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AN EFFECTIVE SHOOTING PIECEWISE ANALYTICAL INTEGRATION METHOD FOR SINGULAR PERTURBATION TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, an effective semi-analytical-numerical method is proposed for solving singular perturbation two-point boundary value problems (SPBVPs). Firstly, the original problem is replaced by an equivalent singular perturbation initial value problems (SPIVPs) of first-order with an unknown initial condition that can be determined iteratively using shooting method. Then, an adaptive one-step explicit piecewise analytical integration scheme over a special non-uniform mesh is presented to integrate these SPIVPs. The accuracy and stability properties of the scheme are investigated and shown to yield at least second-order of accuracy and *L*-stability property. A good estimation of the missed initial condition is obtained and suggested as a starting initial guess to ensure accelerated convergence of the shooting method. To demonstrate the applicability of the method, we have applied it to linear and nonlinear test problems at different values of the perturbation parameter. The method can be extended to higher-order SPBVPs. We have applied it to the well-known third-order Blasius' viscous flow problem for a large suction case. The results indicate that the method approximates the solution very well not only over the boundary layer region but also overall the problem domain. Moreover, the method is more accurate and has a higher computational efficiency compared to other methods in the literature.

1. Introduction

Singular perturbation problems (SSPs) are of common occurrence in all branches of engineering and applied mathematics. These problems are encountered in various fields such as solid mechanics, quantum mechanics, fluid mechanics, fluid dynamics, celestial mechanics, chemical reactor theory, aerodynamics, reaction-diffusion equations, geophysics, and many other problems [1-5]. In these problems, the presence of a small perturbation parameter multiplying the highest derivative of the differential equation results in a sharp boundary and/or interior layers in the solution which prevents us from obtaining satisfactory numerical solutions [1-11, 16-41]. Therefore, more efficient and simpler computational methods are needed.

Many theoretical and numerical studies have been carried out on SPPs and their solutions. For details, one may refer to the books of O'Malley [6], Farrell et al. [7], Doolan et al. [8], Roos et al. [9], Morton [10], and Miller et al. [11]. A common strategy for dealing with singular perturbation boundary value problems (SPBVPs) consists of replacing the original problem with singular perturbation initial value problems (SPIVPs), which are more convenient to handle. Shooting method is an iterative method that solves boundary value problems (BVPs) by replacing them by equivalent initial value problems (IVPs) with some unknown initial conditions that can be obtained iteratively using shooting methods [12-20, 40].

Due to the high stiffness in SPBVPs, using stable integration schemes with the shooting method is essential to obtain an acceptable and accurate solution [16-20, 40]. Moreover, using high-order initial approximation methods [15, 42-44] to improve the shooting method for SPBVPs has quite limited utility due to the degeneration of their order to first-order for ill-conditioned problems [45]. In addition to their high computational cost, more than one differential equation is required to be solved at each step [15].

Shooting method combined with different adapted integration schemes, for layer and outer regions, is presented for solving linear SPBVPs [16-20]. Natesan and Ramanujam [16] presented a shooting method with fourth-order Runge-Kutta method and a classical finite difference scheme for second-order reaction-diffusion Neumann SPBVPs. The same authors [17] extended their work using a shooting method with an exponentially fitted difference scheme and a classical upwind scheme to solve second-order Robin-Dirichlet SPBVPs having a less severe boundary layer. The same problem was solved by Vigo-Aguiar and Natesan [18] using the shooting method with two integration procedures. The first procedure consists of a classical and exponentially fitted difference scheme whereas the second one is an adaptive single-step exponential scheme. Geng and Tang [19] presented a shooting method with a piecewise reproducing kernel scheme for second-order

Dirichlet SPBVPs. Roja and Tamilselvan [20] used the combination of the shooting method, boundary value technique, and a finite difference scheme to find a numerical solution for third-order reaction-diffusion SPBVPs. Liu [40] constructed a Lie-group shooting method for nonlinear SPBVPs to search for a missing initial condition through the finding of a suitable value of a parameter after a coordinate transformation via a rescaling technique.

Most of the present shooting methods in the literature are for linear SPBVPs, result in approximate numerical solutions, use different integration schemes for layer and outer domains, and have a relatively high computational cost. To overcome these drawbacks, in this paper, an effective semi-analytical-numerical method is proposed for solving linear and nonlinear SPBVPs. Firstly, the original problem is replaced by an equivalent singular perturbation system of first-order initial value problems (SPIVPs) with an unknown initial condition that can be determined iteratively using shooting method. Then, an adaptive one-step explicit piecewise analytical integration scheme over a special non-uniform mesh is presented to integrate these SPIVPs. The accuracy and stability properties of the scheme are investigated and shown to yield at least second-order of accuracy and L-stability property. To ensure accelerated convergence of the shooting method, a good estimation of the missed initial condition is obtained and suggested as a starting initial guess. To demonstrate the applicability of the method, we have applied it to linear and nonlinear test problems at different values of the perturbation parameter. The method can be extended to higher-order SPBVPs. We have applied it to the well-known third-order Blasius' viscous flow problem for a large suction case. The results indicate that the method approximates the solution very well not only over the boundary layer region but also overall the problem domain. Moreover, the method is more accurate and has a higher computational efficiency compared to other methods in the literature.

