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ON THE EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTION FOR RAYLEIGH TYPE p-LAPLACIAN EQUATION

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Abstract

In this paper, we study the existence and uniqueness of periodic solution for Rayleigh type p-Laplacian equation

$$(\phi_p(x'(t)))' + f(t, x'(t)) + g(t, x(t)) = e(t).$$

We prove the existence and uniqueness of periodic solution of the given equation provided that there exist constants $a > 0$, $b > 0$ such

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that $|f(t, s)| \le a|s|^{p-1} + b, \forall(t, s) \in \mathbb{R}^2$ or f is bounded below (or above) and g satisfies the monotonicity condition.

1. Introduction

We consider the existence and uniqueness of periodic solution of the following Rayleigh type p-Laplacian equation:

$$(\phi_p(x'(t)))' + f(t, x'(t)) + g(t, x(t)) = e(t), \tag{1.1}$$

where $p > 1$, $\phi_p : \mathbb{R} \to \mathbb{R}$, $\phi_p(s) = |s|^{p-2}s$ for $s \neq 0$, $\phi_p(0) = 0$, $f \in C(\mathbb{R}^2, \mathbb{R})$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $e \in C(\mathbb{R}, \mathbb{R})$, f, g are T-periodic in the first variable t, and e is T-periodic with $T > 0$.

As is well known, the existence of periodic solutions for p-Laplacian differential equations has been studied extensively in literature by using various different mathematical techniques such as phase-plane analysis, continuation theorems, upper and lower solutions methods and variational methods (see [1-5, 7, 8, 10] and the references therein). In [1], Liu studied the periodic problems for the Liénard type p-Laplacian equation

$$(\phi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t). \tag{1.2}$$

Assume the following conditions hold:

(A_1) $g \in C^1(\mathbb{R})$, $g'(x) < 0$, for all $x \in \mathbb{R}$.

(A_2) There exists a constant $d > 0$ such that $x(g(t, x) - e(t)) < 0$, for all $|x| > d$ and $t \in \mathbb{R}$.

It was proved in [1] that equation (1.2) has a unique T-periodic solution. Lately, Wang et al. [7] generalized (A_2) to the condition

(A_2') There exists a constant $d > 0$ such that $x(g(x) - \bar{e}) < 0$, for all $|x| > d$ and $t \in \mathbb{R}$, where $\bar{e} = \frac{1}{T}\int_0^T e(t)dt$.

For other related results, we refer to [3, 9]. In contrast to equation (1.2), equation (1.1) is less studied on the existence and uniqueness of periodic solutions. Since equations (1.1) and (1.2) have different structures, different conditions should be introduced to ensure the existence of periodic solutions for equation (1.1). In [8], Xiong and Shao studied the periodic problem for equation (1.1). Assume that the following assumptions hold:

(B_1) $(x_1 - x_2)(g(t, x_1) - g(t, x_2)) < 0$, for all $t, x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$.

(B_2) There exists a constant $d > 0$ such that

$$g(t, x) - e(t) < 0, \forall x > d, t \in \mathbb{R}, \quad g(x) - e(t) > 0, \forall x \leq 0, t \in \mathbb{R}.$$

(B_3) There exist nonnegative constants m_1, m_2 such that $2^{1-p} m_1 T^p < 1$, and one of the following conditions holds:

(i) $f(t, 0) \equiv 0$, $\forall t \in \mathbb{R}$, $f(t, x) \geq 0$, $\forall (t, x) \in \mathbb{R}^2$, and $g(t, x) \geq -m_1 |x|^{p-1} - m_2$, $\forall t \in \mathbb{R}$, $x \geq 0$;

(ii) $f(t, 0) \equiv 0$, $\forall t \in \mathbb{R}$, $f(t, x) \leq 0$, $\forall (t, x) \in \mathbb{R}^2$, and $g(t, x) \leq m_1 |x|^{p-1} + m_2$, $\forall t \in \mathbb{R}$, $x \leq 0$.

It was proved in [8] that equation (1.1) has a unique positive *T*-periodic solution.

It is noted that the monotonicity condition (B_1) takes the same role as (A_1) in guaranteeing the uniqueness of *T*-periodic solution of related differential equations.

