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SOLITON SOLUTIONS OF 10th ORDER 2-D BOUSSINESQ EQUATION

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Abstract

The 2-D Boussinesq equation of 10th order is derived from its bilinear form. Its soliton solutions are studied in detail using the Hirota's bilinear method. Since the 2-D Boussinesq equation is not completely integrable, we only obtain its 1-soliton and 2-soliton solutions. The equation is solved by the tanh method to reconstruct the 1-soliton solution obtained by the Hirota's method.

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1. Introduction

The study of nonlinear partial differential equations and analysis of their solution is one of the most fascinating research works. Bounded solutions to such equations, in particular, soliton solutions have gained significance after the pioneering works of Hirota, Hietarinta, Malfliet, Wazwaz and many others in the recent literature. It is well recorded that the Boussinesq equation is used to model the propagation of long waves in shallow water [14]. The 2-D Boussinesq equation governs the propagation of gravity waves on water surface [6]. Soliton solutions can be obtained from many standard methods such as: inverse scattering method, G'/G method, tanh method, tanh-coth method, perturbation method, the Hirota's method and so on [1, 2, 12, 14].

In this paper, we derive the 10th order 2-D Boussinesq equation applying Hirota's direct method which expresses it in a bilinear form. Any Hirota's bilinear form possesses 1-soliton and 2-soliton solutions [4, 5]. We also treat the derived equation with tanh method to get the soliton solution which agrees with the 1-soliton solution of Hirota's method.

2. Derivation of 10th Order 2-D Boussinesq Equation

In this section, we derive the 10th order 2-D Boussinesq equation. For that we make use of Hirota's bilinear form and the works of Hietarinta [4].

The (k, m)-order bilinear partial derivative of a function $f(x, t) \cdot f(x, t)$ is defined in the literature [6] as follows:

$$D_t^k D_x^m (f(x, t) \cdot f(x, t))$$

$$= \left[\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m (f(x, t) \cdot f(x, t))\right]_{x'=x, t'=t}.$$

Consider the two dimensional Boussinesq equation [7]:

$$w_{tt} - w_{xx} - 3(w^2)_{xx} - w_{xxxx} - w_{yy} = 0. \tag{1}$$

Integrating (1) twice with respect to x, we obtain

$$\iint w_{tt} dxdx - w - 3w^2 - w_{xx} - \iint w_{yy} dxdx = 0. \quad (2)$$

Using Hirota's bilinear form [6, 12, 14], (2) can be expressed as

$$(D_t^2 - D_x^2 - D_x^4 - D_y^2)(f \cdot f) = 0, \text{ where } w = 2\frac{\partial^2}{\partial x^2} \log f(x, t, y). \quad (3)$$

Now, we extend the idea of Hirota's bilinear form and consider the tenth order bilinear form

$$P(D) = (D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10} - D_y^2)(f \cdot f) = 0, \quad (4)$$

where α, β and γ are real constants and the relation between w and f is same.

On expanding Hirota D-operators, we obtain

$$\begin{aligned}
&2[(f_{tt}f - f_t^2) - (f_{xx}f - f_x^2) - (ff_{4x} - 4f_x f_{3x} + 3f_{xx}^2) \\
&- \alpha(ff_{6x} - 6f_x f_{5x} + 15 f_{xx} f_{4x} - 10 f_{xxx}^2) \\
&- \beta(ff_{8x} - 8f_x f_{7x} + 28 f_{xx} f_{6x} - 56 f_{3x} f_{5x} + 35 f_{4x}^2) \\
&- \gamma(ff_{10x} - 10 f_x f_{9x} + 45 f_{xx} f_{8x} - 120 f_{3x} f_{7x} + 210 f_{4x} f_{6x} - 126 f_{5x}^2) \\
&- (f_{yy} f - f_y^2)] = 0. \quad (5)
\end{aligned}$$

