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OPTIMAL CONTROL OF A NONLINEAR ELLIPTICAL EVOLUTION PROBLEM WITH MISSING DATA

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Abstract

We consider the optimal control of a nonlinear elliptic problem with missing data (so-called ill-posed problems). Using the notion of no-regret and low-regret control, we give a characterization of the control for ill-posed problems. More precisely, we study the control of Cauchy evolution problems via a regularization approach which generates incomplete information. We obtain a singular optimality system characterizing the no-regret control for the Cauchy evolution problems.

1. Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^N$ with bounded $\Gamma = \Gamma_0 \cup \Gamma_1$ of class $\mathcal{C}^2$. For $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma_0 = \Gamma_0 \times (0, T)$, $\Sigma_1 = \Gamma_0 \times (0, T)$. We consider the following controlled nonlinear elliptic problem:

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z - z^3 = 0 & \text{in } Q, \\ z = v_0, \dfrac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \\ z(0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $z \in L^2(Q)$ is unknown on Σ_1 and $v = (v_0, v_1) \in (L^2(\Sigma_0))^2$ is the control.

We come across such a system while dealing with physical phenomenon such as wave propagation and scattering, vibration of the structure, and electromagnetic scattering.

Problem (1) is a Cauchy problem. It is well known that it is ill-posed in the sense that it does not admit a solution in general and that existing solutions (if any) are unstable. To act on such a system via a control, it is worth working with suitable optimization tools. More precisely, we are concerned in this paper with the following problem:

$$\inf_{(v, z) \in U} J(v, z), \tag{2}$$

where

$$U = \left\{ (v, z) \in (L^2(\Sigma_0))^2 \times L^2(Q), \begin{cases} \dfrac{\partial z}{\partial t} - \Delta z - z^3 = 0 & \text{in } Q, \\ z = v_0, \dfrac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \ z(0) = 0 \text{ in } \Omega \end{cases} \right\}$$

is a closed subset of $(L^2(\Sigma_0))^2 \times L^2(Q)$, assuming that $U \neq \varnothing$. We call any control-state pair $(v, z) \in U$ an *admissible couple*. J is a strictly convex cost functional defined for all admissible control-state couples (v, z) by

$$J(v, z) = \| z - z_d \|^2_{L^2(Q)} + N_0 \| v_0 \|^2_{L^2(\Sigma_0)} + N_1 \| v_1 \|^2_{L^2(\Sigma_0)}, \tag{3}$$

where $(N_0, N_1) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $z_d \in L^2(Q)$ is the desired state. We study this problem combining the notion of no-regret control and low-regret control introduced by Lions [10] to control problem with incomplete data with the regularization method of the Laplacian, where we introduce a new data:

$$z = g_0, \quad \frac{\partial z}{\partial \nu} = g_1 \text{ on } \Sigma_1. \tag{4}$$

So we first consider for any $\gamma > 0$, the low-regret problem:

$$\inf_{v \in (L^2(\Sigma_0))^2} \left(\sup_{g \in (L^2(\Sigma_1))^2} (J_\varepsilon(v, g) - J_\varepsilon(0, g) - \gamma \| g_0 \|^2_{L^2(\Sigma_1)} - \gamma \| g_1 \|^2_{L^2(\Sigma_1)}) \right). \tag{5}$$

Then we prove that the low-regret control converges to the no-regret control solution of the problem:

$$\inf_{v \in (L^2(\Sigma_0))^2} \left(\sup_{g \in (L^2(\Sigma_1))^2} (J(v, g) - J(0, g)) \right), \tag{6}$$

that we characterize by giving the corresponding optimality system.

Lions [1] introduced the notions of no-regret control and low-regret control in order to control a parabolic equation governed with an operator free unknown parameter and unknown initial condition. According to Lions, by looking for such a control, one looks for the 'best possible control' v which does 'at least well' and 'not much worse in the worst situation' than doing nothing. The notions were then applied to control some models with incomplete data, including models involving fractional derivative in time. We refer for instance to Lions [2, 3], Nakoulima et al. [4-7], Jacob and Omrane [8], and Mophou [9].

