Advances in Differential Equations and Control Processes

http://www.pphmj.com
http://dx.doi.org/10.17654/0974324323007
Volume 30, Number 2, 2023, Pages 97-116 **P-ISSN: 0974-3243**

EXISTENCE OF FIXED POINT FOR NONLINEAR OPERATOR IN PARTIALLY ORDERED METRIC SPACES

Yan Sun and Ravi P. Agarwal

Department of Mathematics
Shanghai Normal University
Shanghai, 200234, China
e-mail: ysun881@163.com
ysun@shnu.edu.cn
584511027@qq.com

Department of Mathematics
Texas A M University-Kingsville
700 University Blvd., MSC 172
Kingsville, Texas 78363-8202, U.S.A.
e-mail: Ravi.Agarwal@tamuk.edu

Abstract

In this article, first we introduce new notions of a contractive mapping and establish some fixed point theorems for the contractive mapping in the setting of LG-complete LG-metric spaces. Further, we establish a

Received: September 29, 2022; Revised: October 19, 2022; Accepted: November 21, 2022
2020 Mathematics Subject Classification: 34B18, 35A05, 35J65.
Keywords and phrases: weakly regular cone, fixed point, contractive mappings, existence.

How to cite this article: Yan Sun and Ravi P. Agarwal, Existence of fixed point for nonlinear operator in partially ordered metric spaces, Advances in Differential Equations and Control Processes 30(2) (2023), 97-116. http://dx.doi.org/10.17654/0974324323007

Published Online: April 14, 2023

new criterion between weakly regular cone and normal cone, and we also obtain a fixed point result in the same *LG*-complete *LG*-metric spaces by making use of analysis technique. Later, we give some examples to illustrate the valid of our main results.

1. Introduction

Banach contraction mapping principle is one of excellent results in functional analysis fields. Following the well-known results, many researchers have paid their attention to improving, generalizing, extending and enhancing this theory. Most of the authors have devoted to generalize some nice results to various abstract spaces such as quasi-metric spaces, partial order metric spaces, *G*-metric spaces and so on (see, e.g., [1-29]).

It is well known that fixed point theory in ordered metric space is important in many research fields of mathematical problems, sciences, technology, economics etc (see, [3-10]). For decades, the classical theory of fixed points in ordered metric space and *G*-metric space has been brought to light as crucial and powerful tool in the study of nonlinear problems. The existence and uniqueness of fixed point with fixed point iterative techniques have been successfully applied in many research areas such as chemistry, physics and so forth. One refers to see ([10-29]) and the references therein.

Many authors considered fixed point theory by applying simultaneously enriching metric space structure with partial orders afterwards. One can refer [1-29], where many famous researchers improved and generalized some fixed point results. All in all, the results of fixed points of contractive mappings have been a center on rigorous research for a long time.

Recently, many nice fixed point results have been obtained in *G*-metric space, one can see ([10, 11]) and the references therein. Samet et al. showed that some fixed point theorems in the circumstance of a *G*-metric space could be showed (by simple change) applying associate existing results in the ascertained a (quasi-) metric space [27]. It means that, if the contraction condition of the nonlinear operator on *G*-metric space can be split into two variables, then one can give an equivalent nonlinear operator equation in the ascertained usual order metric space. Newly, Karapinar and Agarwal established many nice results [15]. They presented new contraction condition in *G*-metric space. Agarwal et al. considered some excellent results for a class of generalized contractions in ordered metric spaces ([3-5]). Karapinar et al. studied a few results on fixed points [16].

Mustafa and Sims constructed a structure of generalized *G*-metric space and presented certain basic topological properties of *G*-metric spaces (see [18]). Mustafa proved new fixed point theorems of various mappings in *G*-metric spaces. Since then, many authors have studied and expanded fixed point theory on *G*-metric spaces [23], and further results one can see ([7-24]). Popescu gave the definition of multi-valued operators [26]. For more interesting results, one can also refer to ([12-14], [18-21]), in which listing the fixed point theory in *G*-metric space. And some other new fixed point theorems for operators satisfying various contractive conditions in *G*-metric spaces were showed in ([19-29]). Karapinar et al. proposed a number of coupled fixed point results in *G*-metric spaces [17].

From then on, the *G*-metric space has been attracted serious concern of both mathematicians and natural philosopher. Thus, the theory about the *G*-metric space became a very popular topic largely in the sense of perspective of fixed point theory. For other definitions of *G*-metric space, one can refer the author to see [2-7] and [18].

As far as we know that a few of authors have used fixed point theory to study the existence of solutions to a singular and/or nonsingular boundary value problem. The text by Agarwal [2-4], Ćirić and Lakshmikantham [6], Gaba [10, 11] are excellent resources for the use of fixed point theory in the considering the existence of solutions to boundary value problem. While much attention has been focused on the Cauchy problem for differential operator.

In the present article, one of the novelties is the fact that existence of a solution is established in terms of the functional associated to the nonlinear differential operator. It is also found that the fixed point of nonlinear operator is equivalent a solution of nonlinear differential equation boundary value problem. It would be interesting to consider the existence of fixed point for nonlinear operator in partially ordered metric spaces. It is well known that real ecosystems demonstrate patters of complex competitive relations containing multiple species. However, to solve this kind of problem, there is no equivalent theorem for systems. The purpose of this note is to fill this gap. Now we might as well try to put forward the following new definitions on contracting metric space. Denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$.