2. Description of the Method

Consider the nonlinear SPBVP given by

$$\varepsilon y''(x) + p(x, y)y'(x) + q(x, y) = 0, \quad x \in [a, b],$$
$$y(a) = \alpha, \quad y(b) = \beta, \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, a, b, α and β are given constants, $p(x, y)$ and $q(x, y)$ are assumed to be sufficiently differentiable functions, and $p(x, y) \geq M > 0$ for every $x \in [a, b]$, where M is some positive constant. Under these assumptions, SPBVP (1) has a boundary layer of width $O(\varepsilon)$ at $x = a$. The SPBVP (1) can be written as a SPIVP, given by

$$\left.\begin{aligned} v_1' &= v_2,\ v_1(a) = \alpha \\ \varepsilon v_2' &= -p(x, v_1)v_2(x) - q(x, v_1),\ v_2(a) = \varpi \end{aligned}\right\}, \tag{2}$$

where $v_1 = y(x)$, $v_2 = y'(x)$ and ϖ is the missing initial value to be determined. In shooting method, the solution of the SPIVP (2) is denoted by $v_i = v_i(x, \varpi)$, $i = 1, 2$. The method results in a sequence of approximate solutions $v_i^k = v_i^k(x, \varpi^k)$, $i = 1, 2$, where k is the number of shooting iterations. The parameter ϖ is selected in such a way that a residual function Ψ satisfies:

$$\Psi = \lim_{k \to \infty} \psi^k(b, \varpi^k) = v_1(b, \varpi^k) - \beta = 0. \tag{3}$$

Starting the shooting process with an initial guess $\varpi = \varpi^0$ results into a residual function $\psi^0(b, \varpi^0)$. If the residual function $\psi^0(b, \varpi^0)$ is not sufficiently close to *zero*, we choose another elevation, i.e., $\varpi = \varpi^1$, and repeat the process until the residual function $\psi^k(b, \varpi^k)$ is sufficiently close to *zero* at the iteration $k = K$, i.e., $\|\psi^k(b, \varpi^k)\| < tol$, $\varpi \cong \varpi^K$, where *tol* is a user-specified tolerance.

2.1. Adaptive piecewise integration method for SPIVPs

Since the SPIVP (2) is still singular perturbation, applying any conventional method requires a very fine mesh $(h < \varepsilon)$ to satisfy the criterion of absolute stability. In view of being very small in practice, the restriction on the step size would often lead to the prohibitive cost of computational time. Therefore, in order to obtain an acceptable approximate solution to the SPIVP (2), an adaptive explicit piecewise analytical integration scheme over a special non-uniform mesh is suggested as follows:

The SPIVP (2) can be written as

$$\left.\begin{aligned} v_1' &= v_2,\ v_1(a) = \alpha \\ \varepsilon v_2' &= \theta(x, v_1, v_2),\ v_2(a) = \varpi \end{aligned}\right\}, \tag{4}$$

where $\theta(x, v_1, v_2) = -p(x, v_1)v_2(x) - q(x, v_1)$.

Consider the interval $I = [a, b]$ and divide it into $N-1$ nonoverlapping sub-intervals $I_n = [x_n, x_{n+1}]$, $n = 0, 1, 2, \ldots, N-1$ such that $x_0 = a$ and $x_{N-1} = b$, where N denotes the number of grid points. Suppose we have solved numerically the SPIVP (4) up to a point x_n and assuming the localization hypothesis $v_{i,n} = v_{i,n}(x_n)$, $i = 1, 2$, we want to obtain an approximate analytical solution $w_{i,n} = w_{i,n}(x)$, $i = 1, 2$, that is, $w_{i,n}(x) \approx v_{i,n}(x)$, $i = 1, 2$, $\forall x \in I_n$. Then the SPIVP (4) may be approximated by [21, 22]:

$$\left.\begin{aligned} w_{1,n}'(x) &= w_{2,n}(x),\ w_{1,n}(x_n) = v_{1,n} \\ \varepsilon w_{2,n}'(x) &= \theta_n + J_n(x - x_n) + H_n(w_{1,n}(x) - v_{1,n}) \\ &\quad + K_n(w_{2,n}(x) - v_{2,n}),\ w_{2,n}(x_n) = v_{2,n} \end{aligned}\right\}, \quad \forall x \in I_n, \tag{5}$$

where

$$\theta_n = \theta(x_n, v_{1,n}, v_{2,n}),\ J_n = \frac{\partial\theta}{\partial x}(x_n, v_{1,n}, v_{2,n}),\ H_n = \frac{\partial\theta}{\partial v_1}(x_n, v_{1,n}, v_{2,n})$$

and $K_n = \dfrac{\partial\theta}{\partial v_2}(x_n, v_{1,n}, v_{2,n}),$

whose analytical solution may be readily obtained. For example, if $H_n \neq 0$, then we have the following analytical solution:

$$\left.\begin{aligned} w_{1,n}(x) &= \frac{\Gamma\chi + (\sigma\chi - \rho)e^{\frac{(x-x_n)(K_n-\chi)}{2\varepsilon}} + (\sigma\chi + \rho)e^{\frac{(x-x_n)(K_n+\chi)}{2\varepsilon}}}{\chi H_n^2} \\ w_{2,n}(x) &= \frac{-2J_n\chi + (\phi\chi + \varsigma)e^{\frac{(x-x_n)(K_n-\chi)}{2\varepsilon}} + (\phi\chi - \varsigma)e^{\frac{(x-x_n)(K_n+\chi)}{2\varepsilon}}}{2\chi H_n} \end{aligned}\right\}, \quad \forall x \in I_n, \quad (6)$$

