadditivity properties of the nonlinear Rayleigh-Ritz functional operator. The obtained results have applications in the theory of nonlinear infinite-dimensional adaptive dynamical systems for a class of bilinear differential models of higher orders.

## 0. Introduction

The subject of this paper is to elaborate a mathematical language to express one of the most universal natural-science ideas - the idea of *linearity*. For this idea, its most important (central) position is the *principle of linearity of small increments*, when essentially any mathematical operation in small increments is almost always linear. This principle underlies all mathematical analysis and its numerous applications. In addition, after Einstein, it became clear that the surrounding physical space is approximately linear only in the small vicinity of the observer. Fortunately, this small vicinity is quite large, so the 20th century physics significantly expanded the scope of the idea of linearity by adding to the principle of linearity of small increments the methodology of the *superposition principle* postulating the position when the functional dependence of output quantities (trajectories) on input influences (controls) is essentially *linear*.

In this context, the main goal of this paper is to advance the superposition principle into the domain of *bilinear* controlled dynamical processes; the *bilinearity* of many of the most important second-order differential systems encountered in mathematical physics is an immutable fact. Therefore, the proposed work continues the qualitative research [1-4] with the main goal of constructing existential proofs in solvability of the problem of realization (in separable Hilbert space) of operator coefficients of the *invariant linear* regulator (*IL*-regulator) of the second-order nonautonomous differential system; however, its results, obviously, are applicable to stationary models [5-7]. The functional concept of the *IL*-regulator implies: (i) the property of linearity of this regulator, (ii) the differential system (*D*-system) under study (simulated *a posteriori*) must contain in the class of admissible solutions a finite family of controllable trajectory curves of the "*trajectory*, *control*" type, with each such integral curve induced by a bilinear regulator (*BL*-regulator), not excluding the case when these regulators are individual (are different).

## 1. Terminology and Problem Statement

The systems $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z_i, \|\cdot\|_{Z_i})$, $i = 1, 2$ are real separable Hilbert spaces (pre-Hilbertian [8] defines the norms $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$), $U := Y \times Z_1 \times Z_2$ - Hilbert space-production with the norm $\|(y, z_1, z_2)\|_U := \left(\|y\|_Y^2 + \sum_{i=1,2} \|z_i\|_Z^2\right)^{1/2}$, $L(Y, X)$ is a Banach space with operator norm $\|\cdot\|_{L(Y, X)}$ of all linear continuous operators acting from $Y$ to $X$ (similarly $(L(X, X), \|\cdot\|_{L(X, X)})$ and $(L(Z_i, X), \|\cdot\|_{L(Z, X)})$), $X^i$ is the $i$th Cartesian degree of space $X$, $\mathbb{L}(X^i, Z_i)$ is the space of all continuous $i$-linear (linear for $i = 1$ and bilinear for $i = 2$) mappings from $X^i$ to $Z_i$; further, we denote by $\mathbb{O}$ the zero operator from $\mathbb{L}(X^2, Z_2)$.

Let $T := [t_0, t_1]$ be a segment of a number line $R$ with Lebesgue measure $\mu$ and $\wp_\mu$ be the $\sigma$-algebra of all $\mu$-dimensional subsets of $T$. If below $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a Banach space, then we denote by $L_p(T, \mu, \mathbf{B})$, $p \in [1, \infty)$ the Banach factor-space of the $\mu$-equivalence classes of all Bochner [8, p. 137] mappings $\xi : T \to \mathbf{B}$ with the norm

$$\|\xi\|_p := \left(\int_T \|\xi(\tau)\|_{\mathbf{B}}^p \mu(d\tau)\right)^{1/p} < \infty.$$

Thus, $L_\infty(T, \mu, \mathbf{B})$ is the space of all (equivalent classes) strongly μ-dimensional μ-substantially bounded functions from $T$ to **B**. In addition, further, $AC^1(T, X)$ is the set of all functions $\zeta : T \to X$, whose first derivative is a perfectly continuous function on $T$ (with respect to μ), moreover, we assume that

$$\mathbf{\Pi} := AC^1(T, X) \times L_2(T, \mu, Y) \times L_2(T, \mu, Z_1) \times L_2(T, \mu, Z_2).$$

We introduce auxiliary constructions associated with the notation system via

$$H_2 := L_2(T, \mu, Y) \times L_2(T, \mu, Z_1) \times L_2(T, \mu, Z_2)$$

denoting the space-production with the topology induced by the norm

$$\| (w_0, w_1, w_2) \|_H$$

$$:= \left( \int_T \| (w_0(\tau), w_1(\tau), w_2(\tau)) \|_U^2 \mu(d\tau) \right)^{1/2}, \quad (w_0, w_1, w_2) \in H_2;$$

by virtue of the construction $\| \cdot \|_H$ and parallelogram identity [8, p. 52, p. 79] $H_2$ - is a Hilbert space.

Further, as the space of nonstationary operator coefficients of bilinear regulators (*BL*-regulators) of the *D*-system under study, consider a Banach space-production

$$\mathbf{L}_2 := L_2(T, \mu, L(Y, X)) \times L_2(T, \mu, L(Z_1, X)) \times L_2(T, \mu, L(Z_2, X))$$

μ-equivalence classes of ordered 3-cortices of operator-functions with norm

$$\| (B_0, B_1, B_2) \|_{\mathbf{L}}$$

$$:= \left( \int_T \left( \| B_0(\tau) \|_{L(Y, X)}^2 + \sum_{i=1,2} \| B_i(\tau) \|_{L(Z, X)}^2 \right) \mu(d\tau) \right)^{1/2}.$$

In addition, we stipulate that the space of operator coefficients of the $D$-system itself specifies

$$\mathbf{D} := L_1(T,\ \mu,\ L(X,\ X)) \times L_1(T,\ \mu,\ L(X,\ X)) \times (L_\infty(T,\ \mu,\ L(X,\ X))\setminus\{0\}).$$

Now, we define *a priori* the initial operator "*proto-model*" of nonstationary coefficients (at the trajectory, and its derivatives) of the modeled $D$-system in the form of triple operator-functions

$$(A_0,\ A_1,\ A_2) \in \mathbf{D},$$

and also set (in the structure of $BL$-regulators of the $D$-system) the 2-vector $i$-linear mappings

$$(\mathbb{B}_1,\ \mathbb{B}_2) \in \mathbb{L}(X^1,\ Z_1) \times \mathbb{L}(X^2,\ Z_2).$$

Below we distinguish $(x,\ u,\ \mathbb{B}_1(x),\ \mathbb{B}_2(x,\ x)) \in \mathbf{\Pi}$ as an equivalence class (mod μ) from a representative of this class, namely the vector-function $t \mapsto (x(t),\ u(t),\ \mathbb{B}_1(x(t)),\ \mathbb{B}_2(x(t),\ x(t)))$. It is clear that

$$\{\mathbb{B}_1(x) : (x,\ u,\ \mathbb{B}_1(x),\ \mathbb{B}_2(x,\ x)) \in \mathbf{\Pi}\} \subset L_\infty(T,\ \mu,\ Z_1),$$

$$\{\mathbb{B}_2(x,\ x) : (x,\ u,\ \mathbb{B}_1(x),\ \mathbb{B}_2(x,\ x)) \in \mathbf{\Pi}\} \subset L_\infty(T,\ \mu,\ Z_2).$$