2. Low-regret Control and No-regret Control

Due to the ill-posedness of the Cauchy elliptic problem, it is impossible to solve it directly. For this, we introduce the notion of regularization. Our method consists in regularizing (1) into an elliptic problem of incomplete data. For $\varepsilon > 0$, we consider the regularized problem:

$$\begin{cases} \dfrac{\partial z_\varepsilon}{\partial t} - \Delta^2 z_\varepsilon - z_\varepsilon^3 - \varepsilon z_\varepsilon = 0 & \text{in } Q, \\ z_\varepsilon - \dfrac{\partial \Delta z_\varepsilon}{\partial \nu} = v_0; \quad \dfrac{\partial z_\varepsilon}{\partial \nu} + \Delta z_\varepsilon = v_1 & \text{on } \Sigma_0, \\ \varepsilon z_\varepsilon - \dfrac{\partial \Delta z_\varepsilon}{\partial \nu} = \varepsilon g_0; \quad \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \Delta z_\varepsilon = \varepsilon g_1 & \text{on } \Sigma_1, \\ z_\varepsilon(0) = 0 & \text{in } \Omega, \end{cases} \tag{7}$$

where $g = (g_0, g_1) \in (L^2(\Sigma_1))^2$. We then make the following remark:

Remark 1. For every fixed εg_0 and εg_1, we assume the existence of a unique solution to (7). Indeed, in the following subsection, εg_0 and εg_1 are considered as data perturbations.

To come back to the original Cauchy problem, we make the following change of variable $\eta = \Delta z$ and put $\varepsilon = 0$. Then, we obtain

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta\eta - z^3 = 0 & \text{in } Q, \\ \dfrac{\partial \eta}{\partial \nu} = 0; \quad \eta = 0 & \text{on } \Sigma_1, \\ z(0) = 0 & \text{in } \Omega, \end{cases} \tag{8}$$

and

$$z - \frac{\partial \eta}{\partial \nu} = v_0; \quad \frac{\partial z}{\partial \nu} + \eta = v_1 \text{ on } \Sigma_0. \tag{9}$$

From (8), we have $\dfrac{\partial \eta}{\partial \nu} = \eta = 0$ on Σ_1. Using the unique continuation of Mizohata [12], we deduce that $\dfrac{\partial \eta}{\partial \nu} = \eta = 0$ on Σ_0. Hence, condition (9) becomes

$$z = v_0; \quad \frac{\partial z}{\partial \nu} = v_1 \text{ on } \Sigma_0, \tag{10}$$

that is, the same conditions of the original problem.

Lemma 2.1. *Consider the cost functional*:

$$J_\varepsilon(v, g) = \| z_\varepsilon(v, g) - z_d \|^2_{L^2(Q)} + N_0 \| v_0 \|^2_{L^2(\Sigma_0)} + N_1 \| v_1 \|^2_{L^2(\Sigma_0)}. \tag{11}$$

Under the assumptions of the previous section, for all $v \in (L^2(\Sigma_0))^2$ *and* $g \in (L^2(\Sigma_1))^2$,

$$\begin{aligned} J_\varepsilon(v, g) - J_\varepsilon(0, g) = {} & J_\varepsilon(v, 0) - J_\varepsilon(0, 0) \\ & + 2\left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g_0}(v, 0) + \frac{\partial z_\varepsilon}{\partial g_1}(v, 0) \right\rangle_{L^2(Q)} \\ & - 2\left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g_0}(0, 0) + \frac{\partial z_\varepsilon}{\partial g_1}(0, 0) \right\rangle_{L^2(Q)}. \end{aligned} \tag{12}$$

Proof. We refer to [13]. □

We use a method developed in [13] to obtain the following result:

