**Advances in Differential Equations and Control Processes**
© 2023 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324323002
Volume 30, Number 1, 2023, Pages 15-25 **P-ISSN: 0974-3243**

# CHEBYSHEV WAVELETS BASED TECHNIQUE FOR NUMERICAL DIFFERENTIATION

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## Abstract

Numerical differentiation plays a significant role in numerical analysis. In this research paper, Chebyshev wavelets based efficient scheme has been developed to find the numerical differentiation problems arising in numerical analysis. Proposed technique based on the expansion of unknown function into a series of Chebyshev wavelets. Some numerical examples have been performed to find the accuracy of the proposed technique.

Received: October 9, 2022; Revised: December 14, 2022; Accepted: December 28, 2022
2020 Mathematics Subject Classification: 65N99.
Keywords and phrases: Chebyshev's wavelet of second kind, function approximation, numerical differentiation, numerical examples.
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How to cite this article: Inderdeep Singh and Preeti, Chebyshev wavelets based technique for numerical differentiation, Advances in Differential Equations and Control Processes 30(1) (2023), 15-25. http://dx.doi.org/10.17654/0974324323002
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Published Online: January 6, 2023

## 1. Introduction

Numerical analysis is a branch of applied mathematics and various numerical techniques have been used for solving the applications of applied mathematics. Different numerical techniques have been established for numerical analysis such as Newton method, secant method, Regula Falsi method for find the root of equations, Newton's forward and backward formulae for numerical differentiation, Trapezoidal rule and Simpson rule for numerical integration, Taylor's series method and Runge-Kutta methods for solving differential equations. In recent years, wavelet theory has been widely used in different fields of science, engineering and technology. Wavelet analysis is a new branch of mathematics that helps in numerical analysis, image processing and signal analysis. Several algorithms based on Haar wavelet and Hermite wavelets have been established for solving integral and differential equations, which are discussed in [1-9]. Chebyshev wavelets are powerful mathematical tool for solving several applications of sciences and engineering. Due to the simplicity and accuracy, Chebyshev wavelets attracted the attention of researchers, scientists and engineers. In literature, Chebyshev wavelets have been used for various problems and give accurate, reliable, and applicable solutions of the problems in least time interval.

Chebyshev polynomials have been utilized for solving two dimensional linear and nonlinear integral equations in [10]. Chebyshev wavelets have been used for solving linear and nonlinear differential and integral equations in [11-17]. The outline of this research is as follows: In Section 2, introduction to Chebyshev wavelets of the second kind. In Section 3, the proposed scheme is used to approximate the solution of the problems by numerical differentiation. In Section 4, some numerical experiments have been performed to check the accuracy and reliability of the proposed scheme. In Section 5, conclusion is drawn.

## 2. Chebyshev Wavelets of the Second Kind

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the *mother wavelet*. When the dilation parameter $a$ and the translation parameter $b$ vary continuously, we have the following family of continuous wavelets:

$$\varphi_{a,b}(t) = |a|^{-1/2}\varphi\left(\frac{t-b}{a}\right), \quad a, b \in R, \ a \neq 0.$$

The second kind wavelets $\varphi_{n,m} = \varphi(k, n, m, t)$ have four arguments; $k$ is any positive integer, $n = 1, 2, 3, 4, ..., 2^{k-1}$, $m$ is the degree of second kind Chebyshev polynomials and $t$ is normalized time. It is defined on the interval $[0, 1)$ as follows:

$$\varphi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\widetilde{U}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}} \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\widetilde{U}_m(t) = \sqrt{\frac{2}{\pi}}\, U_m(t), \tag{1}$$

where $m = 0, 1, 2, 3, ..., M-1$ and $M$ is fixed integer. In relation given by equation (1) is for orthonormality. Here $U_m(t)$ are the second kind Chebyshev polynomials of degree $m$ which are orthogonal with respect to the weight function $\omega(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$, and satisfy the following recursive formula:

$$U_0(t) = 1, U_1(t) = 2t, U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, 3, \ldots.$$

It is noted that in case of second kind Chebyshev wavelets, the weight function has to be dilated and translated as $\omega_n(t) = \omega(2^k t - 2n + 1)$.