Definition 1.1. Let E be a nonempty set and $L : E \times E \times E \to \mathbb{R}^+$ be a nonnegative function satisfying the following properties:

(C1) $L(u, v, w(\xi, \eta)) \geq 0$, and $L(u, v, w(\xi, \eta)) = 0$ if and only if $u = v = w(\xi, \eta)$, for any $u, v \in E$, $w(\xi, \eta) \in (\mathbb{R} \times \mathbb{R}; E)$;

(C2) $0 < L(u, u, w(\xi, \eta)) \leq L(u, v, w(\xi, \eta))$, for any $u, v \in E$, $w(\xi, \eta) \in (\mathbb{R} \times \mathbb{R}; E)$, $w \neq v$;

(C3) $L(u, v, w(\xi, \eta)) = L(u, w(\xi, \eta), v) = L(v, u, w(\xi, \eta))$

$= L(v, w(\xi, \eta), u) = L(w(\xi, \eta), u, v) = L(w(\xi, \eta), v, u),$

for any $u, v, w \in E$, $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$;

(C4) $L(u, v, w(\xi, \eta)) = L(u, t, t) + L(t, w, w(\xi, \eta));$

$L(u, v, w(\xi, \eta)) \leq L(u, v, t) + L(t, v, w(\xi, \eta));$

$L(u, v, w(\xi, \eta)) \leq L(u, t, w(\xi, \eta)) + L(t, v, w(\xi, \eta)),$

for any $u, v, w, t \in E$ and $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$. (rectangle inequality).

Then the function L is said a *generalized metric*, or, more specifically, a *LG*-metric on *E*, and the pair (E, LG) is said to be a *LG-metric space*.

Noticing that every *LG*-metric on *E* induces a metric d_{LG} on *E* defined by

$$d_{LG}(u, v) = 3\int_0^{L_1+L_2} d\zeta \text{ for all } u, v \in E, \tag{1.1}$$

where $L_1 = L(u, u, v)$, $L_2 = L(v, v, u)$.

In order to get better idea about the subject, we would give the following examples of LG-metrics.

Example 1.2. Let (E, LG) be a metric space. The function $L : E \times E \times E \to [0, +\infty)$, defined by

$$L(u, v, w(\xi, \eta)) = \sup\{d_{LG}(u, v), d_{LG}(v, w(\xi, \eta)), d_{LG}(w(\xi, \eta), u)\} \tag{1.2}$$

for any $u, v \in E$, $w(\xi, \eta) \in (\mathbb{R} \times \mathbb{R}; E)$.

Example 1.3. Let $E = \mathbb{R} = (-\infty, +\infty)$. The function $L : E \times E \times E \to [0, +\infty)$ defined by

$$L(u, v, w(\xi, \eta)) = |u - w(\xi, \eta)| + |v - w(\xi, \eta)| + |u - v| \tag{1.3}$$

for any $u, v \in E$, $w(\xi, \eta) \in (\mathbb{R} \times \mathbb{R}; E)$ is a LG-metric on E.

Example 1.4. Let (E, LG) be a LG-metric space and $\{w(\xi_n, \eta_n)\}_{n=1}^{\infty}$ be a sequence of points of E. We call that $\{w(\xi_n, \eta_n)\}_{n=1}^{\infty}$ is *LG-convergent* to $w^* \in E$ if

$$\lim_{n, m \to \infty} L(w^*, w(\xi_n, \eta_n), w(\xi_m, \eta_m)) = 0 \tag{1.4}$$

that is, for any $\varepsilon > 0$, there exists a positive integer number $N \in \mathbb{N}$ such that

$$L(w^*, w(\xi_n, \eta_n), w(\xi_m, \eta_m)) < \varepsilon$$

for any $n, m \geq \mathbb{N}$. We say that w^* is the *limit* of the sequence and denote

$$w(\xi_n, \eta_n) \to w(\xi_0, \eta_0) \triangleq w^* \text{ or } \lim_{m \to \infty} w(\xi_m, \eta_m) = w(\xi_0, \eta_0) \triangleq w^*.$$

Proposition 1.5. *Let* (E, LG) *be a LG-metric space. Then the following conditions are equivalent*:

(i) $\{w(\xi_n, \eta_n)\}_{n=1}^{\infty}$ *is LG-convergent to* $w(\xi_0, \eta_0) \triangleq w^*$;

(ii) $\lim_{n\to\infty} L(w(\xi_n, \eta_n), w(\xi_n, \eta_n), w^*) = 0$;

(iii) $\lim_{n\to\infty} L(w(\xi_n, \eta_n), w^*, w^*) = 0$;

(iv) $\lim_{n, m\to\infty} L(w(\xi_n, \eta_n), w(\xi_m, \eta_m), w^*) = 0$.

Remark 1.1. If $w(\xi_n, \eta_n) = x_n$, then we would obtain the same properties as some papers (see [2-15]).