where

$$\Gamma = H_n^2 v_{1,n} + (K_n v_{2,n} + (x_n - x)J_n - \theta_n)H_n + K_n J,$$

$$\chi = \sqrt{4\varepsilon H_n + K_n^2},$$

$$\sigma = \left(-\frac{K_n v_{2,n}}{2} + \frac{\theta_n}{2}\right)H_n - \frac{K_n J_n}{2},$$

$$\rho = \varepsilon v_{2,n} H_n^2 + \tau H_n + \frac{1}{2}J_n K_n^2,$$

$$\tau = \frac{1}{2}K_n^2 v_{2,n} + \varepsilon J_n - \frac{1}{2}K_n\theta_n,$$

$$\phi = H_n v_{2,n} + J_n,$$

$$\varsigma = (K_n v_{2,n} - 2\theta_n)H_n + K_n J_n.$$

Starting with $(v_{1,0}, v_{2,0}) = (\alpha, \varpi)$, we get the solution $w_{i,0}(x)$, $i = 1, 2$ for $x \in [a, x_1]$ and then get new approximate initial values $v_{i,1} = w_{i,0}(x_1)$, $i = 1, 2$, to obtain the solution $w_{i,1}(x)$, $i = 1, 2$ for $x \in [x_1, x_2]$. Repeating the procedure along the nodes on the integration interval resulting in an approximate piecewise analytical solution for the SPIVP (5).

Remark 1. At $H_n = 0$, the solution given by (6) is reduced to

$$\left.\begin{aligned} w_{1,n}(x) &= \frac{\overline{\Gamma} e^{\frac{(x-x_n)K_n}{\varepsilon}} + \overline{\sigma} - \overline{\Gamma}}{2K_n^3} \\ w_{2,n}(x) &= \frac{\frac{1}{2\varepsilon}\overline{\Gamma} e^{\frac{(x-x_n)K_n}{\varepsilon}} + \overline{\tau}}{K_n^2} \end{aligned}\right\}, \ \forall x \in I_n, \tag{7}$$

where

$$\overline{\Gamma} = 2\varepsilon^2 J_n + 2\varepsilon K_n \theta_n,$$

$$\overline{\sigma} = 2K_n^3(v_{2,n}(x - x_n) + v_{1,n}) - (J_n(x - x_n) + 2\theta_n)(x - x_n)K_n^2 - 2\varepsilon K_n J_n(x - x_n),$$

$$\overline{\tau} = K_n^2 v_{2,n} - (J_n(x - x_n) + \theta_n)K_n - \varepsilon J_n.$$

2.2. Mesh selection strategy

We form a special mesh in such a way that one wants to get more information about the solution of the SPBVP in the boundary layer region. This is quite natural because one would like to portray the behavior of the solution inside the boundary layer region. The required step size $h_n = x_{n+1} - x_n$ can be determined directly according to the variation of the solution within a time step as follows:

Suppose we have solved numerically SPIVP (5) up to point x_n and we want to determine a point x_{n+1} according to the variation of the solution that

verifies $|(v_{1,n+1} - v_{1,n})| \le r,\ \ r > 0,$ where r is a user's specified constant. Then we have

$$x_{n+1} - x_n \le \min\left\{\left|\frac{r}{|v_{2,n}|^s}\right|,\ r\right\}, \tag{8}$$

where s is a parameter that can be determined from the local truncation error.

2.3. Local truncation error

From (4) and (5), we have

$$\left.\begin{aligned} \theta_n &= \varepsilon w'_{2,n}(x_n) = \varepsilon v'_{2,n} \\ K_n &= -p(x_n,\ v_{1,n}) = -p_n \\ J_n &= \varepsilon v''_{2,n} + p_n v'_{2,n} - H_n v'_{1,n} \end{aligned}\right\}. \tag{9}$$

We consider the functional $l_i,\ i = 1,\ 2$ associated with the integration scheme in (6) and defined by

$$l_i(w_{i,n}(x),\ x - x_n) = w_{i,n}(x) - v_{i,n}, \quad i = 1,\ 2, \quad \forall x \in I_n. \tag{10}$$

After expanding (6) in Taylor series about x_n and taking into account the restriction on the step size in (8), we get

$$l_i(v_{i,n},\ r) \le \begin{cases} \mu_1(v_{1,n})r^4, & i = 1, \\ \mu_2(v_{2,n})r^3, & i = 2, \end{cases} \tag{11}$$

where

$$\left.\begin{aligned} \mu_1(v_{1,n}) &= \frac{1}{24}\frac{v^{(4)}_{1,n}\theta_n + v''_{1,n}(p_n v'''_{1,n} - H_n v''_{1,n})}{\theta_n (v'_{1,n})^{4s}} = O(\varepsilon^{4S-4}) \\ \mu_2(v_{2,n}) &= \frac{1}{6}\frac{v'''_{2,n}\theta_n - v'_{2,n}v''_{2,n}(p_n v_{2,n} - \theta_n)}{(v_{2,n})^4 \theta_n} = O(\varepsilon^{3S-4}) \end{aligned}\right\}. \tag{12}$$

This suggests that for $S = 1,\ 4/3,$ the integration scheme exhibits a uniform convergence with at least third and second-order of accuracy for v_1 and v_2, respectively.

2.4. Stability analysis

As the present scheme in (6) is exact when the right-hand side of the differential equation is linear, that is $\theta(x, v_1, v_2) = \bar{a} + \bar{b}v_1 + \bar{c}v_2$, with constants $\bar{a}$, $\bar{b}$ and $\bar{c}$, the method is trivially *L*-stable.