A function $f(x) \in L^2(\mathbb{R})$ defined on the interval $[0, 1)$ may be expanded by second kind Chebyshev wavelets as follows:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\varphi_{n,m}(x), \tag{2}$$

where

$$c_{n,m} = \langle f(x), \varphi_{n,m}(x)\rangle_{L^2_\omega[0,1)} = \int_0^1 f(x)\varphi_{n,m}(x)\omega_n(x),$$

in which $\langle \cdot, \cdot \rangle_{L^2_\omega[0,1)}$ denotes the inner product in $L^2_\omega[0, 1)$. If the infinite series in equation (2) is truncated, then it can be written as follows:

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\varphi_{n,m}(x) = C^T \varphi(x),$$

where $C$ and $\varphi(x)$ are $2^{k-1}M \times 1$ matrices given as follows:

$$C = (c_{1,0}, \ldots, c_{1,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1})^T,$$

$$\varphi(x) = [\varphi_{1,0}, \ldots, \varphi_{1,M-1}, \ldots, \varphi_{2^{k-1},0}, \ldots, \varphi_{2^{k-1},M-1}]^T. \quad (3)$$

Let $k = 1$, $M = 6$. Then for this, following is six basis functions in $[0, 1]$:

$$\varphi_{1,0}(t) = \frac{2}{\sqrt{\pi}}, \ \varphi_{1,1}(t) = \frac{2}{\sqrt{\pi}}(4t - 2),$$

$$\varphi_{1,2}(t) = \frac{2}{\sqrt{\pi}}(16t^2 - 16t + 3),$$

$$\varphi_{1,3}(t) = \frac{2}{\sqrt{\pi}}(64t^3 - 96t^2 + 40t - 4),$$

$$\varphi_{1,4}(t) = \frac{2}{\sqrt{\pi}}(256t^4 - 512t^3 + 336t^2 - 80t + 5),$$

$$\varphi_{1,5}(t) = \frac{2}{\sqrt{\pi}}(1024t^5 - 2560t^4 + 2304t^3 - 896t^2 + 140t + 6).$$

## 3. Proposed Scheme for Numerical Differentiation

Numerical differentiation is widely used in numerical analysis to find the value of an unknown function in the given domain. Suppose we are given the following set of values of $y = f(x)$ for a set of values of $x$:

| $x$ | $x_0$ | $x_1$ | $\cdots$ | $x_n$ |
|---|---|---|---|---|
| $y = f(x)$ | $y_0$ | $y_1$ | $\cdots$ | $y_n$ |

Thus

$$y(x_0) = y_0,\ y(x_1) = y_1,\ \ldots,\ y(x_n) = y_n. \tag{4}$$

To find the first, second and third derivatives of $y$ at any value of $x$, consider the approximation

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x). \tag{5}$$

Using the conditions, given in (4), we obtain

$$y_0 = y(x_0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x_0)$$

$$y_1 = y(x_1) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x_1)$$

$$\vdots$$

$$y_{n-1} = y(x_{n-1}) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x_{n-1}).$$

Solving the above system of $n$ algebraic equations, we obtain the wavelet coefficients. Differentiating (5), three times w.r.t. $x$, we obtain

$$y'(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi'_{n,m}(x), \tag{6}$$

$$y''(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi''_{n,m}(x), \tag{7}$$

$$y'''(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi'''_{n,m}(x). \quad (8)$$

Substituting the wavelet coefficients into (6), (7) and (8), we obtain the first, second and third derivatives of $y$.

## 4. Numerical Examples

In this section, some numerical examples have been performed to illustrate the accuracy of the proposed technique.

**Example 1.** Consider the following data:

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $y = f(x)$ | 1 | 1.0160 | 1.1280 | 1.4320 | 2.0240 | 3 |

Using the Newton's forward or backward interpolation formula, we obtain the polynomial $y(x) = 2x^3 + 1$.