Definition 1.6. Let (E, LG) be a LG-metric space. A sequence $\{w(\xi_n, \eta_n)\}_{n=1}^{\infty} \subset E$ is said to be a *LG-Cauchy sequence* if

$$\lim_{m, n, s\to\infty} L(w(\xi_n, \eta_n), w(\xi_m, \eta_m), w(\xi_s, \eta_s)) = 0,$$

that is, for any $\varepsilon > 0$, there exists a positive integer number $N \in \mathbb{N}$ such that

$$L(w(\xi_n, \eta_n), w(\xi_m, \eta_m), w(\xi_s, \eta_s)) < \varepsilon \text{ for all } m, n, s \geq \mathbb{N}.$$

Definition 1.7. Let (E, LG) be a LG-metric space. Then the following conclusions are equivalent:

(1) the sequence $\{w(\xi_n, \eta_n)\}_{n=1}^{\infty} \subset E$ is LG-Cauchy;

(2) for any $\varepsilon > 0$, there exists a positive integer number $N \in \mathbb{N}$ such that

$$L(w(\xi_n, \eta_n), w(\xi_m, \eta_m), w(\xi_m, \eta_m)) < \varepsilon \text{ for all } m, n \geq \mathbb{N}.$$

Definition 1.8. A LG-metric space (E, LG) is called *LG-complete* if every LG-Cauchy sequence is LG-convergent in (E, LG).

Definition 1.9. Let (E, LG) be a LG-metric space. A mapping $\mathfrak{K} : E \times E \times E \to E$ is called *LG-completely continuous* if $\mathfrak{K}$ is compact and LG-metric continuous.

Definition 1.10. Let (E, LG) be a LG-metric space. A mapping $\mathfrak{Q} : E \times E \times E \to E$ is said to be *LG-metric continuous* if for any three LG-convergent sequences $\{u_n\}_{n=1}^{+\infty} \subset E$, $\{v_n\}_{n=1}^{+\infty} \subset E$, $\{w(\xi_n, \eta_n)\}_{n=1}^{+\infty} \subset E$ satisfying $\lim\limits_{n\to+\infty} u_n = u^*$, $\lim\limits_{n\to+\infty} v_n = v^*$, $\lim\limits_{n\to+\infty} w(\xi_n, \eta_n) = w^*$, respectively, such that

$$\lim_{n\to+\infty} \mathfrak{Q}(u_n, v_n, w(\xi_n, \eta_n)) = \mathfrak{Q}(u^*, v^*, w^*), \text{ for any } u^*, v^*, w^* \in E.$$

Notice that each LG-metric on E induces a topology $\eth_{LG}$ on E. For any $u \in E$ and $\rho > 0$, $\mathfrak{B}_{LG}(u, \rho) = \{u \in E \,|\, L(u, w(\xi, \eta), w(\xi, \eta)) < \rho, \quad w \in E, (\xi, \eta) \in \mathbb{R} \times \mathbb{R}\}$ is a family of open LG-balls in E. A nonempty set $M \subset E$ is LG-closed in the LG-metric space (E, LG) if $\overline{M} = M$. We see that

$$u \in \overline{M} = \mathfrak{B}_{LG}(u, \rho) \cap M \neq \phi, \text{ for all } \rho > 0. \tag{1.5}$$

Definition 1.11. Let (E, LG) be a LG-metric space and M be a nonempty subset of E. Then M is LG-closed if for any LG-convergent sequence. A mapping $\mathfrak{Q} : E \times E \times E \to E$ is said to be *LG-metric continuous* if for any LG-convergent sequences $\{w(\xi_n, \eta_n)\}_{n=1}^{+\infty} \subset M$ with limit w^*, one has $w^* \in M$.

Motivated and inspired by above excellent articles, we would like to overcome the difficulties caused by compactness, and construct a new contracting operator as well present a new technique to establish the existence and uniqueness of fixed point. In this work, we extend, generalize, improve, enrich, boost, fortify, heighten, promote and enhance the above mentioned fixed points results of nonlinear contraction mapping in partially ordered LG-metric spaces under some weaker conditions. We also present some examples to demonstrate our results. Firstly, we give some new definitions and fixed point results in a *LG*-complete *LG*-metric space. Then the aim of the article is also to propose an original criterion that a weakly regular cone is a normal cone on *LG*-complete *LG*-metric spaces and prove the corresponding results. We should address here that our new results extend and complement some known results.

The rest of the article is organized as follows. In Section 2, some elementary and new definitions are introduced. Then the main results and proofs are presented in Section 3. Some examples are given to demonstrate the application of our main results, and some discussion is provided in Section 4.

2. Preliminaries

In this section, we consider a few of elementary definitions from the asymmetric topology and the order metric space, which are necessary for a good understanding of the work below.

The following definitions and results (cf. [18, 26, 29]) which is studied a comparison result about the *LG*-convergent and some notions of *LG*-metric space. Now let us go to see some corresponding concepts and results of *LG*-metric spaces. In the following paper, we denote $E = (E, LG)$, $J = [a, b] \subset R^1$.