Using $w = 2\frac{\partial^2}{\partial x^2} \log f(x, t, y)$, (5) has the corresponding nonlinear PDE:

$$\begin{aligned}
&w_{tt} - w_{xx} - 3(w^2)_{xx} - w_{xxxx} - \alpha(w_{6x} + 15ww_{4x} + 30w_x w_{3x} + 15(w_{2x})^2 \\
&\quad + 90ww_x^2 + 45w^2 w_{xx}) - \beta(w_{8x} + 28ww_{6x} + 56w_{5x}w_x + 98w_{2x}w_{4x} \\
&\quad + 70w_{3x}^2 + 210w^2 w_{4x} + 840ww_x w_{3x} + 420ww_{xx}^2 + 420w^3 w_{xx} \\
&\quad + 1260w^2 w_x^2 + 420w_{2x}w_x^2) - \gamma(w_{10x} + 45ww_{8x} + 90w_{7x}w_x
\end{aligned}$$

$$+ 255w_{6x}w_{2x} + 420w_{5x}w_{3x} + 210w_{4x}^2 + 4410ww_{2x}w_{4x}$$

$$+ 3150ww_{3x}^2 + 6300w_xw_{2x}w_{3x} + 1575w_{2x}^3 + 630w^2w_{6x}$$

$$+ 2520ww_xw_{5x} + 1260w_{4x}w_x^2 + 3150w^3w_{4x}$$

$$+ 18900w^2w_xw_{3x} + 9450w^2w_{xx}^2 + 18900ww_x^2w_{2x}$$

$$+ 18900w^3w_x^2 + 4725w^4w_{2x}) - w_{yy} = 0. \tag{6}$$

In the next two sections, solutions are worked out for (4) and (6), respectively.

3. The Hirota's Method for 1-soliton and 2-soliton Solution

In this section, the Hirota's direct method [6] is applied to deduce 1-soliton and 2-soliton solutions of the nonlinear PDE (4). Since the multi-soliton beyond 2 for the 2-D Boussinesq equation is not in Hietarinta's list [3], they are not attempted here.

3.1. One soliton solution

In this subsection, we deduce the one soliton solution to (4).

For 1-soliton solution, we consider the auxiliary function f given by $f = 1 + \varepsilon e^{kx+ly+ct}$, where ε, k, l and c are real constants in the tenth order bilinear equation (4):

$$P(D) = (D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - D_x^{10} - D_y^2)(f \cdot f) = 0.$$

By using the formulas of Hirota operators [14] in (4), we obtain

$$2\varepsilon(c^2 - k^2 - k^4 - \alpha k^6 - \beta k^8 - \gamma k^{10} - l^2)e^{kx+ly-ct} = 0.$$

Therefore, $c = \pm\sqrt{k^2 + k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} + l^2}$.

Solution of (6) is given by

$$w = 2\frac{\partial}{\partial x}\left(\frac{f_x}{f}\right)$$

$$= \frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx + ly - ct}{2}\right), \tag{7}$$

where

$$c = \pm\sqrt{k^2 + k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} + l^2}.$$

3.2. 2-soliton solution

In this subsection, we obtain 2-soliton solution of (4).

To deduce the 2-soliton solution, we choose the auxiliary function f in (4) where

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2,$$

$$f_1 = e^{\theta_1} + e^{\theta_2},$$

$$f_2 = a_{12}e^{\theta_1+\theta_2}, \quad \theta_i = k_i x + l_i y - c_i t, \quad i = 1, 2.$$

k_i, l_i, c_i are real constants and the constant a_{12} is to be determined.

For more details, see [6, 13].