Lemma 2.2. *Under the assumptions of the previous section, consider the function* J_ε *defined by* (11). *Then for all* $v \in (L^2(\Sigma_0))^2$ *and* $g \in (L^2(\Sigma_1))^2$,

$$J_\varepsilon(v, g) - J_\varepsilon(0, g)$$

$$= J_\varepsilon(v, 0) - J_\varepsilon(0, 0)$$

$$+ 2\varepsilon\left(\langle \zeta_\varepsilon(v) - \zeta_\varepsilon(0), g_0\rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v) - \frac{\partial \zeta_\varepsilon}{\partial \nu}(0), g_1\right\rangle_{L^2(\Sigma_1)}\right), \quad (13)$$

where $\zeta_\varepsilon(v)$ *is a solution of*:

$$\begin{cases} -\dfrac{\partial \zeta_\varepsilon}{\partial t} - \Delta^2 \zeta_\varepsilon - 3\zeta_\varepsilon(z_\varepsilon)^2 - \varepsilon\zeta_\varepsilon = -(z_\varepsilon(u_\varepsilon, 0) - z_d) & \text{in } Q, \\ \zeta_\varepsilon - \dfrac{\partial \Delta\zeta_\varepsilon}{\partial \nu} = 0; \quad \dfrac{\partial \zeta_\varepsilon}{\partial \nu} + \Delta\zeta_\varepsilon = 0 & \text{on } \Sigma_0, \\ \varepsilon\zeta_\varepsilon - \dfrac{\partial \Delta\zeta_\varepsilon}{\partial \nu} = 0; \quad \varepsilon\dfrac{\partial \zeta_\varepsilon}{\partial \nu} + \Delta\zeta_\varepsilon = 0 & \text{on } \Sigma_1, \\ \zeta_\varepsilon(T, v) = 0 & \text{in } \Omega. \end{cases} \quad (14)$$

Remark 2.

$$\sup_{g \in (L^2(\Sigma_1))^2} (J(v, g) - J(0, g))$$

$$= J_\varepsilon(v, 0) - J_\varepsilon(0, 0)$$

$$+ 2\varepsilon \sup_{g \in (L^2(\Sigma_1))^2}\left(\langle S(v), g_0\rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu}, g_1\right\rangle_{L^2(\Sigma_1)}\right), \quad (15)$$

with $S(v) = \zeta(v) - \zeta(0)$, and

$$\sup_{g \in (L^2(\Sigma_1))^2} \left(\langle S(v),\, g_0 \rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu},\, g_1 \right\rangle_{L^2(\Sigma_1)} \right)$$

$$= \begin{cases} +\infty & \text{if } \left(\langle S(v),\, g_0 \rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu},\, g_1 \right\rangle_{L^2(\Sigma_1)} \right) \neq 0, \\ 0 & \text{if } S(v) \perp g_0 \text{ and } \frac{\partial S(v)}{\partial \nu} \perp g_1,\ \forall g \in (L^2(\Sigma_1))^2. \end{cases}$$

To make sense of the following minimization problem:

$$\inf_{v \in (L^2(\Sigma_0))^2} \left(\sup_{g \in (L^2(\Sigma_1))^2} (J(v,\, g) - J(0,\, g)) \right), \tag{16}$$

we consider the set:

$$\mathcal{O} = \left\{ v \in (L^2(\Sigma_0))^2 \text{ such as } \langle S(v),\, g_0 \rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial S(v)}{\partial \nu},\, g_1 \right\rangle_{L^2(\Sigma_1)} = 0,\ \forall g \in (L^2(\Sigma_1))^2 \right\}. \tag{17}$$

2.1. Low-regret control

To make solving the problem (16) a bit easier, Lions introduced the expressions $-\gamma \| g_0 \|^2_{L^2(\Sigma_1)}$ and $-\gamma \| g_1 \|^2_{L^2(\Sigma_1)}$, where γ denotes a relaxation parameter and is strictly positive $(\gamma > 0)$. Thus

$$\inf_{v \in (L^2(\Sigma_0))^2} \left(\sup_{g \in (L^2(\Sigma_1))^2} (J_\varepsilon(v,\, g) - J_\varepsilon(0,\, g) - \gamma \| g_0 \|^2_{L^2(\Sigma_1)} - \gamma \| g_1 \|^2_{L^2(\Sigma_1)}) \right). \tag{18}$$

The control *u* satisfying (18) is called a *low-regret control*.