Next, we fix (possibly *a posteriori*) a finite "$k$-family" of dynamical processes:

$$N_1 = (x_1,\ u_1,\ \mathbb{B}_1(x_1),\ ...,\ \mathbb{B}_2(x_1,\ x_1)) \in \mathbf{\Pi} : (x_1,\ u_1) \in AC^1(T,\ X) \times L_2(T,\ \mu,\ Y),$$

$$\ldots$$

$$N_k = (x_k,\ u_k,\ \mathbb{B}_1(x_k),\ ...,\ \mathbb{B}_2(x_k,\ x_k)) \in \mathbf{\Pi} : (x_k,\ u_k) \in AC^1(T,\ X) \times L_2(T,\ \mu,\ Y), \tag{1}$$

as variants of behavior of the investigated $D$-system with trajectories $x$, program control $u$ and feedbacks $x \mapsto \mathbb{B}_1(x),\ x \mapsto \mathbb{B}_2(x,\ x)$ ($L$-form, $BL$-form), with $N_i \neq N_j,\ i \neq j$. We assume that the dynamic processes $N_j$,

$j = 1, ..., k$ are induced by solutions of the same second-order differential system, but possibly with different *BL*-regulators for these processes (right-hand sides of equalities of system (2)):

$$\exists(B_{01}, B_{11}, B_{21}) \in \mathbf{L}_2:$$

$$A_2 d^2 x_1/dt^2 + A_1 dx_1/dt + A_0 x_1 = B_{01}u_1 + B_{11}\mathbb{B}_1(x_1) + B_{21}\mathbb{B}_2(x_1, x_1),$$

$$(x_1, u_1, \mathbb{B}_1(x_1), \mathbb{B}_2(x_1, x_1)) = N_1;$$

$$\ldots$$

$$\exists(B_{0k}, B_{1k}, B_{2k}) \in \mathbf{L}_2:$$

$$A_2 d^2 x_k/dt^2 + A_1 dx_k/dt + A_0 x_k = B_{0k}u_k + B_{1k}\mathbb{B}_1(x_k) + B_{2k}\mathbb{B}_2(x_k, x_k),$$

$$(x_k, u_k, \mathbb{B}_1(x_k), \mathbb{B}_2(x_k, x_k)) = N_k; \tag{2}$$

while $N_i \neq N_j, i \neq j,$ where the violation of the condition $(B_{0i}, B_{1i}, B_{2i}) = (B_{0j}, B_{1j}, B_{2j}),$ $i \neq j,$ is unencumbered (admissible). In the analytic construction of the *x*-solution, we follow [9, p. 418]; i.e., the equalities in the differential equations of system (2) are treated as identities in $L_1(T, \mu, X)$.

***Let us consider the problem*:** for a unified bundle $N_+ := \bigcup_{j=1,\ldots,k} N_j$, determine the conditions of existence of an ordered system of operator-functions $(B_0^+, B_1^+, B_2^+) \in \mathbf{L}_2$, (*IL*-regulator) and operator $\mathbb{B}_1^+ \in \mathbb{L}(X^1, Z_1)$, for which the differential realization of a bundle $N_+$ of the form is feasible:

$$A_2 d^2 x/dt^2 + A_1 dx/dt + A_0 x = B_0^+ u + B_1^+ \mathbb{B}_1^+(x) + B_2^+ \mathbb{O}(x, x),$$

$$\forall(x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x)) \in N_+. \tag{3}$$

The solution of the inverse problem (3) leads to a number of theoretical schemes explaining (in operator language) the physical nature of the *IL*-regulator, at the same time, developing a new mathematical intuition in the posterior modeling of hyperbolic systems [5, 10]. In general, the formulation of the problem (3) geometrically can be treated as a synthesis of a general (for nonlinear trajectory processes $N_j,\ j = 1, ..., k$) nonstationary linear vector field [11].

**Remark 1.** There are no structural obstacles to extend the results obtained below to the theory of implementation of an *IL*-regulator which includes polylinear forms (*PL*-forms) from $\mathbb{L}(X^i \times Y, Z_i)$, containing *l*-times $(l \le i)$ trajectory *x* and $(i - l - 1)$-times derivative $dx/dt$, as well as 1-time program control *u*; that is, in this formulation, for any $\mathbb{B}_i \in \mathbb{L}(X^i \times Y, Z_i)$:

$$\mathbb{B}_i(\overbrace{x, ..., x}^{l\text{-times}}, \underbrace{dx/dt, ..., dx/dt, u}_{(i-l-1)\text{-times}}) \in L_2(T, \mu, Z_i);$$

this cannot be said with respect to the structure of the regulator with program-position relations from $\mathbb{L}(X^I \times Y^j, Z_{ij}),\ j \ge 2$, because in this case, i.e., when the operator definition area $\mathbb{B} \in \mathbb{L}(X^i \times Y^j, Z_{ij}),\ \ j \ge 2$ includes the *j*-times the variable *u*, the condition $\mathbb{B}(x, ..., dx/dt, ..., u) \in L_2(T, \mu, Z_{ij})$ may *not be fulfilled*.

If, for system (3), the problem of solvability of realization of *PL*-forms from $\mathbb{L}(X^i \times Y, Z_i),\ i = 1, ..., n$ (see further Remark 5), then the tensor product of Hilbert spaces [3], including the entropic formulation [12], can be the geometric basis of the mathematical apparatus.

## 2. Constructions of Related Mathematical Formalism

Denote by $L(T, \mu, R)$ the space of $\mu$-equivalence classes of all real $\mu$-dimensional functions on *T* and $\le_L$ be a quasi-order in $L(T, \mu, R)$ such, as $\varphi_1 \le_L \varphi_2$, if $\varphi_1(t) \le \varphi_2(t)$ $\mu$-almost everywhere in *T*. We denote by $\sup_L$

the smallest upper bound for a subset $W \subset L(T, \mu, R)$ if this bound exists for a subset $W$ in the partial ordering structure $\leq_L$.

Since $\mathbf{\Pi} = AC^1(T, X) \times H_2$, the vector-function $(q, w_0, w_2) \in \mathbf{\Pi}$ (as a rule) be denoted by $(q, w)$; i.e., in this position, we have

$$q \in AC^1(T, X), \quad w = (w_0, w_2) \in H_2, \quad w \mapsto \| w \|_U \in L_2(T, \mu, R).$$

By virtue of Theorem 2.1 [13] for $q \in AC^1(T, X)$, there is $d^2q/dt^2 \in L_1(T, \mu, X)$, so it is correct.

**Definition 1** [3, 4]**.** Let $(A_0, A_1, A_2) \in \mathbf{D}$. Then a nonlinear operator $\Psi : \mathbf{\Pi} \to L(T, \mu, R)$ of the form

$$\Psi(q, w)(t) := \begin{cases} \| w(t) \|_U^{-1} \| A_2(t) d^2 q(t)/dt^2 + A_1(t) dq(t)/dt + A_0(t) q(t) \|_X, \\ \text{if } w(t) \neq 0 \in U; \\ 0 \in R, \text{ if } w(t) = 0 \in U; \end{cases} \tag{4}$$

is called to be the *Rayleigh-Ritz operator*.

Consider (without reference to the system of nonlinear processes (1)) the dynamic bundle

$$N \subset \{(x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x)) \in \mathbf{\Pi} : (x, u) \in AC^1(T, X) \times L_2(T, \mu, Y)\},$$

$$(\mathbb{B}_1, \mathbb{B}_2) \in \mathbb{L}(X^1, Z_1) \times \mathbb{L}(X^2, Z_2), \quad \text{Card } N \leq \exp \aleph_0,$$

and let $Q$ be an absorbing set in Span $N$; we follow [8] in the geometry of the absorbing set, i.e., $\bigcup \{rQ\}_{r>0} = \text{Span } N$. Fixing the terminology (for motivation, see Theorem 2 [2]), we have

**Definition 2** [2]**.** A dynamic bundle $N$ is *regular* for operator functions $(A_0, A_1, A_2) \in \mathbf{D}$ if and only if the following holds:

$$\operatorname{supp}\| A_2 d^2q/dt^2 + A_1 dq/dt + A_0 q \|_X \subset \operatorname{supp}\| w \|_U \ (\operatorname{mod} \mu),\ (q, w) \in Q.$$

Here (and hereafter) the "supp-bearing" function is defined with exactness to the set of measure zero.