Theorem 1. *The integration scheme defined by* (6) *for the numerical integration of SPIVP* (5) *is L-stable and has at least second-order of accuracy with the following bounded error inequality*:

$$|v_{1,n}(x_{n+1}) - w_{1,n}(x_{n+1})| \le Cr^3, \quad n = 0, 1, \ldots, N-1, \tag{13}$$

where C is independent of ε *and r.*

2.5. Approximation of the initial value ϖ

One of the main problems in applying the shooting methods for solving nonlinear SPBVPs is the wide range of the starting initial guess domain, i.e., $O(\varepsilon^{-1+m})$ [1, 5, 23-30], where *m* is the order of the SPBVP. This wide range hinders obtaining a fast convergent sequence of approximate solutions and increases the computational cost of the method even with using stable and accurate integration schemes. To overcome these drawbacks, a good asymptotic estimation of the starting initial guess is essential to ensure fast convergence and low computational cost of the method.

The reduced problem solution $\bar{v}_1$ of the SPIVP (2) satisfies

$$p(x, \bar{v}_1)\bar{v}_1'(x) + q(x, \bar{v}_1) = 0, \quad \bar{v}_1(b) = \beta, \tag{14}$$

and the SPIVP (2) can be asymptotically approximated by [5, 23-26]

$$\varepsilon v_1' + f(x, v_1) = f(x, \bar{v}_1) + O(\varepsilon) \text{ with } v_1(a) = \alpha, \tag{15}$$

where $f(x, v_1) = \int p(x, v_1)dv_1$.

From equation (15), we have

$$\varpi = v_1'(a) = \frac{f(a, \bar{v}_1(a)) - f(a, \alpha)}{\varepsilon} + O(1). \tag{16}$$

And so, a good asymptotic estimation for the starting initial guess ϖ^0 is given by

$$\varpi^0 = \varpi^* = \frac{f(a, \bar{v}_1(a)) - f(a, \alpha)}{\varepsilon}, \tag{17}$$

with an absolute error $O(1)$ and a relative error $O(\varepsilon)$.

3. Numerical Examples

To show the applicability and efficiency of the proposed method, we have applied it to four test problems at different values of ε, r and at tolerance $tol = 10^{-3}$. The result numerical solution is compared with the exact one. Moreover, the error at the nodal points $E = e_{n+1} = |\, y(x_{n+1}) - y_{n+1} \,| = |\, y(x_{n+1}) - w_1(x_{n+1}) \,|$, $n = 0, 1, \ldots, N-1$ as well as the maximum absolute error $E_{\max} = \| e_{n+1} \|_\infty$, $n = 0, 1, \ldots, N-1$ and the order of convergence are presented.

Example 3.1. Consider the following SPBVP from fluid dynamics for fluid of small viscosity from Geng and Tang [19]:

$$\begin{cases} \varepsilon y''(x) + y'(x) = 1 + 2x; & x \in [0, 1], \\ y(0) = 0, & y(1) = 1. \end{cases} \tag{18}$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{1 - e^{-1/\varepsilon}}.$$

Using the present method with the initial guess $\varpi^0 = \frac{-1}{\varepsilon}$, and at tolerance $tol = 10^{-3}$, the solution with the desired accuracy is obtained through only two iterations. The numerical and the exact solutions of Example 3.1 at $\varepsilon = 10^{-4}$ and $r = 0.1$ are shown in Figure 3.1. The maximum absolute error $E_{\max}$ at different values of ε and r is shown in Table 3.1. Results in Table 3.1 confirm that the method results in exact

solution for linear SPBVPs with constant coefficients and the result error E is due to the round-off error that controlled by the machine precision. Comparisons of the numerical results using the present method and those in [19, 31-34] are presented in Table 3.2 for $\varepsilon = 10^{-3}$, Table 3.3 for $\varepsilon = 10^{-4}$ and Table 3.4 for $\varepsilon = 10^{-5}, 10^{-9}$.

The numerical results show that the present method is accurate and has low computational cost.

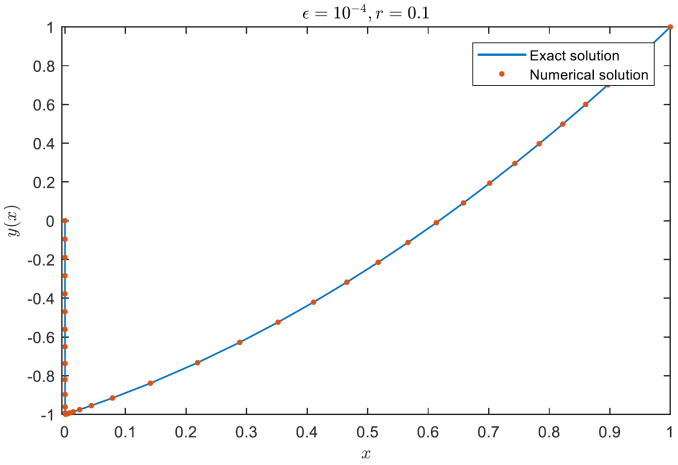

Figure 3.1. Exact and numerical solutions of Example 3.1 at $\varepsilon = 10^{-4}$, $r = 0.1$.