In the present paper, we study the existence and uniqueness of periodic solutions of equation (1.1) under new conditions. On one hand, we assume that g only satisfies the sign condition in dealing with the existence of periodic solutions of equation (1.1). On the other hand, we do not require f to keep a constant sign. In Section 4, we give two examples to illustrate the applications of our results.

Throughout this paper, the usual norm in $L^p(0, T)$, $1 < p < +\infty$, is denoted by $|\cdot|_p$. That is,

$$|x|_p = \left(\int_0^T |x(t)|^p dt\right)^{\frac{1}{p}}, \quad x \in L^p(0, T),$$

and for any continuous T-periodic function $x(t)$, we set

$$|x|_\infty = \max\{|x(t)| : t \in [0, T]\}.$$

Moreover, we set

$$C_T^1 := \{x \in C^1(\mathbb{R}, \mathbb{R}) \mid x(t + T) = x(t), \forall t \in \mathbb{R}\}.$$

2. Preliminary Lemmas

We use a continuation theorem [6] to prove the existence of periodic solutions of equation (1.1). We first restate it here. Consider the boundary value problem

$$(\phi_p(x'))' = h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{2.1}$$

where $h \in C(\mathbb{R}^3, \mathbb{R})$, and h is T-periodic in the first variable.

Lemma 2.1 [6]**.** *Assume that* Ω *is an open bounded set in* C_T^1 *such that the following conditions hold*:

(i) *For each* $\lambda \in (0, 1)$, *the problem*

$$(\phi_p(x'))' = \lambda h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T)$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree of F*:

$$\deg(F, \Omega \cap \mathbb{R}, 0) \neq 0.$$

Then the problem (2.1) *has at least one T-periodic solution on* $\overline{\Omega}$.

Lemma 2.2. *Assume that there exists a constant* $\eta > 0$ *such that*

$$(x_1 - x_2)(g(t, x_1) - g(t, x_2)) < 0, \forall t \in \mathbb{R}, x_1, x_2 \in [-\eta, \eta], x_1 \neq x_2.$$

Then equation (1.1) *has at most one T-periodic solution* $x(t)$ *satisfying* $|x|_\infty \leq \eta$.

We can use the similar method as in [8] to prove Lemma 2.2.

3. Main Results

Theorem 3.1. *Assume that the following conditions are satisfied*:

(h_1) *There exists a constant* $d > 0$ *such that* $x(f(t, 0) + g(t, x) - e(t)) < 0, \forall t \in \mathbb{R}, |x| > d$;

(h_2) $(x_1 - x_2)(g(t, x_1) - g(t, x_2)) < 0$ *for all* $t \in \mathbb{R}, x_1, x_2 \in [-d, d]$, $x_1 \neq x_2$;

(h_3) *f is bounded from below or above*. *That is*, *there exists a constant c such that* $f(t, x) \geq c$ *or* $f(t, x) \leq c, \forall (t, x) \in \mathbb{R}^2$.

Then equation (1.1) *has a unique T-periodic solution*.

Proof. Consider the homotopic equation of equation (1.1):

$$(\phi_p(x'(t)))' + \lambda f(t, x'(t)) + \lambda g(t, x(t)) = \lambda e(t), \quad \lambda \in (0, 1). \tag{3.1}$$

Let $x(t)$ be an arbitrary possible T-periodic solution of equation (3.1). We proceed in three steps:

(1) We first prove that

$$|x|_\infty \leq d, \tag{3.2}$$

where d is defined in (h_1). Let $t_1, t_2 \in [0, T]$ be such that

$$x(t_1) = \max_{t\in[0,T]} x(t), \quad x(t_2) = \min_{t\in[0,T]} x(t).$$

Thus, $x'(t_1) = 0$ and $x'(t_2) = 0$. Now, we prove that $(\phi_p(x'(t_1)))' \le 0$. Assume by contradiction that $(\phi_p(x'(t_1)))' > 0$. Then there exists a sufficiently small positive constant δ such that

$$(\phi_p(x'(t)))' > 0, \quad t \in (t_1 - \delta, t_1 + \delta),$$

which, together with $x'(t_1) = 0$, yields

$$x'(t) > 0, t \in (t_1, t_1 + \delta), \quad x'(t) < 0, t \in (t_1 - \delta, t_1).$$

This contradicts with the fact that $x(t_1)$ is a maximum value. Since

$$(\phi_p(x'(t_1)))' + \lambda f(t_1, x'(t_1)) + \lambda g(t_1, x(t_1)) = \lambda e(t_1),$$

we get $f(t_1, 0) + g(t_1, x(t_1)) - e(t_1) \ge 0$. According to (h_1), we obtain that $x(t_1) \le d$. Similarly, we have that $x(t_2) \ge -d$. Thus, $-d \le x(t_2) \le x(t) \le x(t_1) \le d, \forall t \in [0, T]$. Hence, $|x|_\infty \le d$.