Lemma 2.1. *Let* $E \subset R^3$ *be a subset. If there exists a function* $\varphi : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ *such that, for any* $\sigma > 0$, $d(\sigma, \sigma) > \sigma$ *and*

(i) $\sup\{d(u, v) : (u, v) \in (0, \sigma) \times (0, \sigma)\} \geq d(\sigma, \sigma)$ *and*

(ii) $(u, v, w) \in E$ *and* $(u, v) \in (0, d(\sigma, \sigma)) \times (0, d(\sigma, \sigma))$ *imply* $w \in (0, \sigma)$, *then there exist functions* $g, \xi : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ *such that, for any* $\sigma > 0$, $g(\sigma, \sigma) > \sigma$, $\xi(\sigma, \sigma) < \sigma$, *and* $(u, v, w) \in E$ *with* $(u, v) \in (0, g(\sigma, \sigma)) \times (0, g(\sigma, \sigma))$ *imply* $w < \xi(\sigma, \sigma)$.

Lemma 2.2. *Let* $E \subset R^3$ *be a subset. If there exists an upper semi-continuous function* $\Psi : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ *such that* Ψ *is non-decreasing* $\Psi(u, v) < u$ *for* $(u, v) \in (0, +\infty) \times (0, +\infty)$, *and* $(u, v, w) \in E$ *implies* $w < \Psi(u, v)$, *then there exists a lower semi-continuous function* $\sigma : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ *such that* σ *is non-decreasing, for any* $\rho > 0,\ d(\rho, \rho) > \rho$, *and* $(u, v, w) \in E$ *with*

$$(u, v) \in (0, d(\rho, \rho)) \times (0, d(\rho, \rho))$$

imply $w < \rho$.

Let $E = C[0, 1]$ be endowed with norm $\| z \| = \max_{t \in [0, 1]} | z(t) |,\ z \in E$. Then E is a Banach space.

Theorem 2.3. *Let K be a self-map of Banach space* $(E, \| \cdot \|)$ *and* $b > 0$. *Assume that*

(A_1) $\| Ku - Kv \| < \| u - v \|$, *for any u, v with* $0 < \| u - v \| < b$;

(A_2) *For any* $0 < \eta < b$, *there exists a real number* $\sigma,\ 0 < \sigma < b - \eta$ *such that for any* $u, v \in E, \| u - v \| < \eta + \sigma$ *implies* $\| Ku - Kv \| \leq \eta$. *Then we have*

(1°) *If there exists* $u_0 \in E$ *such that* $\| u_0 - Ku_0 \| < b$, *then the sequence* $\{K^n u_0\}$ *converges to a fixed point of K.*

(2°) *If* $(E, \| \cdot \|)$ *is set of b-chainable points, then K has a unique fixed point* u^*, *and* $K^n u \to u^*$ *for any* $u \in E$.

Theorem 2.4. *Suppose that E is a complete LG-metric space and* $\mathfrak{P} \subset E$ *is a cone. If* $\mathfrak{P}$ *is weakly regular, then* $\mathfrak{P}$ *is a normal cone.*

3. Main Results

Inspired by many nice ideas in [6-12, 16] and [19-29] with some extensive research of the subject (see, e.g., [3-23]). In what follows, we would study weak φ-contractions on *LG*-metric spaces.

Theorem 3.1. *Suppose that* $\{F_n\}_{n=1}^{\infty} \subset E$ *is a family of nonempty LG-closed subsets sequence of a LG-complete LG-metric space* (E, L) *with* $\prod = \bigcup_{n=1}^{\infty} F_n$ *and for all* $n \neq m$, $F_n \cap F_m \neq \emptyset$. *Let* $K : \prod \to \prod$ *be a map satisfying*

$$F_{n+1} \supset KF_n \supset KF_{n+1}, \quad n = 1, 2, \ldots \tag{3.1}$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ *be a positive continuous function with* $\varphi(0) = 0$ *and* $\varphi(\alpha) > 0$ *for* $\alpha > 0$ *such that*

$$L(Ku, Kv, Kw(\xi, \eta)) \leq P(u, v, w(\xi, \eta)) - \varphi(P(u, v, w(\xi, \eta))) \tag{3.2}$$

for all $u, v, w \in F_n$, $(\xi, \eta) \in F_n \times F_n$, $n = 1, 2, \ldots$, *where*

$$P(u, v, w(\xi, \eta))$$

$$= \sup_{(\xi, \eta) \in F_n \times F_n} \{L(u, v, w(\xi, \eta)), L(u, Ku, Ku), L(v, Kv, Kv),$$

$$L(w(\xi, \eta), Kw(\xi, \eta), Kw(\xi, \eta))\}$$

for any $u, v, w \in F_n \times F_n$, $n = 1, 2, \ldots$. *Then K has a unique fixed point in* $\bigcap_{n=1}^{\infty} F_n$.