Table 3.1. Maximum absolute error $E_{\max}$ for Example 3.1 at different values of ε and r

$\varepsilon \backslash r$	0.1	0.08	0.06	0.04	0.02
10^{-3}	1.1125E-15	2.2204e-15	1.2212e-15	2.4425e-15	2.4425e-15
10^{-4}	1.7764e-15	5.5511e-16	2.8866e-15	3.5527e-15	4.2188e-15
10^{-5}	3.3307e-16	9.9920e-16	3.9968e-15	7.7716e-16	2.2204e-15
10^{-6}	9.9920e-16	2.2204e-15	1.7764e-15	1.1102e-15	3.5527e-15
10^{-7}	8.8818e-16	1.9984e-15	1.3323e-15	1.2212e-15	1.7764e-15
10^{-8}	1.5543e-15	1.1102e-15	9.9920e-16	1.7764e-15	2.8866e-15
10^{-9}	4.4409e-16	1.6653e-15	3.4417e-15	1.8874e-15	2.3315e-15

Table 3.2. Numerical solution for Example 3.1 at $\varepsilon = 10^{-3}$, $r = 0.1$

Nodes	Exact solution	Kumar et al. [31]	Geng [32]	Geng and Tang [19]	Reddy and Chakravarthy [33]	Present method
0.000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.001	-0.629857	-0.631119	-0.629684	-0.629852	-1.000997	-0.629857
0.010	-0.987874	-0.989854	-0.987496	-0.987875	-0.991880	-0.987874
0.030	-0.967160	-0.969100	-0.966866	-0.967160	-0.971040	-0.967160
0.100	-0.888200	-0.890000	-0.887957	-0.888200	-0.891800	-0.888200
0.300	-0.608600	-0.610000	-0.608408	-0.608600	-0.611400	-0.608600
0.500	-0.249000	-0.250000	-0.248848	-0.249000	-0.251000	-0.249000
0.700	0.190600	0.190000	0.190706	0.190600	0.189400	0.190600
0.900	0.710200	0.709999	0.710243	0.710200	0.709800	0.710200
1.000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table 3.3. Numerical solution for Example 3.1 at $\varepsilon = 10^{-4}$, $r = 0.1$

Nodes	Exact solution	Kumar et al. [31]	Geng [32]	Geng and Tang [19]	Reddy and Chakravarthy [33]	Present method
0.0000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.0001	-0.631894	-0.632020	-0.633487	-0.631888	-1.000100	-0.631894
0.0010	-0.998753	-0.998953	-0.997079	-0.998754	-0.999199	-0.998753
0.0030	-0.996791	-0.996991	-0.995239	-0.996792	-0.997190	-0.996791
0.1000	-0.889820	-0.890000	-0.888582	-0.889820	-0.890180	-0.889820
0.3000	-0.609860	-0.610000	-0.608887	-0.889820	-0.610140	-0.609860
0.5000	-0.249900	-0.250000	-0.249165	-0.249900	-0.250100	-0.249900
0.7000	0.190060	0.190000	0.190538	0.190060	0.189939	0.190060
0.9000	0.710019	0.709999	0.710201	0.710019	0.709979	0.710019
1.0000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table 3.4. Numerical results for Example 3.1 at $\varepsilon = 10^{-5}$, 10^{-9} and $r = 0.1$

ε	$E_{\max}$		N (Grid points)	
	Kadalbajoo and Kumar [34]	Present method	Kadalbajoo and Kumar [34]	Present method
10^{-5}	1.3900e-4	5.5551E-16	240	36
10^{-9}	1.1900e-4	6.6654E-16	240	39

Example 3.2. Consider the variable coefficient SPBVP from Kevorkian and Cole [4]:

$$\begin{cases} \varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0; & x \in [0, 1], \\ y(0) = 0, & y(1) = 1. \end{cases} \tag{19}$$

The exact solution is approximated by Nayfeh [35] as

$$y(x) = \frac{1}{2 - x} - \frac{1}{2} e^{-\frac{x - \frac{x^2}{4}}{\varepsilon}} + O(\varepsilon).$$

Using the present method with the initial guess $\varpi^0 = \frac{1}{2\varepsilon}$, and at tolerance $tol = 10^{-3}$, the solution with the desired accuracy is obtained through only two iterations. The numerical and the exact solutions of Example 3.2 at $\varepsilon = 10^{-4}$ and $r = 0.1$ are shown in Figure 3.2. Moreover, the absolute error E for different values of ε at $r = 0.1$ is shown in Figure 3.3. The maximum absolute error $E_{\max}$ and the order of convergence of the present method at different values of ε and r are shown in Table 3.5. The numerical results using the present method are compared with [19, 31-35] in Table 3.6 for $\varepsilon = 10^{-3}$ and Table 3.7 for $\varepsilon = 10^{-5}, 10^{-7}, 10^{-9}$ at $r = 0.01$. The numerical results show that the present method results in accurate solution overall the problem domain, and converges uniformly to the exact solution with at least second-order of accuracy and has a low computational cost.

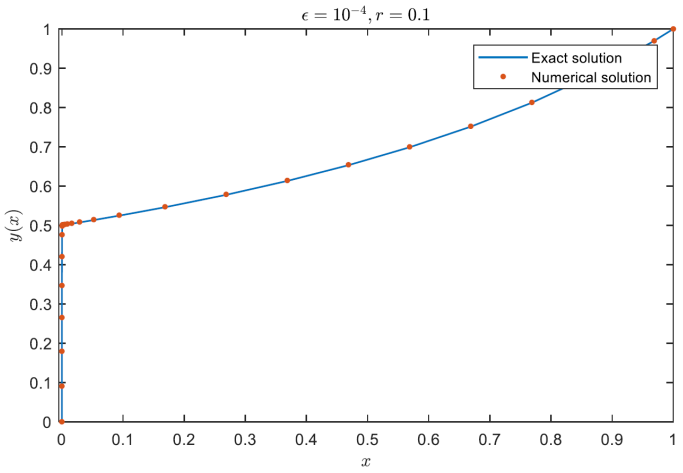

Figure 3.2. Exact and numerical solutions of Example 3.2 at $\varepsilon = 10^{-4}$, $r = 0.1$.