**Proposition 1.** (i) *If the dynamical bundle $N$ is regular, then the contraction $\Psi|\operatorname{Span} N$ has* $($*for any vector-function* $(q, w) \in \operatorname{Span} N)$ *the analytic representation*

$$\Psi(q, w) = \| A_2 d^2q/dt^2 + A_1 dq/dt + A_0 q \|_X (\| w \|_U + \chi_{S_w})^{-1},$$

*where $\chi_{S_w}$ is the set indicator $S_w := T \backslash \operatorname{supp}\| w \|_U$.*

(ii) *In the inverse problem* (3), *the combined dynamic bundle $N_+$ is regular.*

Here are the unencumbered starting points providing point (i) of Proposition 1.

**Proposition 2.** *If* $\ker \mathbb{B}_1 = 0$, *then the bundle $N$ will be regular for all* $(A_0, A_1, A_2) \in \mathbf{D}$.

**Proof.** It is clear that the position of $\ker \mathbb{B}_1 = 0$ entails

$$\operatorname{supp}\| \mathbb{B}_1(x) \|_Z \cup \operatorname{supp}\| \mathbb{B}_2(x, x) \|_Z$$
$$= \operatorname{supp}\| x \|_X (\operatorname{mod} \mu), \quad (x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x)) \in N.$$

Therefore, it suffices to show that for any functions $f \in AC(T, X)$, $g \in L(T, \mu, R)$, the equality $df(t)/dt = 0$ is $\mu$-almost everywhere $T_{fg} := \{t \in T : \| f(t) \|_X + | g(t) | = 0\}$; clearly, in the proof structure of Proposition 2, the pair $(f, df/dt)$ has a dual role as an $(x, dx/dt)$ and as a $(dx/dt, d^2x/dt^2)$.

Let $T_f := \{t \in T : f(t) = 0\}$. Since $T_f \supset T_{fg}$, in the case $\mu(T_f) = 0$ the statement

$$\{t \in T : \| df(t)/dt \|_X = 0\} \supset T_{fg} (\mathrm{mod}\, \mu)$$

is clear. Therefore, we consider the variant $\mu(T_f) \neq 0$.

We denote $T_0 := \{t \in T_f : \exists \delta > 0, \mu((t - \delta, t + \delta) \cap T_f) = 0\}$. We show that $\mu(T_0) = 0$. To do this, choose for each $t \in T_0$ a constant $\delta_t^* > 0$ such that $\mu((t - \delta_t^*, t + \delta_t^*) \cap T_f) = 0$. We find such rational numbers $\delta_t', \delta_t''$, that $\delta_t' \in (t - \delta_t^*, t), \delta_t'' \in (t, t + \delta_t^*)$ and let $I_t := (\delta_t', \delta_t'')$. Then the family of intervals $\{I_t\}_{t \in T_0}$ covers the set $T_0$, and since each interval $I_t$ is open with rational ends, the family $\{I_t\}_{t \in T_0}$ contains a countable subfamily $\{I_{ti}\}_{i=1, 2, ...,}$ also being the coverage of the set $T_0$.

Further, since for any index $i = 1, 2, \ldots$ is valid, $I_{ti} \subset (t_i - \delta_{ti}^*, t_i + \delta_{ti}^*)$, then $\mu(I_{ti} \cap T_\delta) = 0$, so the following chain of $\mu$-relations is valid:

$$\mu(T_0) = \mu\left(T_0 \cap \left(\bigcup_{i=1, 2, \ldots} I_{ti}\right)\right) = \mu\left(\bigcup_{i=1, 2, \ldots} T_0 \cap I_{ti}\right)$$

$$\leq \sum_{i=1, 2, \ldots} \mu(T_0 \cap I_{ti}) = 0,$$

where $\mu(T_0) = 0$. Now, we carry out the final part of the proof.

Let $t \in T_f \backslash T_0$. Then for any $\delta > 0$ correct $\mu((t - \delta, t + \delta) \cap T_f) > 0$, and since $f \in AC(T, X)$, there exists a set $T^* \subset T$, such that $\mu(T^*) = 0$ and $\forall t \in T_f \backslash T^*$ exists $df(t)/dt$. We show that $df(t)/dt = 0$ for $t \in T_f \backslash (T_0 \cup T^*)$. Indeed, for any natural $j$, we have

$$\mu((t - 1/j, t + 1/j) \cap T_f) > 0$$

and hence we find a moment $t_j \neq t$, $|t_j - t| < 1/j$, $t_j \in T_f$. But then the limit transition in the structure of the strong topology will be fulfilled (which, eventually, was required to be shown):

$$\begin{aligned} df(t)/dt &= \lim\{(f(t-\Delta t) - f(t))/\Delta t : \Delta t \to 0\} \\ &= \lim\{(f(t_j) - f(t))(t_j - t)^{-1} = 0 \in X : j \to \infty\} \\ &= 0 \in X. \end{aligned}$$ □

**Corollary 1.** *If* $(x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x)) \in \mathbf{\Pi}$ *and* $\ker \mathbb{B}_1 = 0$, *then* $\wp_\nu \subset \wp_{\nu_-}$, *where* $\wp_\nu, \wp_{\nu_-}$ *are the corresponding Lebesgue augmented* $\sigma$*-algebras of behavioral measures*

$$\nu(S) := \int_S \|(\mu(\tau), \mathbb{B}_1(x(\tau)), \mathbb{B}_2(x(\tau), x(\tau)))\|_U^2 \mu(d\tau), \quad S \in \wp_\mu,$$

$$\nu_-(S)$$

$$:= \int_S \| A_2(\tau) d^2x(\tau)/d\tau^2 + A_1(\tau)dx(\tau)/d\tau + A_0(\tau)x(\tau)\|_X \mu(d\tau), \ S \in \wp_\mu,$$

*where* $\operatorname{Im} \mathbb{B}_1 = Z_1$ *additionally entails* $Z_1 = X$, *and for any triplet* $(A_0, A_1, A_2) \in \mathbf{D}$, *there will be*

$$\|(u, \mathbb{B}_1(x), \mathbb{B}_2(x, x))\|_U^{-1} \| A_2 d^2x/dt^2 + A_1 dx/dt + A_0 x\|_X \in L_2(T, \mu, R)$$

$$\Leftrightarrow \|(u, x, \mathbb{B}_2(x, x))\|_U^{-1} \| A_2 d^2x/dt^2 + A_1 dx/dt + A_0 x\|_X \in L_2(T, \mu, R)$$

$\Leftrightarrow$ *process* $(x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x))$ *has a differential realization* (2).

It follows from the functional construction (4) that the Rayleigh-Ritz operator satisfies simple, but very important[1], relations (below $\chi_\varnothing$ is “zero

[1]Let $P_{\mathbf{\Pi}}$ be a projective space [11] associated with $\mathbf{\Pi}$. Then (5) induces a projectivization mapping $P\Psi : P_{\mathbf{\Pi}} \to L_+(T, \mu, R)$ of the Rayleigh-Ritz operator in which $P\Psi(\gamma) := \Psi[\gamma]$, $\gamma \in P_{\mathbf{\Pi}}$ (i.e., $\gamma = \{r\phi : r \in R\backslash\{0\}\}$, $\phi \in \mathbf{\Pi}$).

function" space $L(T, \mu, R)$):

$$\chi_{\varnothing} \leq_L \Psi(\pm r\phi) = \Psi(\phi), \quad r \in R \setminus \{0\}, \quad \phi \in \mathbf{\Pi}. \tag{5}$$

Now, before we go any further, it is necessary to introduce additional terminology.