(2) We next prove that there exists a constant $M > 0$ such that

$$|x'|_\infty \le M.$$

Integrating equation (3.1) from 0 to T, we get

$$\int_0^T f(t, x'(t))dt + \int_0^T g(t, x(t))dt = \int_0^T e(t)dt.$$

It follows that

$$\left|\int_0^T f(t, x'(t))dt\right| \le \int_0^T |g(t, x(t))|dt + \int_0^T |e(t)|dt \le (C + |e|_\infty)T, \quad (3.3)$$

with $C = \max\{|g(t, x)| : t \in [0, T], |x| \le d\}$.

Let $f^{+}(t, x) = \max\{0, f(t, x)\}$, $f^{-}(t, x) = \min\{0, f(t, x)\}$. Then $f(t, x) = f^{+}(t, x) + f^{-}(t, x)$. Thus

$$\int_0^T f(t, x'(t))dt = \int_0^T f^{+}(t, x'(t))dt + \int_0^T f^{-}(t, x'(t))dt. \tag{3.4}$$

We now assume that $f(t, x)$ is bounded from below. In this case, $f^{-}(t, x)$ is bounded and then $\int_0^T f^{-}(t, x'(t))dt$ is bounded. It follows from (3.3) and (3.4) that $\int_0^T f^{+}(t, x'(t))dt$ is also bounded. Since

$$\begin{aligned}\int_0^T |f(t, x'(t))|dt &= \int_0^T |f^{+}(t, x'(t)) + f^{-}(t, x'(t))|dt \\ &\le \int_0^T |f^{+}(t, x'(t))|dt + \int_0^T |f^{-}(t, x'(t))|dt \\ &= \int_0^T f^{+}(t, x'(t))dt - \int_0^T f^{-}(t, x'(t))dt,\end{aligned}$$

we obtain that $\int_0^T |f(t, x'(t))|\, dt$ is bounded. Hence, there exists a constant $M_1 > 0$ such that $\int_0^T |f(t, x'(t))|dt \le M_1$.

As $x(t)$ is T-periodic, there exists a $t_0 \in [0, T]$, such that $x'(t_0) = 0$. Then we get that, for $t \in [0, T]$,

$$\begin{aligned}|\phi_p(x'(t))| &= \left|\int_{t_0}^t \phi_p'(x'(t))dt\right| \\ &= \left|\int_{t_0}^t [-\lambda f(t, x'(t)) - \lambda g(t, x(t)) + \lambda e(t)]dt\right| \\ &\le \int_0^T [|f(t, x'(t))| + |g(t, x(t))| + |e(t)|]dt \\ &\le M_1 + T(C + |e|_\infty).\end{aligned}$$

Furthermore, we obtain that

$$|x'(t)| \le [M_1 + T(C + |e|_\infty)]^{\frac{1}{p-1}} := M.$$

Consequently,

$$|x'|_\infty \le M. \tag{3.5}$$

The case when *f* is bounded from above can be treated similarly.

(3) To use Lemma 2.1 to prove the existence of *T*-periodic solution of equation (1.1), we set

$$h(t, x(t), x'(t)) = -f(t, x'(t)) - g(t, x(t)) + e(t).$$

Then equation (3.1) is equivalent to the equation as follows:

$$(\phi_p(x'(t)))' = \lambda h(t, x(t), x'(t)).$$

Choose a constant $K = M + d + 1$ and set

$$\Omega = \{x \in C_T^1 : |x(t)| < K, |x'(t)| < K\}.$$

In what follows, we check that all conditions of Lemma 2.1 are satisfied. It follows from (3.2) and (3.5) that equation (3.1) has no *T*-periodic solutions on $\partial\Omega$. By the definition of *F* in Lemma 2.1, we have

$$F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt = \frac{1}{T}\int_0^T (e(t) - f(t, 0) - g(t, a))dt.$$