Proof. Since F_1 is a nonempty subset of *E*, choose an arbitrary $u_0 \in F_1$ and define the sequence $\{u_n\}$ as

$$u_n = Ku_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{3.3}$$

It follows from (3.1) that $u_1 = Ku_0 \in F_2$, $u_2 = Ku_1 \in F_3, \ldots$. If there exists $n^* \in \mathbb{N}$ such that $u_{n^*} = u_{n^*+1}$, we see that $Ku_{n^*} = u_{n^*+1} = u_{n^*}$. Then, we easily know that u_{n^*} is a fixed point of *K*. Without loss of generality, we assume that $u_{n+1} = u_n$ for any $n \in \mathbb{N}$. Thus, for any $n \in \mathbb{N}$, by making use of (3.2), we know that

$$L(Ku_n, Ku_{n+1}, Ku_{n+1}) = L(u_{n+1}, u_{n+2}, u_{n+2})$$
$$\leq P(u_n, u_{n+1}, u_{n+1}) - \varphi(P(u_n, u_{n+1}, u_{n+1})), \quad (3.4)$$

where

$$P(u_n, u_{n+1}, u_{n+1}) = \max\{L(u_n, u_{n+1}, u_{n+1}), L(u_n, Ku_n, Ku_n),$$
$$L(u_{n+1}, Ku_{n+1}, Ku_{n+1}), L(u_{n+1}, u_{n+2}, u_{n+2})\}$$
$$= \max\{L(u_n, u_{n+1}, u_{n+1}), L(u_{n+1}, u_{n+2}, u_{n+2})\}. \quad (3.5)$$

If

$$L(u_n, u_{n+1}, u_{n+1}) \leq L(u_{n+1}, u_{n+2}, u_{n+2}),$$

then

$$P(u_n, u_{n+1}, u_{n+1}) \leq L(u_{n+1}, u_{n+2}, u_{n+2}).$$

Hence, (3.4) induces that

$$L(u_{n+1}, u_{n+2}, u_{n+2}) \leq L(u_{n+1}, u_{n+2}, u_{n+2}) - \varphi(L(u_{n+1}, u_{n+2}, u_{n+2})) \quad (3.6)$$

which contradicts the assumption $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Consequently we see that $L(u_n, u_{n+1}, u_{n+1}) > L(u_{n+1}, u_{n+2}, u_{n+2})$. Thus $P(u_n, u_{n+1}, u_{n+1}) = L(u_n, u_{n+1}, u_{n+1})$. From (3.4), we get that

$$L(u_{n+1}, u_{n+2}, u_{n+2}) \leq L(u_n, u_{n+1}, u_{n+1}) - \varphi(L(u_n, u_{n+1}, u_{n+1}))$$
$$= L(u_n, u_{n+1}, u_{n+1}). \quad (3.7)$$

Thus, we see that the sequence $\{L(u_n, u_{n+1}, u_{n+1})\}$ is a nonnegative nonincreasing sequence which converges to $D \geq 0$. Letting $n \to \infty$ in (3.7), we obtain

$$D \leq D - \varphi(D). \quad (3.8)$$

It follows from that $\varphi(D) = 0.$ So $D = 0.$ Thus, we have

$$\lim_{n\to\infty} L(u_n, u_{n+1}, u_{n+1}) = 0. \tag{3.9}$$

We claim that $\{u_n\}_{n=1}^{\infty}$ is a LG-Cauchy sequence in $(E, L).$ If not, there exists $\nu_0 > 0$ and corresponding subsequences $\{n(s)\}_{s=1}^{\infty}$ and $\{m(s)\}_{s=1}^{\infty}$ of $\mathbb{N}$ satisfying $n(s) > m(s) > s$ for which

$$L(u_{m(s)}, u_{n(s)}, u_{n(s)}) \geq \nu_0, \tag{3.10}$$

where $n(s)$ is chosen as the smallest integer satisfying (3.10), that is

$$L(u_{m(s)}, u_{n(s)-1}, u_{n(s)-1}) < \nu_0. \tag{3.11}$$

By making use of (3.10) and (3.11) with the rectangle inequality, it is easy to see that

$$\begin{aligned}\nu_0 &\leq L(u_{m(s)}, u_{n(s)}, u_{n(s)})\\ &\leq L(u_{m(s)}, u_{n(s)-1}, u_{n(s)-1}) + L(u_{n(s)-1}, u_{n(s)}, u_{n(s)})\\ &< \nu_0 + L(u_{n(s)-1}, u_{n(s)}, u_{n(s)}).\end{aligned} \tag{3.12}$$

Taking limit as $n \to \infty$ in (3.12) and applying (3.9) we get that

$$\lim_{n\to\infty} L(u_{m(s)}, u_{n(s)}, u_{n(s)}) = \nu_0. \tag{3.13}$$

Observe that for every $s \in \mathbb{N},$ there exists $r(s)$ satisfying $0 \leq r(s) \leq i$ such that

$$n(s) - m(s) + r(s) = m(i). \tag{3.14}$$

Therefore, for large enough values of s we get $h(s) = m(s) - r(s) > 0$ and $u_{h(s)}$ with $u_{n(s)}$ lie in the consecutive sets F_j and $F_{j+1},$ respectively, for some $0 \leq j \leq i,$ we substitute $u = u_{h(s)}$ and $v = w(\xi, \eta) = u_{n(s)}$ in (3.2) to get that

$$L(Ku_{h(s)}, Ku_{n(s)}, Ku_{n(s)})$$

$$\leq P(u_{h(s)}, u_{n(s)}, u_{n(s)}) - \varphi(P(u_{h(s)}, u_{n(s)}, u_{n(s)})), \tag{3.15}$$