The concept of $\ll$ low-regret control $\gg$ depends on γ and the norm $\| g \|$.

It is interpreted as an approximation of no-regret control.

Indeed, with low-regret control, we admit the possibility of making a control choice u slightly catastrophic than the ground state with a margin of error not exceeding $\gamma\| g \|^2_{L^2(\Sigma_1)}$.

Lemma 2.3. *Consider the function* J_ε *given by* (11). *Then for all* $v \in (L^2(\Sigma_0))^2$, *the relaxed problem* (16) *becomes*:

$$\inf_{v\in (L^2(\Sigma_0))^2}\left(J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{\varepsilon^2}{\gamma}\| S(v)\|^2_{L^2(\Sigma_1)} + \frac{\varepsilon^2}{\gamma}\left\| \frac{\partial S(v)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}\right). \tag{19}$$

Proof. Using (13),

$$J_\varepsilon(v, g) - J_\varepsilon(0, g)$$

$$= J_\varepsilon(v, 0) - J_\varepsilon(0, 0)$$

$$+ 2\left(\langle \zeta^\gamma_\varepsilon(v) - \zeta^\gamma_\varepsilon(0), \varepsilon g_0\rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial \zeta^\gamma_\varepsilon}{\partial \nu}(v) - \frac{\partial \zeta^\gamma_\varepsilon}{\partial \nu}(0), \varepsilon g_1 \right\rangle_{L^2(\Sigma_1)}\right).$$

So,

$$\sup_{g\in (L^2(\Sigma_1))^2} (J_\varepsilon(v, g) - J_\varepsilon(0, g) - \gamma\| g_0 \|^2_{L^2(\Sigma_1)} - \gamma\| g_1 \|^2_{L^2(\Sigma_1)})$$

$$= J_\varepsilon(v, 0) - J_\varepsilon(0, 0)$$

$$+2 \sup_{g\in (L^2(\Sigma_1))^2}\left(\langle \zeta^\gamma_\varepsilon(v) - \zeta^\gamma_\varepsilon(0), \varepsilon g_0\rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial \zeta^\gamma_\varepsilon}{\partial \nu}(v) - \frac{\partial \zeta^\gamma_\varepsilon}{\partial \nu}(0), \varepsilon g_1 \right\rangle_{L^2(\Sigma_1)} \right.$$

$$\left. - \frac{\gamma}{2}\| g_0 \|^2_{L^2(\Sigma_1)} - \frac{\gamma}{2}\| g_1 \|^2_{L^2(\Sigma_1)} \right).$$

According to the Fenchel transformation,

$$\sup_{g \in (L^2(\Sigma_1))^2} \left(\langle \zeta_\varepsilon^\gamma(v) - \zeta_\varepsilon^\gamma(0),\, \varepsilon g_0 \rangle_{L^2(\Sigma_1)} + \left\langle \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu}(v) - \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu}(0),\, \varepsilon g_1 \right\rangle_{L^2(\Sigma_1)} \right.$$

$$\left. - \frac{\gamma}{2} \| g_0 \|^2_{L^2(\Sigma_1)} - \frac{\gamma}{2} \| g_1 \|^2_{L^2(\Sigma_1)} \right)$$

$$= \frac{\varepsilon^2}{2\gamma} \| \zeta(v) - \zeta(0) \|^2_{L^2(\Sigma_1)} + \frac{\varepsilon^2}{2\gamma} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu}(v) - \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu}(0) \right\|^2_{L^2(\Sigma_1)}.$$

We thus obtain the result. □

Finally, we can formulate the problem (16) as follows:

For all $\gamma > 0$, find $u_\varepsilon^\gamma \in (L^2(\Sigma_0))^2$ such that:

$$J_\varepsilon^\gamma(u^\gamma) = \inf_{v \in (L^2(\Sigma_0))^2} J_\varepsilon^\gamma(v), \tag{20}$$

with

$$J_\varepsilon^\gamma(v) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{\varepsilon^2}{\gamma} \| S(v) \|^2_{L^2(\Sigma_1)} + \frac{\varepsilon^2}{\gamma} \left\| \frac{\partial S(v)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}.$$

Problem (20) is a low-regret problem and its solution is called *low-regret control*.