Since $K > d$, we obtain from (h_1) that $F(K) > 0$, $F(-K) < 0$. Consequently, $F(a) = 0$ has no solution on $\partial\Omega \cap \mathbb{R} = \{K, -K\}$. Finally, it is easy to check that $\deg(F, \Omega \cap \mathbb{R}, 0) = 1$. Therefore, it follows from Lemma 2.1 that equation (1.1) has at least one *T*-periodic solution. The uniqueness of *T*-periodic solution of equation (1.1) is guaranteed by (h_2) and Lemma 2.1 because we know from the proof of step (1) that each *T*-periodic solution $x(t)$ of equation (1.1) satisfies $|x|_\infty \le d$. This completes the proof.

Suppose that f is not bounded from below or above, but there exist two constants $a > 0,\ b > 0$ such that

(h_4) $|f(t, s)| \leq a|s|^{p-1} + b, \forall t \in [0, T], s \in \mathbb{R}.$

Then, we have the following result.

Theorem 3.2. *Assume that conditions* (h_i) $(i = 1, 2, 4)$ *hold. Then equation* (1.1) *has a unique T-periodic solution.*

Proof. Let $x(t)$ be an arbitrary possible T-periodic solution of equation (3.1). It follows from the proof of Theorem 3.1 that $|x|_\infty \leq d$. Next, we prove that there exists a constant $M' > 0$ such that $|x'|_\infty \leq M'$. Multiplying both sides of equation (3.1) by $x(t)$ and integrating it over the interval $[0, T]$, we get

$$\int_0^T (\phi_p(x'(t)))' x(t)dt + \int_0^T \lambda f(t, x'(t))x(t)dt + \int_0^T \lambda g(t, x(t))x(t)dt$$

$$= \lambda \int_0^T e(t)x(t)dt. \tag{3.6}$$

In terms of (3.6) and (h_4), we have

$$\int_0^T |x'(t)|^p dt$$

$$= -\int_0^T (\phi_p(x'(t)))' x(t)dt$$

$$= \int_0^T \lambda f(t, x'(t))x(t)dt + \int_0^T \lambda g(t, x(t))x(t)dt - \lambda \int_0^T e(t)x(t)dt$$

$$\leq \int_0^T |f(t, x'(t))||x(t)|dt + \int_0^T |g(t, x(t))||x(t)|dt + \int_0^T |e(t)||x(t)|dt$$

$$\leq d\int_0^T |f(t, x'(t))|dt + d\int_0^T |g(t, x(t))|dt + d\int_0^T |e(t)|dt$$

$$\leq ad\int_0^T |x'(t)|^{p-1}dt + Td(b + C + |e|_\infty),$$

where $C = \max\{|g(t, x)| : t \in [0, T], |x| \leq d\}$. By the Hölder's inequality, we get

$$\int_0^T |x'(t)|^p dt \leq adT^{\frac{1}{p}}\left(\int_0^T (|x'|^p)dt\right)^{\frac{1}{q}} + Td(b + C + |e|_\infty).$$

Therefore,

$$|x'|_p^p \leq adT^{\frac{1}{p}}|x'|_p^{p-1} + Td(b + C + |e|_\infty).$$

Consequently, there exists a constant $M_2 > 0$ such that

$$|x'|_p \leq M_2. \tag{3.7}$$

Since $x(0) = x(T)$, there exists $t_0 \in [0, T]$ such that $x'(t_0) = 0$. Thus we have for $t \in [0, T]$,

$$|\phi_p(x'(t))| = \left|\int_{t_0}^t (\phi_p(x'(t)))' dt\right|$$

$$= \left|\int_{t_0}^t [-\lambda f(t, x'(t)) - \lambda g(t, x(t)) + \lambda e(t)]dt\right|$$