where

$$P(u_{h(s)}, u_{n(s)}, u_{n(s)})$$

$$= \max\{L(u_{h(s)}, u_{n(s)}, u_{n(s)}), L(u_{h(s)}, u_{n(s)+1}, u_{n(s)+1}),$$

$$L(u_{n(s)}, u_{n(s)+1}, u_{n(s)+1})\}. \tag{3.16}$$

Employing rectangle inequality repeatedly we obtain that

$$L(u_{h(s)}, u_{n(s)}, u_{n(s)})$$

$$\leq L(u_{h(s)}, u_{h(s)+1}, u_{h(s)+1}) + L(u_{h(s)+1}, u_{n(s)}, u_{n(s)})$$

$$\leq L(u_{h(s)}, u_{h(s)+1}, u_{h(s)+1}) + L(u_{h(s)+1}, u_{h(s)+2}, u_{h(s)+2})$$

$$+ L(u_{h(s)+2}, u_{n(s)}, u_{n(s)})$$

$$\leq \cdots \leq \sum_{j=h}^{m-1} [L(u_{j(s)}, u_{j(s)+1}, u_{j(s)+1})] + L(u_{m(s)}, u_{n(s)}, u_{n(s)}) \tag{3.17}$$

or equivalently

$$0 \leq L(u_{h(s)}, u_{n(s)}, u_{n(s)}) - L(u_{m(s)}, u_{n(s)}, u_{n(s)})$$

$$\leq \sum_{j=h}^{m-1} L(u_{j(s)}, u_{j(s)+1}, u_{j(s)+1}). \tag{3.18}$$

Notice that the sum on the right-hand side of inequality (3.18) consists of finite $r - 1 \leq i$ number of terms, and applying (3.9) each term of this sum convergent to 0 as $s \to \infty$. Therefore

$$\lim_{s\to\infty} L(u_{h(s)}, u_{n(s)}, u_{n(s)}) = \lim_{s\to\infty} L(u_{m(s)}, u_{n(s)}, u_{n(s)}) = \nu_0. \tag{3.19}$$

By making use of rectangle inequality, we get

$$
\begin{aligned}
0 &\le L(u_{h(s)+1}, u_{n(s)+1}, u_{n(s)+1}) \\
&\le L(u_{h(s)+1}, u_{n(s)}, u_{n(s)}) + L(u_{n(s)}, u_{n(s)+1}, u_{n(s)+1}) \\
&\le L(u_{h(s)+1}, u_{h(s)}, u_{h(s)}) + L(u_{h(s)}, u_{n(s)}, u_{n(s)}) \\
&\quad + L(u_{n(s)}, u_{n(s)+1}, u_{n(s)+1}) \qquad (3.20)
\end{aligned}
$$

from which we obtain

$$\lim_{s\to\infty} L(u_{n(s)+1}, u_{n(s)+1}, u_{n(s)+1}) = \nu_0. \qquad (3.21)$$

Letting $s \to \infty$ and employing (3.9), (3.15) and (3.19) with (3.21) we have

$$\nu_0 \le \max\{\nu_0, 0, 0\} - \varphi(\max\{\nu_0, 0, 0\}) = \nu_0 - \varphi(\nu_0) \qquad (3.22)$$

and hence $\varphi(\nu_0) = 0$. We claim that $\nu_0 = 0$ which contradicts the assumption that u_n is not LG-Cauchy. Thus, the sequence $\{u_n\}$ is LG-Cauchy, and since (E, L) is LG-complete space; it is LG-convergent to a limit, call $z \in E$. It can be easily gotten that $z \in \bigcap_{s=1}^{i} F_s$. Since $u_0 \in F_1$, the subsequence $\{u_{m(n-1)}\}_{n=1}^{\infty} \subset F_1$, the subsequence $\{u_{mn-1}\}_{n=1}^{\infty} \subset F_m$. All the m subsequences are LG-convergent in the LG-closed sets F_s and therefore, they all convergent to the same limit $z \in \bigcap_{s=1}^{i} F_s$.

To show that the limit of the Picard sequence is the fixed point of K, that is, $z = Kz$. Due to (3.4) with $u = u_n$, $v = w(\xi, \eta) = z$. This induces to

$$L(Ku_n, Kz, Kz) = P(u_n, z, z) - \varphi(P(u_n, z, z)), \qquad (3.23)$$

where

$$P(u_n, z, z) = \max\{L(u_n, z, z), L(u_n, u_{n+1}, u_{n+1}), L(z, Kz, Kz)\}. \qquad (3.24)$$

Letting $n \to \infty$, we have

$$L(z, Kz, Kz) = L(z, Kz, Kz) - \varphi(L(z, Kz, Kz)). \tag{3.25}$$

Thus $\varphi(L(z, Kz, Kz)) = 0$ and hence, $L(z, Kz, Kz) = 0$, that is $z = Kz$.