Now, by simplifying f and by equating the coefficients of ε^2 in $P(D)(f \cdot f) = 0$, we obtain

$$P(D)(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0,$$

$$(D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 - \beta D_x^8 - \gamma D_x^{10} - D_y^2)(2(f_2 \cdot 1) + (f_1 \cdot f_1)) = 0,$$

$$2[a_{12}[(c_1 + c_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4 - \alpha(k_1 + k_2)^6 - \beta(k_1 + k_2)^8$$

$$- \gamma(k_1 + k_2)^{10} - (l_1 + l_2)^2] + (c_1 - c_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4$$

$$- \alpha(k_1 - k_2)^6 - \beta(k_1 - k_2)^8 - \gamma(k_1 - k_2)^{10} - (l_1 - l_2)^2] = 0.$$

Also,

$$a_{12} = -\frac{\begin{array}{l}(c_1 - c_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4 - \alpha(k_1 - k_2)^6 \\ - \beta(k_1 - k_2)^8 - \gamma(k_1 - k_2)^{10} - (l_1 - l_2)^2\end{array}}{\begin{array}{l}(c_1 + c_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4 - \alpha(k_1 + k_2)^6 \\ - \beta(k_1 + k_2)^8 - \gamma(k_1 + k_2)^{10} - (l_1 + l_2)^2\end{array}}$$

$$= -\frac{P(k_1 - k_2,\, c_1 - c_2,\, l_1 - l_2)}{P(k_1 + k_2,\, c_1 + c_2,\, l_1 + l_2)}.$$

Now, we obtain f as

$$f = 1 + \varepsilon(e^{\theta_1} + e^{\theta_2}) + \varepsilon^2 a_{12} e^{\theta_1 + \theta_2}$$

$$= 1 + \varepsilon(e^{\theta_1} + e^{\theta_2}) - \varepsilon^2 \frac{P(k_1 - k_2,\, c_1 - c_2,\, l_1 - l_2)}{P(k_1 + k_2,\, c_1 + c_2,\, l_1 + l_2)} e^{\theta_1 + \theta_2}.$$

Using the above f in $w = 2\frac{\partial^2}{\partial x^2} \log f(x, t, y)$, we obtain 2-soliton solution of (4).

4. Tanh Method

In this section, we transform the PDE (6) into an ODE and solve the resulting differential equation using tanh method.

To convert the PDE (6) into an ODE, we introduce a variable $z = kx + ly - ct$ and $w(x, t, y) = W(z)$. Then the PDE (6) reduces to the following ODE:

$$(c^2 - k^2 - l^2)W_{zz} - \{6k^2 WW_{zz} + 6k^2 W_z^2 + k^4 W_{4z}\}$$

$$- \alpha\{15k^4 WW_{4z} + 30k^4 W_z W_{zzz} + 15k^4 W_{2z}^2 + 90k^2 WW_z^2 + 45k^2 W^2 W_{2z}$$

$$+ k^6 W_{6z}\} - \beta\{28k^6 WW_{6z} + 56k^6 W_z W_{5z} + 98k^6 W_{2Z} W_{4z} + 70k^6 W_{3z}^2$$

$$+ 210k^4 W^2 W_{4z} + 840k^4 WW_z W_{3z} + 420k^4 WW_{2z}^2 + 420k^2 W^3 W_{2z}$$

$$+ 1260k^2W^2W_z^2 + 420k^4W_{2z}W_z^2 + k^8W_{8z}\} - \gamma\{k^{10}W_{10z}$$

$$+ 45k^8WW_{8z} + 90k^8W_{7z}W_z + 255k^8W_{6z}W_{2z} + 420k^8W_{5z}W_{3z}$$

$$+ 210k^8W_{4z}^2 + 4410k^6WW_{2z}W_{4z} + 3150k^6WW_{3z}^2 + 6300k^6W_zW_{2z}W_{3z}$$

$$+ 1575k^6W_{2z}^3 + 630k^6W^2W_{6z} + 2520k^6WW_zW_{5z} + 1260k^6W_{4z}W_z^2$$

$$+ 3150k^4W^3W_{4z} + 18900k^4W^2W_zW_{3z} + 9450k^4W^2W_{2z}^2$$

$$+ 18900k^4WW_z^2W_{2z} + 18900k^2W^3W_z^2 + 4725k^2W^4W_{2z}\} = 0, \tag{8}$$

where $W_z = \frac{dW}{dz}$.

The tanh method [9-11] admits the finite series expansion in terms of tanh series given by $W = \sum_{k=0}^{M} a_k Y^k$, where $Y = \tanh \frac{z}{2}$, $M \in \mathbb{N}$ need to be determined.