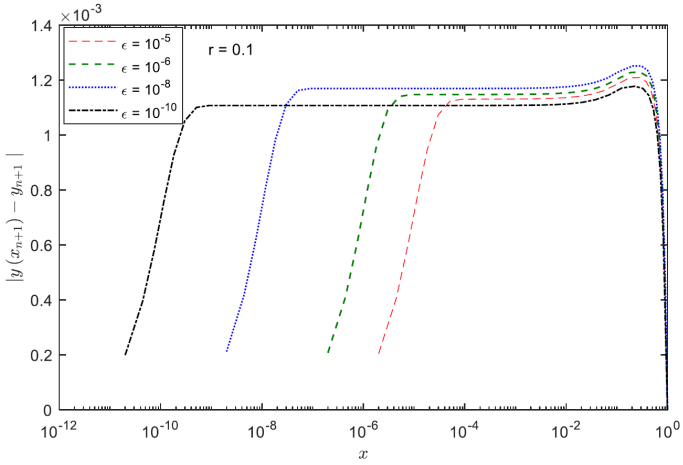

Figure 3.3. Error distribution for the solution of Example 3.2 for different values of ε at $r = 0.1$.

Table 3.5. Maximum absolute errors and the computed order of convergence for Example 3.2 at different values of ε and r

r	$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$		$\varepsilon = 10^{-10}$	
	E_{max}	Order	E_{max}	Order	E_{max}	Order
0.1	1.2781e-03	2.3377	1.2792e-03	2.3463	1.2823e-03	2.0962
0.08	7.5859e-04	1.9111	7.5782e-04	1.8110	8.0326e-04	2.1196
0.06	4.3776e-04	1.8517	4.5009e-04	1.9209	4.3656e-04	2.1475
0.04	2.0662e-04	2.0129	2.0655e-04	2.0015	1.8276e-04	1.8590
0.02	5.1197e-05		5.1584e-05		5.0381e-05	

Table 3.6. Numerical solution for Example 3.2 at $\varepsilon = 10^{-3}$, $r = 0.01$

Nodes	Nayfeh [35]	Kumar et al. [31]	Geng and Tang [19]	Reddy and Chakravarthy [33]	Present method
0.000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.001	0.3163104	0.3162644	0.3163075	0.4997493	0.3163357
0.010	0.5024899	0.5024893	0.5024898	0.5020117	0.5024898
0.020	0.5050505	0.5050505	0.5050505	0.5045496	0.5050505
0.100	0.5263158	0.5263158	0.5263158	0.5258163	0.5263168
0.300	0.5882353	0.5882353	0.5882353	0.5877497	0.5882353
0.500	0.6666667	0.6666667	0.6666667	0.6662214	0.6666667
0.700	0.7692308	0.7692308	0.7692308	0.7688746	0.7692308
0.900	0.9090909	0.9090909	0.9090909	0.9089253	0.9090909
1.000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 3.7. Numerical results for Example 3.2 at $\varepsilon = 10^{-5}, 10^{-7}, 10^{-9}$ and $r = 0.01$

ε	E_{max}		N (Grid points)	
	Kadalbajoo and Kumar [34]	Present method	Kadalbajoo and Kumar [34]	Present method
10^{-5}	3.7600e-3	2.0415e-05	240	156
10^{-7}	3.8000e-3	1.2812e-05	240	158
10^{-9}	3.8000e-3	1.2843e-05	240	160

Example 3.3. Consider the following nonlinear SPBVP from Bender and Orszag [1]:

$$\begin{cases} \varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; & x \in [0, 1], \\ y(0) = 0, & y(1) = 0. \end{cases} \tag{20}$$

The problem has a uniformly valid approximation for comparison [1] given by

$$y(x) = -\log\left(\frac{x+1}{2}\right) + \log\left(\frac{1}{2}\right) + e^{-\frac{2x}{\varepsilon}} + O(\varepsilon).$$

Using the present method with the starting initial guess $\varpi^0 = -\frac{2}{\varepsilon}\ln\left(\frac{1}{2}\right)$, and at tolerance $tol = 10^{-3}$ results in accurate solution over the layer region as well as the outer region. The numerical and the exact solutions of Example 3.3 at $\varepsilon = 10^{-4}$, $r = 0.1$ are shown in Figure 3.4. The absolute error E for different values of ε at $r = 0.1$ is shown in Figure 3.5. The maximum absolute error $E_{\max}$ and the computed order of convergence of the present method at different values of ε and r are shown in Table 3.8.

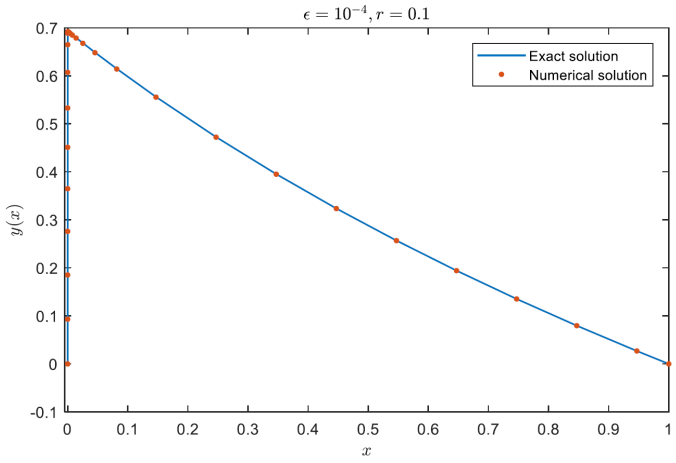

Figure 3.4. Exact and numerical solutions of Example 3.3 at $\varepsilon = 10^{-4}$, $r = 0.1$.