Finally, we prove that the fixed point is unique. Assume that $y \in E$ is another fixed point of K such that $y \neq z$. Then, since both y and z belong to $\bigcap_{s=1}^{i} F_s$, we set $u = y$ and $v = w(\xi, \eta) = z$ in (3.4) which yields

$$L(y, z, z) = P(y, z, z) - \varphi(P(y, z, z)), \tag{3.26}$$

where

$$P(y, z, z) = \max\{L(y, z, z), L(y, Ky, Ky), L(z, Kz, Kz)\} = L(y, z, z). \tag{3.27}$$

Then (3.26) becomes

$$L(y, z, z) \leq L(y, z, z) - \varphi(L(y, z, z)) \tag{3.28}$$

and clearly, $L(y, z, z) = 0$, so we claim that $y = z$, that is, the fixed point of K is unique. This completes the proof of the theorem. □

4. Examples and Discussions on the Conditions of Theorems

To illustrate the φ-contractions on LG-metric spaces, we give the following example.

Example 4.1. Let $E = \{(x, y) | x^2 + y^2 \leq 1\}$ and $K : E \times E \to E$ be given as $Kx = -\frac{1}{4}x$. Let

$$A = \{(x, y) | x^2 + y^2 \leq 1, -1 \leq x \leq 0, -1 \leq y \leq 1\}$$

and

$$B = \{(x, y) | x^2 + y^2 \leq 1, 0 \leq x \leq 1, -1 \leq y \leq 1\}.$$

Define the function $L : E \times E \times E \to [0, +\infty)$ as

$$L(x, y, z) = [(x - y)^2 + (y - z)^2 + (z - x)^2]^{\frac{1}{2}}. \tag{4.1}$$

Clearly, the function L is a LG-metric on E. Define also $\varphi : [0, +\infty) \to [0, +\infty)$ as $\varphi(t) = \lambda^2 t$, $\lambda \in \left(0, \frac{\sqrt{15}}{4}\right)$. Obviously, the map K has a unique fixed point $(x^*, y^*) = (0, 0) \in A \cap B = \{(x, y) | x = 0, -1 \le y \le 1\}$.

It can be easily shown that the map K satisfies the condition (3.2). Indeed, notice that

$$\begin{aligned} L(Kx, Ky, Kz) &= \sqrt{(Kx - Ky)^2 + (Ky - Kz)^2 + (Kz - Kx)^2} \\ &= \frac{1}{16}\sqrt{(x - y)^2 + (y - z)^2 + (z - x)^2} \\ &= \frac{1}{16} L(x, y, z), \end{aligned}$$

$$\begin{aligned} P(x, y, z) = \max\{&\sqrt{(x - y)^2 + (y - z)^2 + (z - x)^2}, \\ &L(x, Kx, Kx), L(y, Ky, Ky), L(z, Kz, Kz)\}, \end{aligned}$$

$$L(x, Kx, Kx) = \frac{5\sqrt{2}}{4}|x|, \; L(y, Ky, Ky) = \frac{5\sqrt{2}}{4}|y|,$$

$$L(z, Kz, Kz) = \frac{5\sqrt{2}}{4}|z|,$$

$$P(x, y, z)$$

$$= \max\left\{\sqrt{(x - y)^2 + (y - z)^2 + (z - x)^2}, \frac{5\sqrt{2}}{4}|x|, \frac{5\sqrt{2}}{4}|y|, \frac{5\sqrt{2}}{4}|z|\right\},$$

$$\varphi(P(x, y, z)) = \lambda^2 P(x, y, z).$$

Then

$$P(x, y, y) = \varphi(P(x, y, z)) = (1 - \lambda^2) P(x, y, z) \tag{4.2}$$

and clearly

$$L(Kx, Ky, Kz) = \frac{1}{16}\sqrt{(x-y)^2 + (y-z)^2 + (z-x)^2}$$

$$< (1-\lambda^2)\sqrt{(x-y)^2 + (y-z)^2 + (z-x)^2}$$

$$\leq (1-\lambda^2) P(x, y, z).$$

Hence, K has a unique fixed point by Theorem 3.1. □

It is also found that the differential equations boundary value problems and Cauchy problems provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. With these features, various differential equations models become apparent in modeling electrical and mechanical properties of real materials. It is necessary to present the fixed point theory in a complete metric space.

We discuss the conditions in this paper. It is easy to see that the functions satisfying the conditions of the theorems are rather wide. For example, we can obtain the following corollary:

Corollary 4.2. *Let E be a complete LG-metric space and $K : E \to E$ be a φ-contractive single-valued operator with $L(w(\xi, \eta), w(\xi, \eta), Ku) \leq P(w(\xi, \eta), w(\xi, \eta), Ku)$ for $u, v, w \in E$, $(\xi, \eta) \in E$ implies*

$$L(Ku, Ku, Kv) \leq P(u, u, v), \tag{4.3}$$

where

$$L(u, u, v) = \max\{L(u, u, v), L(u, u, Ku), L(v, v, Kv)\}.$$

Then K has a fixed point.

In Section 2, we have shown how to construct the weakly regular cone. We also have provided the simplest way and technique to prove the original conclusion on the *LG*-metric space. In Section 3, by using monotone theory, the existence and uniqueness of fixed point theory is established. First we transform the original metric space to the complete metric space by constructing each *LG*-Cauchy sequence which is convergent. Then we compute the approximate subsequence and get the unique limit point on the *LG*-metric space.

Remark 4.3. From above discussions, it is clear that our results improve and extend the results in [6, 11] and [12] with [23].