Remark 3. In contrast to the linear case, the function J_ε^γ is not convex. Therefore, we do not necessarily have the uniqueness of u_ε. Moreover, we are not sure that u_ε converges in $\mathcal{O}$. Thus, we use the penalization-adapted method defined by Lions in [11] for the search for low-regret control.

2.2. Existence of adapted low-regret control

In this part, we are interested in finding a solution of the following minimization problem:

$$\inf_{v \in (L^2(\Sigma_0))^2} J^{\gamma}_{\varepsilon a}(v), \tag{21}$$

or

$$J^{\gamma}_{\varepsilon a}(v) = J_{\varepsilon}(v, 0) - J_{\varepsilon}(0, 0) + \| v_0 - \tilde{u}_0 \|^2_{L^2(\Sigma_1)} + \| v_1 - \tilde{u}_1 \|^2_{L^2(\Sigma_1)}$$

$$+ \frac{\varepsilon^2}{\gamma} \| S(v) \|^2_{L^2(\Sigma_1)} + \frac{\varepsilon^2}{\gamma} \left\| \frac{\partial S(v)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}, \tag{22}$$

with $\tilde{u} \in (L^2(\Sigma_1))^2$ a no regret control, and the control u^{γ}_{ε} solution of (16) is called an *adapted low-regret control*.

The following proposition shows the existence of an adapted low-regret control.

Proposition 2.1. *There is at least an adapted low-regret control* $u^{\gamma}_{\varepsilon} \in (L^2(\Sigma_0))^2$ *solution of* (16).

Proof. See [13]. □

2.3. Characterization of the adapted low-regret control

Proposition 2.2. *The adapted low-regret control* $u^{\gamma}_{\varepsilon} = (u^{\gamma}_{0\varepsilon}, u^{\gamma}_{1\varepsilon})$ *as a solution of* (21) *is characterized by the unique solution* $\{\zeta^{\gamma}_{\varepsilon}, z^{\gamma}_{\varepsilon}, \beta^{\gamma}_{\varepsilon}, \phi^{\gamma}_{\varepsilon}\}$ *of*:

$$\begin{cases} -\dfrac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial t} - \Delta^2 \zeta^{\gamma}_{\varepsilon} - 3\zeta^{\gamma}_{\varepsilon}(z^{\gamma}_{\varepsilon})^2 - \varepsilon \zeta^{\gamma}_{\varepsilon} = -(z^{\gamma}_{\varepsilon} - z_d) & \text{in } Q, \\ \zeta^{\gamma}_{\varepsilon} - \dfrac{\partial \Delta \zeta^{\gamma}_{\varepsilon}}{\partial \nu} = 0; \quad \dfrac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial \nu} + \Delta \zeta^{\gamma}_{\varepsilon} = 0 & \text{on } \Sigma_0, \\ \varepsilon \zeta^{\gamma}_{\varepsilon} - \dfrac{\partial \Delta \zeta^{\gamma}_{\varepsilon}}{\partial \nu} = 0; \quad \varepsilon \dfrac{\partial \zeta^{\gamma}_{\varepsilon}}{\partial \nu} + \Delta \zeta^{\gamma}_{\varepsilon} = 0 & \text{on } \Sigma_1, \\ \zeta^{\gamma}_{\varepsilon}(T, v) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta^2 z_\varepsilon^\gamma - (z_\varepsilon^\gamma)^3 - \varepsilon z_\varepsilon^\gamma = 0 & in\ Q, \\ z_\varepsilon^\gamma - \dfrac{\partial \Delta z_\varepsilon^\gamma}{\partial \nu} = u_{0\varepsilon}^\gamma; \quad \dfrac{\partial z_\varepsilon^\gamma}{\partial \nu} + \Delta z_\varepsilon^\gamma = u_{1\varepsilon}^\gamma & on\ \Sigma_0, \\ \varepsilon z_\varepsilon^\gamma - \dfrac{\partial \Delta z_\varepsilon^\gamma}{\partial \nu} = 0; \quad \varepsilon \dfrac{\partial z_\varepsilon}{\partial \nu} + \Delta z_\varepsilon^\gamma = 0 & on\ \Sigma_1, \\ z_\varepsilon^\gamma(0) = 0 & in\ \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial \beta_\varepsilon^\gamma}{\partial t} - \Delta^2 \beta_\varepsilon^\gamma - 3\beta_\varepsilon^\gamma (z_\varepsilon^\gamma)^2 - \varepsilon \beta_\varepsilon^\gamma = 0 & in\ Q, \\ \beta_\varepsilon^\gamma - \dfrac{\partial \Delta \beta_\varepsilon^\gamma}{\partial \nu} = 0; \quad \dfrac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \Delta \beta_\varepsilon^\gamma = 0 & on\ \Sigma_0, \\ \varepsilon \beta_\varepsilon^\gamma - \dfrac{\partial \Delta \beta_\varepsilon^\gamma}{\partial \nu} = -\dfrac{\varepsilon^2}{\gamma}(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0)) & on\ \Sigma_1, \\ \varepsilon \dfrac{\partial \beta_\varepsilon^\gamma}{\partial \nu} + \Delta \beta_\varepsilon^\gamma = -\dfrac{\varepsilon^2}{\gamma}\left(\dfrac{\partial(\zeta_\varepsilon(u_\varepsilon^\gamma) - \zeta_\varepsilon(0))}{\partial \nu} \right) & on\ \Sigma_1, \\ \beta_\varepsilon^\gamma(0) = 0 & in\ \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial \phi_\varepsilon^\gamma}{\partial t} - \Delta^2 \phi_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \phi_\varepsilon^\gamma - \varepsilon \phi_\varepsilon^\gamma = z_\varepsilon^\gamma - z_d - \beta_\varepsilon^\gamma & in\ Q, \\ \phi_\varepsilon^\gamma - \dfrac{\partial \Delta \phi_\varepsilon^\gamma}{\partial \nu} = 0; \quad \dfrac{\partial \phi_\varepsilon^\gamma}{\partial \nu} + \Delta \phi_\varepsilon^\gamma = 0 & on\ \Sigma_0, \\ \varepsilon \phi_\varepsilon^\gamma - \dfrac{\partial \Delta \phi_\varepsilon^\gamma}{\partial \nu} = 0; \quad \varepsilon \dfrac{\partial \phi_\varepsilon^\gamma}{\partial \nu} + \Delta \phi_\varepsilon^\gamma = 0 & on\ \Sigma_1, \\ \phi_\varepsilon^\gamma(T, v) = 0 & in\ \Omega, \end{cases}$$

$$\phi_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma = \tilde{u}_0 - u_{0\varepsilon}^\gamma + \tilde{u}_1 - u_{1\varepsilon}^\gamma \quad in \quad (\Sigma_0),$$

or

$$z_\varepsilon^\gamma := z_\varepsilon(u_\varepsilon^\gamma, 0).$$

Proof. See [13]. □

2.4. Singular optimality system

In this part, we give the optimality system for the least regret control of the system (1).