To find M, we balance the highest order W_{10z} and the exponent of nonlinear term $W^3W_z^2$. This, in turn, implies that $M + 10 = 3M + 2(M + 1)$, so $M = 2$.

On substituting $M = 2$ in the above series, we obtain

$$w(x, t, y) = W(Y) = a_0 + a_1Y + a_2Y^2. \tag{9}$$

Using (9) in the ODE (8), and collecting the coefficients of Y, we obtain $a_1 = 0$. Now, fix $a_0 = \frac{k^2}{2}$ and $a_2 = -\frac{k^2}{2}$, to get the desired solution

$$W = \frac{k^2}{2}\left[1 - \tanh^2\left(\frac{z}{2}\right)\right] = \frac{k^2}{2}(1 - Y^2).$$

In order to examine the value c as in (7), we simplify the terms of the ODE (8) as follows:

$$(c^2 - k^2 - l^2)W_{zz} = \frac{k^2}{4}(c^2 - k^2 - l^2)(3Y^2 - 1)(1 - Y^2),$$

$$6k^2WW_{zz} + 6k^2W_z^2 + k^4W_{4z} = \frac{k^6}{4}(3Y^2 - 1)(1 - Y^2),$$

$$15k^4WW_{4z} + 30k^4W_zW_{zzz} + 15k^4W_{2z}^2 + 90k^2WW_z^2$$

$$+ 45k^2W^2W_{2z} + k^6W_{6z}$$

$$= \frac{k^8}{4}(3Y^2 - 1)(1 - Y^2),$$

$$28k^6WW_{6z} + 56k^6W_zW_{5z} + 98k^6W_{2z}W_{4z} + 70k^6W_{3z}^2$$

$$+ 210k^4W^2W_{4z} + 840k^4WW_zW_{3z} + 420k^4WW_{2z}^2$$

$$+ 420k^2W^3W_{2z} + 1260k^2W^2W_z^2 + 420k^4W_{2z}W_z^2 + k^8W_{8z}$$

$$= \frac{k^{10}}{4}(3Y^2 - 1)(1 - Y^2),$$

$$k^{10}W_{10z} + 45k^8WW_{8z} + 90k^8W_{7z}W_z + 255k^8W_{6z}W_{2z}$$

$$+ 420k^8W_{5z}W_{3z} + 210k^8W_{4z}^2 + 4410k^6WW_{2z}W_{4z}$$

$$+ 3150k^6WW_{3z}^2 + 6300k^6W_zW_{2z}W_{3z} + 1575k^6W_{2z}^3$$

$$+ 630k^6W^2W_{6z} + 2520k^6WW_zW_{5z} + 1260k^6W_{4z}W_z^2$$

$$+ 3150k^4W^3W_{4z} + 18900k^4W^2W_zW_{3z} + 9450k^4W^2W_{2z}^2$$

$$+ 18900k^4WW_z^2W_{2z} + 18900k^2W^3W_z^2 + 4725k^2W^4W_{2z}$$

$$= \frac{k^{12}}{4}(3Y^2 - 1)(1 - Y^2).$$

Using the above simplification in the ODE (8), we obtain

$$c = \pm\sqrt{k^2 + k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} + l^2}.$$

Finally, the solution to (8) is precisely (9):

$$w = \frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx + ly - ct}{2}\right)$$

with $c = \pm\sqrt{k^2 + k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} + l^2}$, which agrees with the 1-soliton solution in (7).

Thus, we have successfully constructed the 1-soliton solution of nonlinear PDE (6) by two efficient methods, namely, Hirota's method and tanh method.

5. Discussion

It is shown that the derived 10th order 2-D Boussinesq equation admits 1-soliton and 2-soliton solutions. It is worth noting that 2-D Boussinesq equation is not completely integrable but it exhibits periodic solutions [8]. Further work on finding other types of solutions of 10th order 2-D Boussinesq equations such as periodic solutions, singular solutions and shock wave solutions can be attempted if they exist.

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