References

[1] M. Abbas, T. Nazir and B. Rhoades, Fixed points of multivalued mapping satisfying ciric type contractive conditions in *G*-metric spaces, Hacettepe J. Math. Stat. 42(1) (2013), 21-29.

[2] Ravi P. Agarwal, E. Karapinar and A.-F. Roldán-López-de-Hierro, Fixed point theorem in quasi-metric space and applications to multidimensional fixed point theorem on *G*-metric *s*-paces, J. Nonlinear Convex Anal. 16(9) (2015), 1787-1816.

[3] Ravi P. Agarwal, E. Karapinar and D. O'Regan, Fixed point theory in metric type spaces, Springer Int. Publishing, 2016.

[4] Ravi P. Agarwal and E. Karapinar, Remarks on some coupled fixed point theorems in *G*-metric spaces, Fixed Point Theory Appl. 2013(2) (2013), 1-33.

[5] Ravi P. Agarwal, E. Karapinar, D. O'Regan and A.-F. Roldán-López-de-Hierro, Further fixed point results on *G*-metric spaces, Fixed Point Theory in Metric Type Spaces 2015 (2015), 107-173.

[6] L. Ćirić and V. Lakshmikantham, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70(12) (2009), 4341-4349.

[7] P. Charoensawan and C. Thangthong, On coupled coincidence point theorems on partially ordered *G*-metric spaces without mixed *g*-monotone, J. Inequalities Appl. 2014(150) (2014), 1-17.

[8] B. S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Model. 54(1-2) (2011), 73-79.

[9] M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc. 37 (1962), 74-79.

[10] Y. U. Gaba, An order theoretic approach in fixed point theory, Math. Sci. 8(3) (2014), 87-93.

[11] Y. U. Gaba, Fixed point theorems in *G*-metric spaces, J. Math. Anal. Appl. 455 (2017), 528-537.

[12] T. Gnana-Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(7) (2006), 1379-1393.

[13] N. Hussain, E. Karapinar, P. Salimi and P. Vetro, Fixed point results for G^m-Meir-Keeler contractive and G-(α, ψ)-Meir-Keeler contractive mappings, Fixed Point Theory Appl. 2013(1) (2013), 1-14.

[14] J. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, J. Math. Anal. Appl. 194(1) (1995), 293-303.

[15] E. Karapinar and Ravi P. Agarwal, Further fixed point results on *G*-metric spaces, Fixed Point Theory Appl. 2013(154) (2013), 1-14.

[16] E. Karapinar, B. Yildiz-Ulus and İ. M. Erhan, Cyclic contractions on *G*-metric spaces, Abstract Applied Anal. 2012(1) (2012), 1-15. Article ID 182947. DOI: 10.1155/2012/182947

[17] E. Karapinar, Billûr Kaymakcalan and K. Tas, On coupled fixed point theorems on partially ordered *G*-metric spaces, J. Inequalities Appl. 200 (2012), 1-13. DOI: 10.1186/1687-1812-2012-174

[18] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7(2) (2006), 289-297.

[19] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete *G*-metric spaces, Fixed Point Theory Appl. 2009(1) (2009), 1-10. DOI: 10.1155/2009/917175

[20] Z. Mustafa, F. Awawdeh and W. Shatanawi, Fixed point theorem for expansive mappings in *G*-metric spaces, Int. J. Contemp. Math. Sci. 5(50) (2010), 2463-2472.

[21] Z. Mustafa and H. Obiedat, A fixed point theorem of Reich in *G*-metric spaces, Cubo A Math. J. 12(1) (2019), 83-93. DOI: 10.4067/S0719-06462010000100008

[22] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete *G*-metric spaces, Fixed Point Theory Appl. 2008(2) (2008), 1-12. DOI: 1155/2008/189870

[23] Z. Mustafa, M. Arshad, S. Khan, J. Ahmad and M. Jaradat, Common fixed points for multivalued mappings in *G*-metric spaces with applications, J. Nonlinear Sci. Appl. 10(5) (2017), 2550-2564. DOI: 10.22436/jnsa.010.05.23

[24] Z. Mustafa, Z. M. Khandagji and W. Shatanawi, Fixed point results on complete *G*-metric spaces, Studia Scientiarum Mathematicarum Hungarica 48(3) (2011), 304-319. DOI: 10.1556/SScMath.48.2011.3.1170

[25] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in *G*-metric spaces, Int. J. Math. Sci. 2009 (2009), 1-10. DOI: 10.1155/2009/283028

[26] O. Popescu, A new type of contractive multivalued operators, Bull. Sci. Math. 137 (2013), 30-44.

[27] B. Samet, C. Vetro and F. Vetro, Remarks on *G*-metric spaces, Int. J. Anal. 2013 (2013), 1-6. DOI: 10.1155/2013/917158

[28] Y. Sun and Chenglin Zhao, Fixed point results for multi-valued mappings in *G*-metric spaces, Dynamic Systems Appl. 30(9) (2021), 1463-1478.

[29] N. Tahat, H. Aydi, E. Karapinar and W. Shatanawi, Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in *G*-metric spaces, Fixed Point Theory Appl. 2012 (2012), 1-9.