Lemma 2.4. *There is a constant* $C > 0$ *such that*

$$\begin{cases} \| u_{0\varepsilon}^{\gamma} \|_{L^2(\Sigma_0)} \leq C, \\ \| u_{1\varepsilon}^{\gamma} \|_{L^2(\Sigma_0)} \leq C, \\ \| z_{\varepsilon}^{\gamma} \|_{L^2(Q)} \leq C, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \| \zeta_{\varepsilon}^{\gamma} \|_{L^2(\Sigma_1)} \leq C, \\ \dfrac{\varepsilon}{\sqrt{\gamma}} \left\| \dfrac{\partial \zeta_{\varepsilon}^{\gamma}}{\partial \nu} \right\|_{L^2(\Sigma_1)} \leq C. \end{cases} \tag{23}$$

Proof. u_{ε}^{γ} being a solution of (20), we have

$$J_{a\varepsilon}^{\gamma}(u_{\varepsilon}^{\gamma}) \leq J_{a\varepsilon}^{\gamma}(v), \quad \forall v \in (L^2(\Sigma_0))^2.$$

Taking $v = 0$, we get

$$\| z_{\varepsilon}^{\gamma} - z_d \|_{L^2(Q)}^2 + N_0 \| u_{0\varepsilon}^{\gamma} \|_{L^2(\Sigma_0)}^2 + N_1 \| u_{1\varepsilon}^{\gamma} \|_{L^2(\Sigma_0)}^2$$

$$+ \| u_{0\varepsilon}^{\gamma} - \tilde{u} \|_{L^2(\Sigma_1)}^2 + \frac{\varepsilon^2}{\gamma} \| \zeta_{\varepsilon}^{\gamma} \|_{L^2(\Sigma_1)}^2 + \frac{\varepsilon^2}{\gamma} \left\| \frac{\partial \zeta_{\varepsilon}^{\gamma}}{\partial \nu} \right\|_{L^2(\Sigma_1)}^2$$

$$\leq \| z_d \|_{L^2(\Sigma_1)}^2 = c,$$

hence the result. □

Theorem 2.1. *The low-regret control* u^{γ} *of problem* (1) *is characterized by* $\{\zeta^{\gamma}, z^{\gamma}, \beta^{\gamma}, \phi^{\gamma}\}$,

$$\begin{cases} -\dfrac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma - 3\zeta^\gamma (z^\gamma)^2 = -(z^\gamma - z_d) & in\ Q, \\ \zeta^\gamma - \dfrac{\partial \Delta \zeta^\gamma}{\partial \nu} = 0; \quad \dfrac{\partial \zeta^\gamma}{\partial \nu} + \Delta \zeta^\gamma = 0 & on\ \Sigma_0, \\ \dfrac{\partial \Delta \zeta^\gamma}{\partial \nu} = 0; \quad \Delta \zeta^\gamma_\varepsilon = 0 & on\ \Sigma_1, \\ \zeta^\gamma(T, v) = 0 & in\ \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial z^\gamma}{\partial t} - \Delta z^\gamma - (z^\gamma)^3 - \varepsilon z^\gamma = 0 & in\ Q, \\ z^\gamma - \dfrac{\partial \Delta z^\gamma}{\partial \nu} = u_0^\gamma; \quad \dfrac{\partial z^\gamma}{\partial \nu} + \Delta z^\gamma = u_1^\gamma & on\ \Sigma_0, \\ \dfrac{\partial \Delta z^\gamma}{\partial \nu} = 0; \quad \Delta z^\gamma = 0 & on\ \Sigma_1, \\ z^\gamma(0) = 0 & in\ \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial \beta^\gamma}{\partial t} - \Delta \beta^\gamma - 3\beta^\gamma (z^\gamma)^2 = 0 & in\ Q, \\ \beta^\gamma - \dfrac{\partial \Delta \beta^\gamma}{\partial \nu} = 0; \quad \dfrac{\partial \beta^\gamma}{\partial \nu} + \Delta \beta^\gamma = 0 & on\ \Sigma_0, \\ \dfrac{\partial \Delta \beta^\gamma}{\partial \nu} = 0; \quad \Delta \beta^\gamma = 0 & on\ \Sigma_1, \\ \beta^\gamma(0) = 0 & in\ \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial \phi^\gamma}{\partial t} - \Delta \phi^\gamma - 3(z^\gamma)^2 \phi^\gamma = z^\gamma - z_d - \beta^\gamma & in\ Q, \\ \phi^\gamma - \dfrac{\partial \Delta \phi^\gamma}{\partial \nu} = 0; \quad \dfrac{\partial \phi^\gamma}{\partial \nu} + \Delta \phi^\gamma = 0 & on\ \Sigma_0, \\ \dfrac{\partial \Delta \phi^\gamma}{\partial \nu} = 0; \quad \Delta \phi^\gamma = 0 & on\ \Sigma_1, \\ \phi^\gamma(T, v) = 0 & in\ \Omega, \end{cases}$$