**Definition 3** [2, 14]**.** We call a Rayleigh-Ritz operator *semiadditive* with weight $p > 0$ on a set $\mathbb{E} \subset \mathbf{\Pi}$, if for any pair $(\phi_1, \phi_2) \in \mathbb{E} \times \mathbb{E}$, it is true[2]

$$\Psi(\phi_1 + \phi_2) \leq_L p\Psi(\phi_1) + p\Psi(\phi_2).$$

**Lemma 1.** *The semiadditivity* (*with fixed weight*) *of the Rayleigh-Ritz operator is a finite character property for subsets of set* $\mathbf{\Pi}$.

**Proof.** Let on some set $\mathbb{E} \subset \mathbf{\Pi}$ the operator $\Psi$ be semiadditive with some weight *p*. Then this operator will be semiadditive with this weight on any finite subset of $\mathbb{E}$. On the other hand, if $\Psi$ is semiadditive with the same weight on any finite subset of $\mathbb{E}$, then for any pair of vector-functions $(\phi_1, \phi_2) \in \mathbb{E} \times \mathbb{E}$,

$$\Psi(\phi_1 + \phi_2) \leq_L p\Psi(\phi_1) + p\Psi(\phi_2),$$

because on a finite subset $\{\phi_1, \phi_2\} \subset \mathbb{E}$ the operator $\Psi$ is semiadditive with weight *p*. □

The relation between Lemma 1 and Tukey's lemma[3] [15, p. 55] leads to an important geometric characteristic of semiadditivity of the Rayleigh-Ritz operator, namely, in a linear manifold $\mathbf{\Pi}$, there exist maximal sets on which the operator (4) is semi-additive with some weight $p > 0$, and these sets cannot be *linear* in the case $p \in (0, 1)$; to be sure, it is enough to consider

[2]This property, when the weight of semiadditivity is 1, is akin to the property of "sub-linearity" [20, p. 400].

[3]Recall that the Tukey's lemma is an alternative form of the axiom of choice [15, p. 55].

the action of $\Psi$ for $\mathbb{E} = \{\phi, 0\} \subset \mathbf{\Pi}$, $\phi \neq 0$ (except for the trivial variant $\mathbb{E} = \{0\}$), which is why it is assumed below in Lemma 2 (and by default further on) that the semiadditivity weight of the operator $\Psi$ is represented by some fixed constant $p \in [1, \infty)$.

The following lemma, important in the context of Theorem 2, justifies Definition 3 and Lemma 1.

**Lemma 2.** *Let* $p \in [1, \infty)$. *Then in* $\mathbf{\Pi}$ *there exists a* (*not the only*) *maximal linear set* $\mathbb{E}$ *on which the Rayleigh-Ritz operator is semiadditive with weight* $p$.

**Proof.** Let $\phi_1$ be a nonzero element in $\mathbf{\Pi}$. Then, by virtue of (5), the operator $\Psi$ is semiadditive with weight $p$ on the linear set $\mathbb{E}_1 := \{r\phi_1 : r \in R\}$. Further, let $\phi_2 \in \mathbf{\Pi}$, $\phi_2 \notin \mathbb{E}_1$ and $\Psi$ is semiadditive on $\mathbb{E}_1 \cup \{\phi_2\}$ with weight $p$; it is not difficult to establish[4], that such a vector-function $\phi_2$ exists for any weight $p \in [1, \infty)$. We choose in $\mathbb{E}_1 + \mathbb{E}_2$, where $\mathbb{E}_2$ is a linear envelope over $\phi_2$, an arbitrary element

$$r_1\phi_1 + r_2\phi_2,$$

where $r_1, r_2 \in R$, $r_2 \neq 0$. In this formulation, the relations will be fulfilled according to (5):

$$\begin{aligned}\Psi(r_1\phi_1 + r_2\phi_2) = \Psi(r_1 r_2^{-1}\phi_1 + \phi_2) &\leq_L p\Psi(r_1 r_2^{-1}\phi_1) + p\Psi(\phi_2)\\ &= p\Psi(r_1\phi_1) + p\Psi(r_2\phi_2),\end{aligned}$$

where it follows that the operator $\Psi$ is semiadditive on the linear manifold $\mathbb{E}_1 + \mathbb{E}_2$ with weight $p$. By reasoning similarly, it can be shown that in the previous constructions, $\mathbb{E}_1$ can be replaced by any nonzero linear subset of $\mathbf{\Pi}$, on which $\Psi$ is semiadditive with weight $p$ (see Example 1 further).

---

[4]Let $\phi_1 = (q_1, w_1)$. Then it is enough to take the vector-function $\phi_2 = (q_2, w_2)$, for which $t \mapsto \langle w_1(t), w_2(t)\rangle_U = \chi_\varnothing$, where $\langle \cdot, \cdot \rangle_U$ is the scalar product in the Hilbert space $U$.

The remaining constructions will concern circuits, so let $\mathbb{P}$ be the family of all ordered pairs $(\mathbb{E}', p')$, where $\mathbb{E}'$ is a nonzero linear set in $\mathbf{\Pi}$ and $p' \in [1, \infty)$, and the Rayleigh-Ritz operator is semiadditive on $\mathbb{E}'$ with weight $p'$. We introduce a partial ordering $\prec$ in $\mathbb{P}$, considering

$$(\mathbb{E}', p') \prec (\mathbb{E}'', p'') \Leftrightarrow \mathbb{E}' \subset \mathbb{E}'', \quad p' = p''.$$

By Hausdorff's theorem (Hausdorff's maximality principle [15, p. 54]), there exists $\Omega$ - is a maximal chain (maximal linearly ordered set) in the family $\mathbb{P}$ containing the chain $(\mathbb{E}_1, p) \prec (\mathbb{E}_1 + \mathbb{E}_2, p)$. Let $\mathcal{L}$ be the set of all linear sets $\mathbb{E}_\gamma$ in $\mathbf{\Pi}$, such as $(\mathbb{E}_\gamma, p) \in \Omega$. Then $\mathcal{L}$ will be linearly ordered with respect to the set-theoretic inclusion, hence, the union $\mathbb{E} := \bigcup\{\mathbb{E}_\gamma : \mathbb{E}_\gamma \in \mathcal{L}\}$ forms (in a trivial way) a linear manifold in $\mathbf{\Pi}$.

Further, if $(\phi_1, \phi_2) \in \mathbb{E} \times \mathbb{E}$, then $(\phi_1, \phi_2) \in \mathbb{E}_\gamma \times \mathbb{E}_\gamma$ for some set $\mathbb{E}_\gamma \in \mathcal{L}$, whence we arrive at $\Psi(\phi_1 + \phi_2) \leq_L p\Psi(\phi_1) + p\Psi(\phi_2)$ and so $(\mathbb{E}, p) \in \Omega$. If the manifold $\mathbb{E}$ were not maximal in $\mathbf{\Pi}$, on which our operator $\Psi$ is semiadditive with weight $p$, then the construction would obtain in the family $\mathbb{P}$ an element $(\mathbb{E}^*, p)$, for with $\mathbb{E}^*$ strictly contains $\mathbb{E}$, but this would contradict the maximal chain $\Omega$ in the family $\mathbb{P}$. □

**Example 1.** We construct (on the basis of a modification of Theorem 2 [1]) a triple $(\Psi, \widetilde{N}, p)$, for which operator (4) is semi-additive with weight $p = 1$ on an infinite-dimensional linear manifold $\widetilde{N} \subset \mathbf{\Pi}$, closed (due to Theorem 31.D [9, p. 111]) in space

$$(L_2(T, \mu, X), \|\cdot\|_2) \times (H_2, \|\cdot\|_H)$$

with the product topology.