Consider the approximation

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x).$$

Letting $k = 1$, $M = 6$,

$$y(x) = c_{1,0}\varphi_{1,0}(x) + c_{1,1}\varphi_{1,1}(x) + \cdots + c_{1,5}\varphi_{1,5}(x). \quad (9)$$

Substituting $x = 0, 0.2, 0.4, 0.6, 0.8, 1,$ respectively, in (9), we obtain

$$y(0) = c_{1,0}\varphi_{1,0}(0) + c_{1,1}\varphi_{1,1}(0) + \cdots + c_{1,5}\varphi_{1,5}(0), \quad (10)$$

$$y(0.2) = c_{1,0}\varphi_{1,0}(0.2) + c_{1,1}\varphi_{1,1}(0.2) + \cdots + c_{1,5}\varphi_{1,5}(0.2), \quad (11)$$

$$y(0.4) = c_{1,0}\varphi_{1,0}(0.4) + c_{1,1}\varphi_{1,1}(0.4) + \cdots + c_{1,5}\varphi_{1,5}(0.4), \quad (12)$$

$$y(0.6) = c_{1,0}\varphi_{1,0}(0.6) + c_{1,1}\varphi_{1,1}(0.6) + \cdots + c_{1,5}\varphi_{1,5}(0.6), \quad (13)$$

$$y(0.8) = c_{1,0}\varphi_{1,0}(0.8) + c_{1,1}\varphi_{1,1}(0.8) + \cdots + c_{1,5}\varphi_{1,5}(0.8), \tag{14}$$

$$y(1) = c_{1,0}\varphi_{1,0}(1) + c_{1,1}\varphi_{1,1}(1) + \cdots + c_{1,5}\varphi_{1,5}(1). \tag{15}$$

From (10)-(15), we obtain a system of 5 algebraic equations. Solving such a system of algebraic equations, we obtain wavelet coefficients:

$$c_{1,0} = 1.2740e + 000,\ c_{1,1} = 3.8772e - 001,\ c_{1,2} = 1.6617e - 001,$$

$$c_{1,3} = 2.7695e - 002,\ c_{1,4} = -2.4980e - 016,\ c_{1,5} = 1.3878e - 016.$$

Substituting these wavelet coefficients into (9), we obtain

$$y(x) = 1.2740.\varphi_{1,0}(x) + \cdots + 1.3878e - 016.\varphi_{1,6}(x). \tag{16}$$

Differentiating (16), twice w.r.t. $x$, we obtain

$$y'(x) = 1.2740.\varphi'_{1,0}(x) + \cdots + 1.3878e - 016.\varphi'_{1,6}(x), \tag{17}$$

$$y''(x) = 1.2740.\varphi''_{1,0}(x) + \cdots + 1.3878e - 016.\varphi''_{1,6}(x). \tag{18}$$

**Table 1.** Comparison of results of first and second derivatives of Example 1

| $x$ | $y'(x)$ | | | $y''(x)$ | | |
|---|---|---|---|---|---|---|
| | Exact | Chebyshev wavelets | Absolute errors | Exact | Chebyshev wavelets | Absolute errors |
| 0.2 | 0.240 | 0.240 | 0.047e-013 | 2.400 | 2.400 | 0.0502e-012 |
| 0.4 | 0.960 | 0.960 | 0.014e-013 | 4.800 | 4.800 | 0.0462e-012 |
| 0.6 | 2.160 | 2.160 | 0.040e-013 | 7.200 | 7.200 | 0.0036e-012 |
| 0.8 | 3.840 | 3.840 | 0.026e-013 | 9.600 | 9.600 | 0.0462e-012 |
| 1.0 | 6.000 | 6.000 | 0.026e-013 | 12.000 | 12.000 | 0.0728e-012 |

Results are exactly same as the results obtained with forward or backward interpolation formulae. Table 1 shows the comparison of exact and solutions obtained by Chebyshev wavelet of second kind of Example 1.

**Example 2.** Consider the following data:

| $x$ | 0 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $y = f(x)$ | 1 | 0.8151 | 0.7706 | 0.8809 | 1.1585 |

Using the Newton's forward or backward interpolation formula, we obtain the polynomial $y(x) = x^2 - \sin x + 1$.