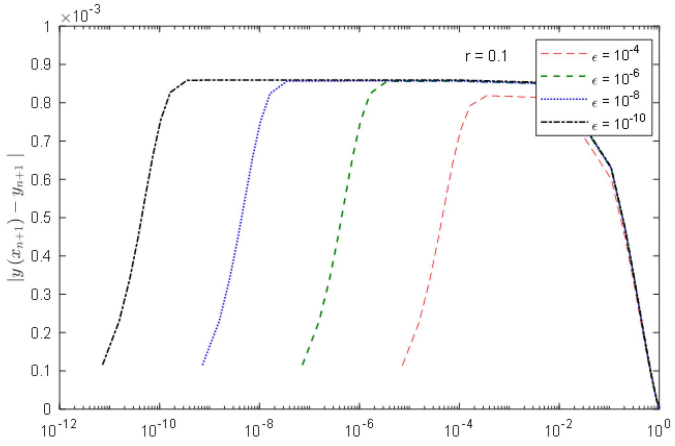

Figure 3.5. Error distribution for the solution of Example 3.3 for different values of ε at $r = 0.1$.

Table 3.8. Maximum absolute errors and the computed order of convergence for Example 3.3 at different values of ε and r

r	$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$		$\varepsilon = 10^{-10}$	
	$E_{\max}$	Order	$E_{\max}$	Order	$E_{\max}$	Order
0.1	8.5649e-04	2.1766	8.5693e-04	2.1755	8.5943e-04	2.1646
0.08	5.2698e-04	2.0195	5.2738e-04	2.0174	5.3020e-04	2.0763
0.06	2.9476e-04	2.0011	2.9516e-04	1.9985	2.9176e-04	1.9828
0.04	1.3095e-04	1.9753	1.3127e-04	1.9593	1.3058e-04	1.8837
0.02	3.3301e-05		3.3755e-05		3.5313e-05	

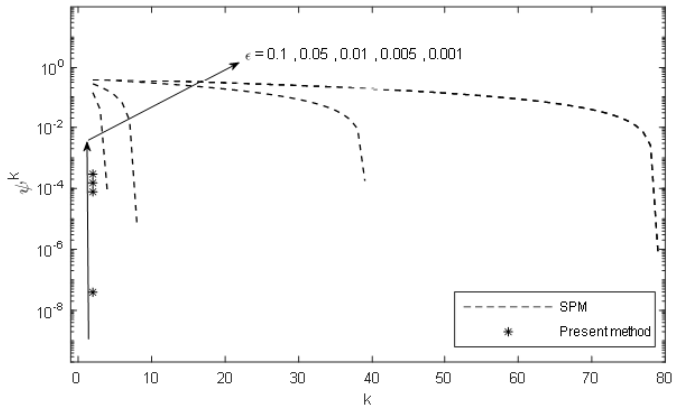

Figure 3.6. Residual function ψ^k versus the number of iterations k for the present method and SPM $(\varpi^0 = 1)$ for Example 3.3 at different values of ε at $r = 0.1$.

Table 3.9. Numerical results of SPM and the present method for Example 3.3 at different values of ε at $r = 0.1$

ε	Residual function ψ^k		Number of iterations k		CPU time (sec)	
	SPM	Present method	SPM	Present method	SPM	Present method
1.0e-01	6.4873e-05	3.9135e-08	5	2	0.722	0.557
1.0e-02	1.7559e-04	1.5092e-04	40	2	1.257	0.562
1.0e-03	3.5558e-03	6.6016e-04	410	2	4.672	0.514
1.0e-04	2.9208e-01	4.5138e-04	900	2	6.995	0.614
1.0e-06	-	4.3030e-04	-	2	-	0.597
1.0e-08	-	4.3009e-04	-	2	-	0.691
1.0e-10	-	4.2920e-04	-	2	-	0.525

The sign '-' refers to no convergence

Table 3.9 shows the CPU computational time and the number of iterations k required to satisfy the residual function ψ^k, with $tol = 10^{-3}$, $k < 900$ for Shooting Piecewise Method (SPM) and the present method with the estimated initial guess ϖ^* in (17). Figure 3.6 shows the residual function

ψ^k versus the number of iterations k for SPM $(\varpi^0 = 1)$ and the present method $(\varpi^0 = \varpi^*)$ for different values of ε at $r = 0.1$. Results in Table 3.9 and Figure 3.6 show that, for SPM, as the value of the perturbation parameter ε decreases, the number of iterations required to satisfy the residual function ψ^k increases. While for the present method, the residual function ψ^k is satisfied using only two iterations for all the considered values of the perturbation parameter ε. The results shown in Table 3.9 confirm that the present method has a fast convergent sequence of approximate solutions compared to the SPM. Moreover, the results shown in Table 3.9 confirm that the present method ensures fast convergence and low computational cost compared to the SPM.