Let in formula (4) the operator-functions $A_0, A_1, A_2$ have the following representations:

$$A_0 = A_1 = 0 \in L_1(T, \mu, L(X, X)), \quad A_2 = \Gamma \in L(X, X),$$

where the operator $\Gamma$ realizes nonzero homothety [20, p. 87] on $X$ and let $\widetilde{N}$ be a linear dynamical bundle induced by all solutions of $D$-system (2) with operators $A_0, A_1, A_2$ and $B_1 = id_X$, $\mathbb{B}_1 = \Gamma$, $\mathbb{B}_2 = 0 \in \mathbb{L}(X^2, Z_2)$, $u(\cdot) = 0 \in L_2(T, \mu, Y)$. In this formulation, the operator $\Psi$ is semi-additive on $\widetilde{N}$ with weight $p = 1$, since (as is easy to establish) $\Psi(\phi) = \chi_T$ for any vector-function $\phi \in \widetilde{N}$.

Now, we consider the very important question of the continuity [16] of the Rayleigh-Ritz nonlinear functional operator. To do this, we introduce a vector topology in the space $L(T, \mu, R)$ generated by convergence on the measure $\mu$. It is well known (Theorem 14 [17, p. 64]) that this topology is generated by an invariant[5] metric (in the original terminology [18, p. 63] quasi-norm):

$$\rho(\xi, \zeta) := \int_T (1 + |\xi(\tau) - \zeta(\tau)|)^{-1} |\xi(\tau) - \zeta(\tau)| \mu(d\tau), \ \xi, \zeta \in L(T, \mu, R);$$

note that $(L(T, \mu, R), \rho)$ is a complete separable metric space (Theorem 15 [17, p. 65]).

The following lemma is motivated by an extension of Theorem 2 [3] in the context of Example 1 [3].

**Lemma 3.** *Let* $(\mathbf{\Pi}, \rho')$, $(\mathbf{\Pi}, \rho'')$ *be metric spaces with* $(\rho', \rho'')$*-metrics of the form*

$$\rho'((g, w), (\hat{g}, \hat{w})) := \rho(\| g \|_X, \| \hat{g} \|_X) + \rho(\| w \|_U, \| \hat{w} \|_U),$$

---

[5]Invariance implies $\rho(\xi + \gamma, \zeta + \gamma) = \rho(\xi, \zeta)$ for any $\xi, \zeta, \gamma \in L(T, \mu, R)$. We should not think that the property of invariance is not essential. It excludes certain "pathologies", in particular, if two metrics on $L(T, \mu, R)$ induce the same topology, and these metrics are invariant, then they cannot define different stocks of Cauchy sequences in $L(T, \mu, R)$ (the reverse is possible if one of the metrics is invariant).

$$\rho''((g, w), (\hat{g}, \hat{w})) := \rho'((g, w), (\hat{g}, \hat{w})) + \rho^{\mu}(w, \hat{w}),$$

$$\rho^{\mu}(w, \hat{w}) := 2^{-1}\mu(\operatorname{supp}\| w\|_U \Delta \operatorname{supp}\| \hat{w}\|_U),$$

$$(g, w), (\hat{g}, \hat{w}) \in AC^1(T, X) \times H_2,$$

*where* $\Delta$ *is a symmetric difference of sets and let* $\mathcal{T}', \mathcal{T}''$ *be topologies of spaces* $(\mathbf{\Pi}, \rho'), (\mathbf{\Pi}, \rho'')$.

*Moreover, suppose that* $\mathbb{E}^{\#} := \{(g, w) \in \Pi : \operatorname{supp}\| w\|_U = T\}$. *Then the following statements are valid*:

(i) *the metric* $\rho'$ *is invariant, with the topology* $\mathcal{T}'$ - *vector* (*the operations of vector space* $\mathbf{\Pi}$ *are continuous in this topology*);

(ii) $\rho^{\mu}$ *is not invariant pseudo-metric in* $H_2$, $\rho''$ *is not invariant metric in* $\mathbf{\Pi}$, *with topology* $\mathcal{T}''$ *not being vector* (*algebraic operations in* $\mathbf{\Pi}$ *are not* $\mathcal{T}''$ *continuous*);

(iii) *the topology* $\mathcal{T}'$ *is strictly weaker than the topology* $\mathcal{T}''$, *with* $\mathbf{\Pi}\backslash\mathbb{E}^{\#} \in \mathcal{T}''$;

(iv) *the identical mapping* $\mathrm{id}_{\mathbf{\Pi}} : (\mathbf{\Pi}, \rho') \to (\mathbf{\Pi}, \rho'')$ *is discontinuous at points of the region* $\mathbf{\Pi}\backslash\mathbb{E}^{\#}$ *whereas the mapping* $\mathrm{id}_{\mathbf{\Pi}} : (\mathbf{\Pi}, \rho'') \to (\mathbf{\Pi}, \rho')$ *is continuous*;

(v) *if* $(\mathbb{E}, \mathcal{T}'')$ *and* $(\mathbb{E}, \mathcal{T}')$ *are some topological subspaces of the spaces* $(\mathbf{\Pi}, \mathcal{T}'')$ *and* $(\mathbf{\Pi}, \mathcal{T}')$, *respectively, and* $(\mathbb{E}, \mathcal{T}'')$ *is compact, then* $\mathcal{T}'' = \mathcal{T}'$;

(vi) *the Rayleigh-Ritz operator* $\Psi : (\mathbf{\Pi}, \rho'') \to (L(T, \mu, R), \rho)$ *is continuous*; *and*

(vii) *the contraction* $\Psi : (\mathbb{E}^{\#}, \rho') \to (L(T, \mu, R), \rho)$ *is continuous*.

**Proof.** We restrict ourselves to establishing statement (ii), since the derivation of propositions (i), (iii)-(vii) can be omitted in view of their transparency by virtue of Theorem 4 [16] and Corollary 3 [16].

Let

$$A,\ B \in \wp_\mu, \quad A \cap B = \varnothing, \quad \mu(A) \neq 0 \neq \mu(B), \quad h \in U, \quad \| h \|_U > 0,$$

$$\varphi^{\#} := \chi_T h, \quad \omega^{\#} := -\chi_T h, \quad \varphi_n^{\#} := \varphi^{\#} + n^{-1}\varphi^{\#},$$

$$\omega_n^{\#} := \omega^{\#} - n^{-1}\omega^{\#}, \quad n = 1, 2, \ldots,$$

where $\chi_T$ is the characteristic function of the interval $T$ (similarly below for the sets $A, B, \varnothing \in \wp_\mu$).