Taking $k = 1$, $M = 5$, the wavelet coefficients are as follows:

$c_{1,0} = 7.5145e - 001$, $c_{1,1} = 3.1143e - 002$, $c_{1,2} = 6.8454e - 002$,

$c_{1,3} = 1.9866e - 003$, $c_{1,4} = -7.0160e - 005$.

At different values of $x$, we obtain the following data:

**Table 2.** Comparison of exact and Chebyshev wavelet solutions of Example 2

| $x$ | $y'(x)$ | | | $y''(x)$ | | |
|---|---|---|---|---|---|---|
| | Exact | Chebyshev wavelets | Absolute errors | Exact | Chebyshev wavelets | Absolute errors |
| 0.00 | 1.000 | 0.999 | 7.000e-004 | 2.000 | 1.988 | 1.186e-002 |
| 0.25 | 0.468 | 0.469 | 1.209e-004 | 2.247 | 2.199 | 1.529e-003 |
| 0.50 | 0.122 | 0.122 | 2.159e-004 | 2.479 | 2.390 | 9.220e-005 |
| 0.75 | 0.768 | 0.768 | 2.111e-004 | 2.681 | 2.563 | 2.305e-003 |
| 1.00 | 1.459 | 1.459 | 6.897e-005 | 2.841 | 2.715 | 7.462e-003 |

Results are exactly same as the results obtained with forward or backward interpolation formulae. Table 2 shows the comparison of exact and Chebyshev wavelet solutions of first and second derivatives of Example 2.

**Example 3.** Consider the following data:

| $x$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|---|
| $y = f(x)$ | 1.00001 | 1.00032 | 1.00243 | 1.01024 | 1.03125 | 1.07776 |

Using the Newton's forward or backward interpolation formula, we obtain the polynomial $y(x) = x^5 + 1$. Taking $k = 1$, $M = 6$, the wavelet coefficients are as follows:

$$c_{1,0} = 9.9008e - 001,\ c_{1,1} = 1.4280e - 001,\ c_{1,2} = 9.5200e - 002,$$

$$c_{1,3} = 3.8080e - 002,\ c_{1,4} = 8.6546e - 003,\ c_{1,5} = 8.6546e - 004.$$

At different values of $x$, we obtain the following data:

Results are exactly same as the results obtained with forward or backward interpolation formulae. Table 3 shows the comparison of exact and Chebyshev wavelet solutions of first and second derivatives of Example 3.

**Table 3.** Comparison of exact and Chebyshev wavelet solutions of Example 3

| $x$ | $y'(x)$ | | | $y''(x)$ | | |
|---|---|---|---|---|---|---|
| | Exact | Chebyshev wavelets | Absolute errors | Exact | Chebyshev wavelets | Absolute errors |
| 0.1 | 0.0005 | 0.0005 | 3.727e-013 | 0.0200 | 0.0200 | 3.640e-012 |
| 0.2 | 0.0080 | 0.0080 | 7.725e-014 | 0.1600 | 0.1600 | 2.256e-012 |
| 0.3 | 0.0405 | 0.0405 | 7.961e-014 | 0.5400 | 0.5400 | 9.093e-013 |
| 0.4 | 0.1280 | 0.1280 | 1.149e-013 | 1.2800 | 1.2800 | 1.236e-013 |
| 0.5 | 0.3125 | 0.3125 | 7.494e-014 | 2.5000 | 2.5000 | 5.626e-013 |
| 0.6 | 0.6480 | 0.6480 | 3.141e-014 | 4.3200 | 4.3200 | 1.279e-013 |

## 5. Conclusion

From the above numerical data, it is concluded that Chebyshev wavelet of the second kind is a powerful mathematical tool for computing numerical differentiation arising in numerical analysis. This technique will be applicable to find higher order derivatives of a function.

## Acknowledgement

We are grateful to the anonymous reviewers for their valuable comments which led to the improvement of the research paper.

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