Example 3.4. Consider the well-known third-order Blasius equation [36, 37]

$$\begin{cases} f'''(\eta) + f(\eta) f''(\eta) = 0; & \eta \in [0, \infty), \\ f(0) = -\alpha,\ f'(0) = -\beta, & \lim\limits_{\eta \to \infty} f'(\eta) = \gamma, \end{cases} \tag{21}$$

where α, β and γ are constants, and α represents a suction/injection parameter. In practice, we can find the numerical solution only over a finite interval. For this reason, we introduce a one-parameter family of problems related to Blasius problem on a finite interval $(0, L)$ and define

$$\begin{cases} f_L'''(\eta) + f_L(\eta) f_L''(\eta) = 0; & \eta \in [0, L), \\ f_L(0) = -\alpha,\ f_L'(0) = -\beta, & f_L'(L) = \gamma. \end{cases} \tag{22}$$

For moderate-to-large values of the suction/injection parameter α, the problem (22) can be written as a SPBVP in the form [26]

$$\varepsilon y''(x) + p(x) y'(x) = 0; \quad y(0) = -\beta,\ y(1) = \gamma, \tag{23}$$

where $\varepsilon = -\delta/\alpha$, $\delta = \dfrac{1}{L}$, $x = \delta\eta$, $p(x) = -f(\eta)/\alpha$, and $y(x) = f'(\eta)$.

Using the present method with a starting initial guess $\varpi_0 = -\alpha(\beta + \gamma)$ and at tolerance $tol = 10^{-3}$ results into an accurate solution for the problem (21) over the layer region as well as the outer region.

Figure 3.7 shows the numerical solution of (21), using bvp4c $(Atol = 10^{-6}, Rtol = 10^{-3})$ in Matlab environment and the present method at $\alpha \neq 0,\ \beta = 0,\ \gamma = 1$.

According to Weyl [38], "the value $[f''(0)]$ is the essential factor in the formula for the skin friction along the immersed plate". Due to its importance, the numerical results using the present method are compared with [26, 39] in Tables 3.10 for $\alpha \neq 0,\ \beta = 0,\ \gamma = 1$. Results in Table 3.10 show that the present method gives more accurate results compared to the results in [26, 39]. Moreover, as the suction value increases, the numerical error decreases.

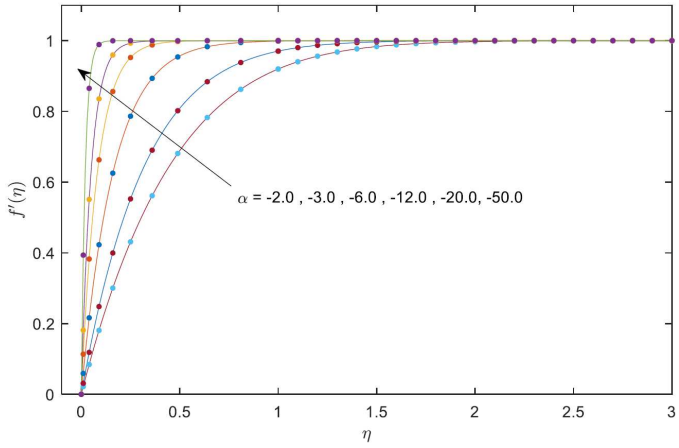

Figure 3.7. Solution comparison of $f'(\eta)$, bvp4c solution (solid lines) and present method solution (doted lines) for different values of α at $\beta = 0$ and $\gamma = 1$.

Table 3.10. Relative error of $f''(0)$ for $\alpha \neq 0$, $\beta = 0$, $\gamma = 1$

α	Numerical (bvp4c)	ADM [39]	Relative error %	El-Zahar and EL-Kabeir [26]	Relative error %	Present method	Relative error %
-1.5	1.7319	2.50943	44.9	1.5	13.389	1.7313	0.0346
-2.0	2.1945	3.00028	36.7	2.0	08.867	2.1933	0.0547
-2.5	2.6666	1.49962	43.8	2.5	06.244	2.6662	0.0150
-3.0	3.1451	-	-	3.0	04.613	3.1441	0.0318
-6.0	6.0799	-	-	6.0	01.314	6.0793	0.0099
-12.0	12.0412	-	-	12.0	00.342	12.0413	0.0008
-20.0	20.0101	-	-	20.0	00.050	20.0100	0.0005
-50.0	50.001	-	-	50.0	00.002	50.0000	0.0002

The sign '-' means not available

4. Conclusion

In this paper, by combining the shooting technique with an adaptive piecewise analytical integration scheme and a good estimated initial guess, an effective semi-analytical numerical method is presented for solving SPBVPs. Firstly, the original SPBVP is replaced by an equivalent first-order SPIVP with an unknown initial condition that can be determined iteratively using shooting method. Then, an adaptive one-step explicit piecewise analytical integration scheme over a special non-uniform mesh is presented to integrate these SPIVPs. The accuracy and stability properties of the scheme are investigated and shown to yield at least second-order of accuracy and *L*-stability property. A good estimation of the missed initial condition is obtained and suggested as a starting initial guess to ensure accelerated convergence of the method. The method results in a piecewise analytical solution that enables us from obtaining the solution overall the problem domain and not only a numerical solution at specific grid points. To demonstrate the applicability of the method, we have applied it to linear and nonlinear test problems at different values of the perturbation parameter. The method results in the exact solution for linear SPBVPs with constant coefficients and a high accurate solution for nonlinear problems through

only two iterations for all the considered values of the perturbation parameter. The method can be extended to higher-order SPBVPs. We have applied it to the well-known third-order Blasius' viscous flow problem for a large suction case. The results indicate that the method approximates the solution very well not only over the boundary layer region but also overall the problem domain. Moreover, the results confirm that the method is more accurate and has a higher computational efficiency compared to other methods in the literature. For future work, we are interested in applying the present method to many real-world problems and how to extend it to fractional SPBVPs.

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