Since $A \Delta B \subset (A \Delta C) \cup (B \Delta C)$, $(H_2, \rho^\mu)$ is pseudo-metric space, and

$$\begin{aligned} &\rho^\mu(\chi_A h + \varphi^{\#},\ \chi_B h + \varphi^{\#}) \\ &= 2^{-1}\mu(\operatorname{supp}\| \chi_A h + \varphi^{\#} \|_U \Delta \operatorname{supp}\| \chi_B h + \varphi^{\#} \|_U) \\ &= 0 \neq 2^{-1}\mu(A) + 2^{-1}\mu(B) \\ &= 2^{-1}\mu(\operatorname{supp}\| \chi_A h \|_U \Delta \operatorname{supp}\| \chi_B h \|_U) \\ &= \rho^\mu(\chi_A h,\ \chi_B h), \end{aligned}$$

i.e., the pseudo-metric $\rho^\mu$ is not invariant, hence, the metric $\rho''$ is not invariant either. Further[6]

$$\rho^\mu(\varphi_n^{\#},\ \varphi^{\#}) = \rho^\mu((1 + n^{-1})\chi_T h,\ \chi_T h) \xrightarrow[n \to \infty]{} 0,$$

[6]For the metric space $(\mathbf{\Pi}, \rho')$, see Proposition 2 [18, p. 53] as well as Example 2 [18, p. 63].

$$\rho^{\mu}(\omega_n^{\#}, \omega^{\#}) = \rho^{\mu}(-(1-n^{-1})\chi_T h, -\chi_T h) \xrightarrow[n\to\infty]{} 0,$$

$$\rho^{\mu}(\varphi_n^{\#} + \omega_n^{\#}, \varphi^{\#} + \omega^{\#}) = \rho^{\mu}(2n^{-1}\chi_T h, \chi_{\varnothing} h) \xrightarrow[n\to\infty]{} 2^{-1}\mu(T),$$

$$\rho^{\mu}(n^{-1}\varphi_n^{\#}, 0 \cdot \varphi^{\#}) = \rho^{\mu}((n^{-1} + n^{-2})\chi_T h, \chi_{\varnothing} h) \xrightarrow[n\to\infty]{} 2^{-1}\mu(T),$$

where we conclude that the operations (adding vectors and multiplying them by a scalar) of vector space $\mathbf{\Pi}$ are not continuous in the topology induced by the metric $\rho'' = \rho' + \rho^{\mu}$. □

**Corollary 2.** *Let* $\mathbf{\Pi}^*$ *be a finite-dimensional subspace in* $\mathbf{\Pi}$. *Then*

(i) *the metric space* $(\mathbf{\Pi}^*, \rho'')$ *is complete, with the class of fundamental sequences from* $(\mathbf{\Pi}^*, \rho'')$ *contained (strictly) in the class of sequences of Cauchy space* $(\mathbf{\Pi}^*, \mathcal{T})$, *where* $\mathcal{T}$ *is the topology induced in* $\mathbf{\Pi}^*$ *from space* $(L_2(T, \mu, X), \|\cdot\|_2) \times (H_2, \|\cdot\|_H)$; *and*

(ii) *if* $\mathbf{\Pi}^* \subset \mathbb{E}^{\#}$, *then the metric* $\rho''$ *will be invariant and the topology* $\mathcal{T}^*$ *in* $\mathbf{\Pi}^*$, *induced by* $\rho''$, *will be vector, with* $\mathcal{T}^* = \mathcal{T}$, $\Psi[\mathbf{\Pi}^*]$ *is compact in the metric space* $(L(T, \mu, R), \rho)$.

**Proof.** Clause (i) follows from Theorem 15 [17, p. 65], whereas clause (ii) is a direct consequence of Proposition 2 [18, p. 53] and Theorem 2 [3]. □

**Theorem 1.** *If conditions of* (ii) *of Corollary* 2 *are fulfilled, and the statement is true*:

$$\exists \sup_L \Psi[\mathbf{\Pi}^*] \Leftrightarrow \rho(\sup_L W_n, \sup_L W_m) \xrightarrow[n,m\to\infty]{} 0,$$

*then* $W_n := \bigcup\{V_j : j = 1, \ldots, n\}$, $\{V_j\}_{j=1,2,\ldots}$ *is countable family finite* $j^{-1}$-*nets in* $\Psi[\mathbf{\Pi}^*]$.

**Remark 2.** Let $W_n = \{\xi_1, ..., \xi_k\}$. Then $\sup_L W_n = \xi_1 \vee \cdots \vee \xi_k$, where $\xi' \vee \xi'' := 2^{-1}(\xi' + \xi'' + |\xi' - \xi''|)$.

**Proof of Theorem 1** (Establishing $\cdots \Rightarrow \cdots$)**.** It is clear that the set $W_\infty := \bigcup\{V_j : j = 1, 2, ...\}$ is everywhere dense in $\Psi[\mathbf{\Pi}^*]$, and obviously (by virtue of the initial condition for $\cdots \Rightarrow \cdots$ and Theorem 17 [17, p. 68]), there exists $\sup_L W_\infty \in L(T, \mu, R)$. Thus, it suffices to show that

$$\rho(\sup_L W_n, \sup_L W_m)$$

$$\leq \rho(\sup_L W_n, \sup_L W_\infty) + \rho(\sup_L W_m, \sup_L W_\infty) \xrightarrow[n, m \to \infty]{} 0.$$

We reason contrariwise: let in $\{W_n\}_{n=1,2,...}$ there be a countable subfamily $\{W_k\}_{k \in \mathcal{K}}$, for which $\forall k \in \mathcal{K} : \rho(\sup_L W_k, \sup_L W_\infty) \leq \text{const} = d_+$. Then, since there is a monotone chain

$$\sup_L W_1 \leq_L \sup_L W_2 \leq_L \cdots \leq_L \sup_L W_n \leq_L \sup_L W_{n+1} \leq_L \cdots \leq_L \sup_L W_\infty,$$

the monotonicity of this chain entails,

$$\rho(\sup_L W_n, \chi_\varnothing) \leq \rho(\sup_L W_{n+1}, \chi_\varnothing),$$

from which we arrive at

$$\exists d \in (0, d_+] : \rho(\sup_L W_n, \sup_L W_\infty) \xrightarrow[n \to \infty]{} d,$$

which contradicts the following equality:

$$t \mapsto \sup\{\sup_L W_i(t) : i = 1, 2, ...\} = \sup_L W_\infty;$$

this equality means the point-wise convergence of $\{\sup_L W_n\}$ to $\sup_L W_\infty$, and hence convergence on measure $\mu$, which in turn, is equivalent to convergence $\rho(\sup_L W_n, \sup_L W_\infty) \xrightarrow[n \to \infty]{} 0$.

(Establishment $\cdots \Leftarrow \cdots$). The proof is transparent by virtue of the clause

$$\chi_{\varnothing} \leq_L f, \quad f \in \{\sup_L W_n : n = 1, 2, ...\} \subset L(T, \mu, R),$$

of Proposition 2 [18, p. 508] and Theorem 17 [17, p. 68]. The theorem is completely proved. □

Now, armed with and formulas (1)-(3) from [17, p. 646], provide

$$\mathbb{B}_1(x) = \mathbb{B}_1\mathbb{B}^{-1}\mathbb{B}(x) \in L_\infty(T, \mu, Z_1),$$

$$\mathbb{B}_2(x, x) = \mathbb{B}_2(x, \mathbb{B}^{-1}\mathbb{B}(x)) \in L_\infty(T, \mu, Z_2),$$

where $\mathbb{B}, \mathbb{B}^{-1} \in L(X, X)$ and $x \in AC^1(T, X)$, we formulate the first of the main results.