$$\phi^\gamma + N_0 u_0^\gamma + N_1 u_1^\gamma = \tilde{u}_0 - u_0^\gamma + \tilde{u}_1 - u_1^\gamma \ in\ L^2(\Sigma_0). \tag{24}$$

Proof. See the proof of Theorem 2.1 of [13]. □

2.5. Characterization of the no-regret control

We now give the optimality system of no-regret control.

By passing to the limit when $\gamma \to 0,$ we obtain the optimality system of no-regret control given by the following:

Theorem 2.2. *The no-regret control* $\tilde{u} = (\tilde{u}_0, \tilde{u}_1)$ *of problem* (1) *is characterized by the unique solution* $\{\zeta, z, \beta, \phi\}$ *of*:

$$\begin{cases} -\dfrac{\partial \zeta}{\partial t} - \Delta\zeta - 3z^2\zeta = -(z - z_d) & \text{in } Q, \\ \zeta = 0; \quad \dfrac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \zeta(T, v) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z - z^3 = 0 & \text{in } Q, \\ z = \tilde{u}_0; \quad \dfrac{\partial z}{\partial \nu} = \tilde{u}_1 & \text{on } \Sigma_0, \\ z(0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial \beta}{\partial t} - \Delta\beta - 3z^2\beta = 0 & \text{in } Q, \\ \beta = 0; \quad \dfrac{\partial \beta}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \beta(0) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial \phi}{\partial t} - \Delta\phi - 3z^2\phi = z - z_d - \beta & \text{in } Q, \\ \phi = 0; \quad \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \Sigma_0, \\ \phi(T, v) = 0 & \text{in } \Omega, \end{cases}$$

$$\phi + N_0\tilde{u}_0 + N_1\tilde{u}_1 = 0 \text{ in } L^2(\Sigma_0).$$

Proof. From Theorem 2.1 and passing to the limit when $\gamma \to 0,$ we obtain

$$\begin{cases} \zeta^{\gamma} \rightharpoonup \zeta = 0, \\ \beta^{\gamma} \rightharpoonup \beta = 0, \\ \phi^{\gamma} \rightharpoonup \phi = 0 \end{cases} \tag{25}$$

on Σ_0.

Likewise, from (24),

$$(u_0^{\gamma}, u_1^{\gamma}) \rightharpoonup (\tilde{u}_0, \tilde{u}_1) \text{ in } L^2(\Sigma_0) \times L^2(\Sigma_0), \tag{26}$$

as a result:

$$\begin{cases} z^{\gamma} \rightharpoonup z = \tilde{u}_0, \\ \dfrac{\partial z^{\gamma}}{\partial \nu} \rightharpoonup \dfrac{\partial z}{\partial \nu} = \tilde{u}_1 \end{cases} \tag{27}$$

on Σ_0.

Of all the above, there is a unique control $\tilde{u}$ characterized by the solution $\{\zeta, z, \beta, \phi\}$ of the system (1). □

3. Concluding Remarks

In this paper, we have examined an ill-posed problem with missing data by combining the method of regularization and adapted least regret control. Due to the regularization method, we were able to generate information on Σ_1 without which the control of the system was complicated. As the system is nonlinear, we used the penalization-adapted method which allowed us to determine the characterization of the no regret control of the system.

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