$$\leq \int_0^T [|f(t, x'(t))| + |g(t, x(t))| + |e(t)|]dt. \tag{3.8}$$

From (h_4), (3.7) and (3.8), we obtain

$$|\phi_p(x'(t))| \leq \int_0^T |f(t, x'(t))|dt + T(C + |e|_\infty)$$

$$\leq a\int_0^T |x'(t)|^{p-1}dt + T(b + C + |e|_\infty)$$

$$\leq aT^{\frac{1}{p}}\left(\int_0^T |x'(t)|^p dt\right)^{\frac{1}{q}} + T(b + C + |e|_\infty)$$

$$\leq aT^{\frac{1}{p}} M_2^{p-1} + T(b + C + |e|_\infty),$$

which implies that $|x'(t)| \leq [aT^{\frac{1}{p}} M_2^{p-1} + T(b + C + |e|_\infty)]^{\frac{1}{p-1}} := M'$. Hence $|x'|_\infty \leq M'$. The reminder is similar to the proof of Theorem 3.1.

Remark 3.3. Obviously, if (h_2) in Theorems 3.1 and 3.2 is replaced by

$$g_x(t, x) < 0, \quad \forall t \in \mathbb{R}, \quad x \in [-d, d],$$

then still the conclusions of Theorems 3.1 and 3.2 hold.

4. Examples

In this section, we give two examples to illustrate the applications of Theorems 3.1 and 3.2.

Example 4.1. Consider the following Rayleigh type p-Laplacian equation:

$$(\phi_p(x'(t)))' + \sin(t + x'(t)) - (3 + \cos t)(2x + \sin x^2) = \cos t, \tag{4.1}$$

where $p > 1$. Here we have $f(t, x) = \sin(t + x)$, $g(t, x) = -(3 + \cos t) \cdot (2x + \sin x^2)$, $e(t) = \cos t$. Then $f(t, 0) = \sin t$, $g_x(t, x) = -(3 + \cos t) \cdot (2 + 2x\cos x^2)$. It is easy to check that

$$g(t, x) < -2, \forall x > 1, t \in \mathbb{R}, \text{ and } g(t, x) > 2, \forall x < -1, t \in \mathbb{R},$$

and so $x(f(t, 0) + g(t, x) - e(t)) < 0$, for $|x| > 1$, $t \in \mathbb{R}$. Moreover, we have

$$g_x(t, x) < 0, \quad \forall x \in [-1, 1], \quad t \in \mathbb{R},$$

and

$$\liminf_{|x|\to+\infty} g_x(t, x) = -\infty, \quad \limsup_{|x|\to+\infty} g_x(t, x) = +\infty. \tag{4.2}$$

Thus conditions (h_1), (h_2) and (h_3) of Theorem 3.1 are satisfied. Therefore, equation (4.1) has a unique 2π-periodic solution.

Example 4.2. Consider the following Rayleigh type p-Laplacian equation:

$$(\phi_p(x'(t)))' + \cos t \ln(1 + |x'(t)|) - e^{\sin t} x(t) = \sin t, \tag{4.3}$$

where $p > 1$. Here we have $f(t, x) = \cos t \ln(1 + |x|)$, $g(t, x) = -e^{\sin t} x$, $e(t) = \sin t$. Then $f(t, 0) = 0$, $g_x(t, x) = -e^{\sin t}$. It is easy to see that $x(f(t, 0) + g(t, x) - e(t)) = -x(e^{\sin t} x + \sin t) < 0, \quad \forall |x| > e, \quad t \in \mathbb{R}$ and $g_x(t, x) < 0, \forall (t, x) \in \mathbb{R}^2$. Meanwhile, we have

$$\liminf_{|x|\to+\infty} f(t, x) = -\infty, \quad \limsup_{|x|\to+\infty} f(t, x) = +\infty \tag{4.4}$$

and

$$\lim_{|x|\to+\infty} \frac{\ln(1 + |x|)}{|x|^{p-1}} = 0,$$

which imply that there exists a constant $c_1 > 0$ such that

$$|f(t, x)| \le |x|^{p-1} + c_1, \quad \forall (t, x) \in \mathbb{R}^2.$$

Thus conditions (h_1), (h_2) and (h_4) of Theorem 3.2 are satisfied. Therefore, equation (4.3) has a unique 2π-periodic solution.

Remark 4.1. It follows from (4.2) and (4.4) that all results obtained in [8, 10] and the references cited therein cannot be applicable to equations (4.1) and (4.3).

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