**Theorem 2.** *Let* $\{N_j : j = 1, ..., k\}$ *be dynamic processes* (1), (2) *and* $Z_i = X, i = 1, 2.$ *Then we find a linear continuous operator* $\mathbb{B}_1^+ \in \mathbb{L}(X, X)$ *for which the representation is valid*:

$\exists(B_{01}^*, B_{11}^*, B_{21}^*) \in \mathbf{L}_2 :$

$A_2 d^2 x_1/dt^2 + A_1 dx_1/dt + A_0 x_1 = B_{01}^* u_1 + B_{11}^* \mathbb{B}_1^+(x_1) + B_{21}^* \mathbb{O}(x_1, x_1),$

$(x_1, u_1, \mathbb{B}_1(x_1), \mathbb{B}_2(x_1, x_1)) = N_1;$

…

$\exists(B_{0k}^*, B_{1k}^*, B_{2k}^*) \in \mathbf{L}_2 :$

$A_2 d^2 x_k/dt^2 + A_1 dx_k/dt + A_0 x_k = B_{0k}^* u_k + B_{1k}^* \mathbb{B}_1^+(x_k) + B_{2k}^* \mathbb{O}(x_k, x_k),$

$(x_k, u_k, \mathbb{B}_1(x_k), \mathbb{B}_2(x_k, x_k)) = N_k;$ (6)

*where it is not excluded that* $(B_{0i}^*, B_{1i}^*, B_{2i}^*) \neq (B_{0j}^*, B_{1j}^*, B_{2j}^*), i \neq j.$

Summing up the "intermediate" result, we see that the following statement is true.

**Corollary 3.** *Let* $\mathbb{B} \in \mathbb{L}(X, X)$ *have a bounded inverse. Then the solution of the differential realization problem* (3) *can be constructed based on the behavior of dynamical processes of the form*:

$$\tilde{N}_1 = (x_1, u_1, \mathbb{B}_1^+(x_1), \mathbb{O}(x_1, x_1)),$$

$$\ldots$$

$$\tilde{N}_k = (x_k, u_k, \mathbb{B}_1^+(x_k), \mathbb{O}(x_k, x_k)),$$

$$\mathbb{B}_1^+ = \mathbb{B}, \tilde{N}_+ := \bigcup_{j=1,\ldots,k} \tilde{N}_j. \tag{7}$$

## 3. Analysis of the Existence of the *IL*-regulator in Rayleigh-Ritz Operator Constructions

The subject of this section is the solvability of the problem of realization on nonlinear trajectory bundles (1), (2) of operator-functions of invariant linear regulator of differential system (3). We formulate it in the form of a compact theorem. Simple sufficient conditions are its consequences; all parts of their proofs have in fact already been prepared by us, and it only remains to connect them together.

**Theorem 3.** *Let* $\tilde{N}_1, \tilde{N}_2$ *be dynamic bundles* (6), (7). *Then the problem* (3) *is solvable for* $k = 2$ *if the Rayleigh-Ritz operator is semiadditive with weight p on the linear manifold* Span $\tilde{N}_1$ + Span $\tilde{N}_2$.

**Remark 3.** The equivalence of Theorem 3 to Theorem 3 [2], the solution of the operator realization of the *IL*-regulator in terms of the angular metric of subspaces of Hilbert space, remains open. In this case, Theorem 3 points to the importance (in the spirit of *a majore ad minus*) of the construction of the weight of semiadditivity of operator (4) in the discussion of the extension of bundles of dynamical processes admitting the realization of (3), essentially developing in passing a qualitative geometric theory (in the infinite-dimensional setting) of vector fields [11, p. 275], conjugate to the given polylinear trajectory manifolds, in particular, in the statement [19] with the minimal operator norm $\|\cdot\|_{\mathbf{L}}$ for operator-functions $(B_0^+, ..., B_n^+) \in \mathbf{L}_2$ from (3).

**Proof of Theorem 3.** Since the linear shells Span $\widetilde{N}_1$ and Span $\widetilde{N}_2$ are absorbing sets in themselves, by virtue of the system of equations (2) of Theorem 2 [2], there will be two functions $\varphi_1, \varphi_2 \in L_2(T, \mu, R)$, for which the following two functional inequalities are satisfied:

$$\sup_L \Psi[\text{Span } \widetilde{N}_1] \leq_L \varphi_1, \quad \sup_L \Psi[\text{Span } \widetilde{N}_2] \leq_L \varphi_2.$$

We choose the manifold $\text{Span } \widetilde{N}_1 + \text{Span } \widetilde{N}_2$ as its absorbing set this manifold itself. Then, by virtue of the semiadditivity of $\Psi$ (with weight $p$) on $\text{Span } \widetilde{N}_1 + \text{Span } \widetilde{N}_2$, we obtain

$$\sup_L \Psi[\text{Span } \widetilde{N}_1 + \text{Span } \widetilde{N}_2]$$

$$\leq_L p \sup_L \Psi[\text{Span } \widetilde{N}_1] + p \sup_L \Psi[\text{Span } \widetilde{N}_2] \leq_L p(\varphi_1 + \varphi_2),$$

whence, based on Theorem 2 [2], it follows (given $\text{Span } \widetilde{N}_1 \cup \widetilde{N}_2 = \text{Span } \widetilde{N}_1 + \text{Span } \widetilde{N}_2$ and clause (ii) of Proposition 1) that the set of processes $N_1 \cup N_2$ (1) has a differential realization (3). □

**Corollary 4.** (i) *An IL-regulator of the differential system* (3) *exists if*

$$\exists p \in [1, \infty), \forall \phi_1, \phi_2 \in \text{Span } \widetilde{N}_+ : \Psi(\phi_1 + \phi_2) \leq_L p\Psi(\phi_1) + p\Psi(\phi_2).$$

*Moreover*, *if* $\text{Span } \widetilde{N}_+ \subset \mathrm{E}^{\#}$, $\dim \text{Span } \widetilde{N}_+ < \aleph_0$, *and* $\mathfrak{T}$ *the topology in* $\text{Span } \widetilde{N}_+$, *induced from the space* $(L_2(T, \mu, X), \|\cdot\|_2) \times (H_2, \|\cdot\|_H)$, *then the additional propositions are valid*:

(ii) *the contraction* $\Psi | (\text{Span } \widetilde{N}_+, \mathfrak{T})$ *is continuous*;

(iii) $\Psi[\mathrm{Span}\ \widetilde{N}_+]$ *is compact in the metric space* $(L_2(T, \mu, R), \rho)$,

$$\sup_L \Psi[\mathrm{Span}\ \widetilde{N}_+] = \lim_\rho \{\zeta_i\}_{i=1,2,\dots} \in L_2(T, \mu, R),$$

*where* $\zeta_i := \xi_1 \vee \cdots \vee \xi_{m(i)}$, $\{\xi_1, \dots, \xi_{m(i)}\}$ *is some* (*i.e.*, *any*) *finite* $i^{-1}$*-net in* $\Psi[\mathrm{Span}\ \widetilde{N}_+]$, $\lim_\rho\{\cdots\}$ *is the limit of the* $\rho$*-fundamental sequence*.

**Remark 4.** Under the conditions of proposition (iii), there exists (see Theorem 17 [17, p. 68]) a countable $Q_+ \subset \mathrm{Span}\ \widetilde{N}_+$ such that the function $\phi_+ := \sup_L \Psi[\mathrm{Span}\ \widetilde{N}_+]$ is performed by the following sup-construction:

$$t \mapsto \phi_+(t) = \sup\{\Psi(q, w)(t) \in R : (q, w) \in Q_+\}.$$

**Remark 5.** Corollary 4 allows us to construct [1] an algebra of dynamical bundles with unit $N_+$, all elements of which have realization with the differential model (3). In this case, by virtue of Theorem 1 [2], the question of the "individual" characteristic feature of the differential realization (2) for each individual process $N_j (j = 1, \dots, k)$ is constructively solved (by symbolic calculations) on the *k*-family of dynamic processes $N_j = \{(x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x))_j\}$:

$$\Psi((x, u, \mathbb{B}_1(x), \mathbb{B}_n(x, x))_j) \in L_2(T, \mu, R), \quad j = 1, \dots, k.$$

If these inclusions (or some of them) are not satisfied, then we can set by means of computer algebra [21] the problem of synthesizing *PL*-forms $\mathbb{B}_i \in \mathbb{L}(X^i, Z_i)$, $i = 1, \dots, n$ from functions $\varphi_{ip} : X \to X$, as well as the "source of influence" $u^*(t)$ providing the mentioned conditions, i.e.,

$$\Psi((x, u^*, \mathbb{B}_1(\varphi_{11}(x)), \dots, \mathbb{B}_n(\varphi_{n1}(x), \dots, \varphi_{nn}(x)))_i) \in L_2(T, \mu, R), i = 1, \dots, k.$$

**Example 2.** We illustrate the statements of Remark 5 by considering two constructions (a) and (b):

(a) Let $n = 2$, $T = [0, 10]$ and

$$Z_1 = Z_2 = Y = X,\ A_0 = A_1 = 0 \in L(X, X), A_2 = \mathbb{B}_1 = \mathrm{id}_X,\ \mathbb{B}_2 = \langle \cdot, \cdot \rangle_X e,$$

where $\langle \cdot, \cdot \rangle_X$ is the scalar product in $X$, $e \in X$, $\| e \|_X = 1$, $t \mapsto x(t) = (t \sin t)e$, $t \mapsto u(t) = 0 \in L_2(T, X)$. In this formulation, the structure of the $BL$-form from (1) would be $\mathbb{B}_2(x, x) = \langle x, x \rangle_X e$.

Then (see Figure 1) the function

$$f := \Psi((x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x)))$$

$$= \| d^2x/dt^2 \|_X (\| x \|_X^2 + \| \mathbb{B}_2(x, x) \|_X^2)^{-1/2}$$

does not belong to the functional class $L_2(T, \mu, R)$ and, hence, according to Corollary 1, the differential realization (2) for an unguided process $(x, u, \mathbb{B}_1(x), \mathbb{B}_2(x, x))|_{u=0}$ does *not exist*.

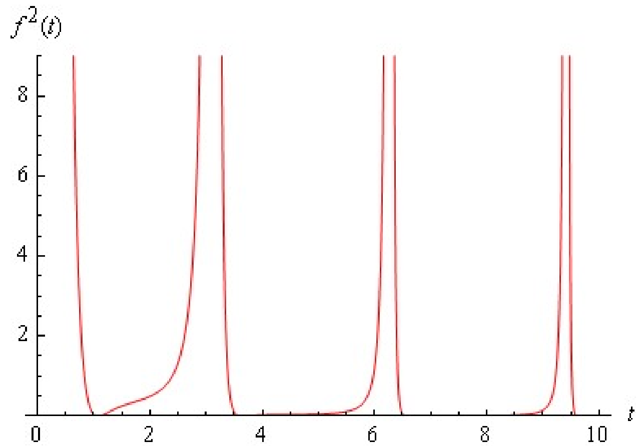


**Figure 1.** $f^2(t) = (2\cos t - t\sin t)^2((t\sin t)^2 + (t\sin t)^4)^{-1}$.

(b) We modify the formulation of the inverse problem (a) by taking

$$t \mapsto u^*(t) = (t\sin^2 t + 2^{-1}t^2 \sin 2t + \cos t)e.$$

Then (see Figure 2) taking into account the choice (in the context of Remark 2) of the *BL*-form of the form $\mathbb{B}_2(x, dx/dt) = \langle x, dx/dt\rangle_X e,$ we obtain the following value of the Rayleigh-Ritz operator (3) [1]:

$$f := \Psi((x, u^*, \mathbb{B}_1(x), \mathbb{B}_2(x, dx/dt)))$$

$$= \| d^2x/dt^2 \|_X (\| x \|_X^2 + \| \mathbb{B}_2(x, dx/dt) \|_X^2 + \| u^* \|_Y^2)^{-1/2} \in L_2(T, \mu, R),$$

and hence the second-order bilinear differential realization for the controlled dynamical process

$$(x, u^*, \mathbb{B}_1(x), \mathbb{B}_2(x, dx/dt))$$

*exists*; it is not difficult to find that

$$d^2x/dt^2 = -x - 2\mathbb{B}_2(x, dx/dt) + 2u^*.$$

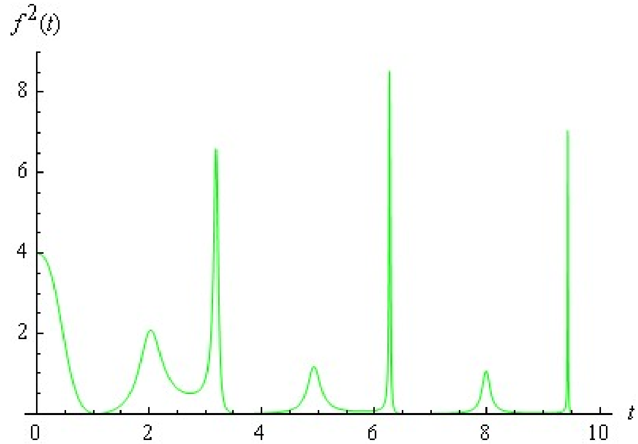


**Figure 2.** $f^2(t) = (2\cos t - t\sin t)^2((t\sin t)^2 + (t\sin t)^2(\sin t + t\cos t)^2 + (t\sin^2 t + 2^{-1}t^2 \sin 2t + \cos t)^2)^{-1}.$

## 4. Conclusion

The modern theory of nonlinear differential realization in separable Hilbert space has reached the most natural level of generality, which allows to state its principles and mathematical constructions with the greatest transparency, simultaneously providing them with the widest application to the concepts of the general theory of inverse problems of mathematical physics [10]. Such a profound development of the theory of realization of infinite-dimensional dynamical systems was largely achieved through the introduction of the Rayleigh-Ritz functional operator [16]. The construction of this operator and its numerous modifications played a significant role in the qualitative formation of differential realization as an independent quite original mathematical theory, which stands out not only for its internal integrity and simplicity, but also for new applications in nonlinear inverse problems of system analysis [22, 23]. With its help, bright theoretical-systems' results have been obtained, while the technical level of research and its intensity have increased considerably. It should be noted that this was not accompanied by an increase in the dissociation of these theoretical investigations and a loss of naturalness in mathematical formulations of their theoretical and applied problems; on the contrary, a profound qualitative result from one research area shed new light on other sections, as a rule, emphasizing their organic unity.

In the present paper, the topological and algebraic properties of the Rayleigh-Ritz operator were studied in the light of the notions about the geometry of infinite-dimensional nonstationary bilinear vector fields [24]. On this methodological basis, the questions of solvability of the inverse problem in the field of the operator realization of the invariant linear regulator (realization of the "*superposition principle*")[7] of a second-order (in particular, hyperbolic) nonstationary differential system containing a finite bundle of fixed integral curves (with program-position bilinear control) in a separable Hilbert space as admissible solutions were studied. This approach can be involved in an extension of the "*quasi-linearization*" method

[7]The term "*superposition principle*" is borrowed from Quantum Mechanics. In Quantum Mechanics, "superposition" means that the state space is linear, whereas in Systems Theory, "superposition" means that the dependence of *output* quantities on *input* influences is linear [25, p. 18].

[25, p. 168] when solving the optimal control problem using the sequential approximation technique, also known as the Picard method [25, p. 173].

As a follow-up to similar studies, we note the more general case, when the family of simulated guided trajectory beams lies in a uniformly convex Banach space [18] and the power of the family of beams $\aleph_0$, is much more complicated. As a sub-problem, it contains the geometric theory of closed partitioned dihedrons of Banach space [9], which has not yet been sufficiently developed. It should be taken into account that in this context it is not productive to make the initial assumption (see the proof of Theorem 1 [26]) that any closed subspace of the Banach space under study is complementary, since then this space will actually be isomorphic to some Hilbert space (Theorem 3 [17, p. 203]).

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