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TWO-STEP ORDER $\frac{3}{2}$ STRONG METHOD FOR APPROXIMATING STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we consider two-step order strong scheme for getting numerical solutions of stochastic differential equations (SDEs) of order $\frac{3}{2}$. It follows a new technique based on replacing stochastic integrals I_α by random variables. Thus we do not need to calculate I_α. We employ Itô-Taylor expansion and Runge-Kutta method to get

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the approximate solutions of the desired order. The experimental results of the approximation method and its error are provided to confirm the validity of the method.

1. Introduction

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, a d-dimensional Wiener martingale $W = (W_t)_{t\geq 0}$ with respect to $\mathcal{F}$ and stochastic differential equations of Itô type

$$d\hat{x}_t = a(t, \hat{x}_t)dt + \sum_{j=1}^{d} b_j(t, \hat{x}_t)dW_t^j, \quad \hat{x}_0 = \hat{x}, \quad 0 \leq t \leq T, \tag{1.1}$$

where a and b are Borel measurable functions defined on $[0, \infty)\times\mathbb{R}^d$ with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, respectively.

For simplicity, we use schemes based on equidistant time discretization points $t_n = t_0 + nh$ with step size $h = \frac{T - t_0}{N}$, $N = 1, 2, ...$ to fix the numerical method on a given time interval $[t_0, T]$. The trajectories of the solutions are computed at discrete time steps since that numerical methods for SDEs are recursive methods. Because of having two types of schemes (strong or weak), our attention in this paper is on the scheme which converges in the strong sense. We say that a sequence of approximation processes $(\hat{x}(jh))_{j=1}^{j=N}$ converges strongly with order p to the solutions $\hat{x}^{(j)}$ of SDE (1.1) at time T if there exist a constant $C > 0$ and some $\delta_0 > 0$ such that for each $h \in (0, \delta_0]$,

$$\left(\mathbb{E}\left(\max_{j=1}^{N} |\hat{x}^{(j)} - \hat{x}(jh)|^2\right)\right)^{\frac{1}{2}} \leq Ch^p. \tag{1.2}$$

For SDEs (1.1), the basic scheme which we have is Euler-Maruyama

$$\hat{x}_i^{(j+1)} = \hat{x}_i^{(j)} + a_i(t_j, \hat{x}_i^{(j)})h + \sum_{k=1} b_{ik}(t, \hat{x}^{(j)})\Delta W_k^{(j)}. \tag{1.3}$$

In order to get Milstein scheme, we add the quadratic terms

$$\hat{x}_i^{(j+1)} = \hat{x}_i^{(j)} + a_i(t_j, \hat{x}_i^{(j)})h + \sum_{k=1} b_{ik}(t, \hat{x}^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(t_j, \hat{x}^{(j)})I_{kl}^{(j)}, \tag{1.4}$$

where

$$\Delta W_k^{(j)} = W_k((j+1)h) - W_k(jh),$$

$$I_{kl}^{(j)} = \int_{jh}^{(j+1)h} \{W_k(t) - W_k(jh)\}dW_l(t)$$

and $\rho_{ikl}(t, \hat{x}) = \sum_{m=1}^{q} b_{mk}(t, \hat{x})\frac{\partial b_{il}}{\partial \hat{x}_m} b_{ml}(t, \hat{x})$ [6]. In general, the analytic solutions of SDEs usually are not available, so we need to develop stochastic numerical method which can give higher-order approximate solutions of the SDEs (1.1). The aim of this paper is to develop a new method that does not require a computational cost. Because of getting higher-order of approximate solutions of SDEs requires adding extra terms from Itô-Taylor expansion, this leads to have difficulty with computing iterated stochastic integrals I_α. In the recent years, there are several efficient methods which have been presented to overcome this issue such as methods in [1-9, 11-14] but still these methods do not meet our needs. The key idea was to use the perturbation and coupling techniques in the numerical scheme to obtain higher-order approximation of order $\frac{3}{2}$.

1.1. Perturbation method

The basic idea of this method is to overcome the difficulty that we might have if we want to simulate a random variable which contains two independent random variables, either of which is hard to generate. In order to clarify the concept of this method, read the following: suppose we wish to simulate $U = X + \varepsilon Y$, where X and Y are independent, X has a smooth density, and ε is small. In addition, suppose that X is easy to generate while Y is hard to generate. Then, generating U by X and Y will be hard. Alternatively, let us suppose another random variable which is easy to generate to overcome this dilemma, so we assume Z which is independent of X. The random variable Z has the same moments up to order $m-1$ as Y (i.e., $E(Z^k) = E(Y^k)$ for $k = 1, 2, 3, ..., m-1$). Hence, it can be proved that $V = X + \varepsilon Z$ is an approximation to U with error of order ε^m [6]. The justification for this can be seen by writing f_X for the density of X. So we have

$$f_U(\hat{x}) = E f_X(\hat{x} - \varepsilon Y) = f(\hat{x}) + \sum_{k=1}^{m-1} \frac{-\varepsilon}{k!} f^{(k)}(\hat{x}) E(Y^k) + O(\varepsilon^m).$$

Due to Z having the same moments, we get the same expression for $f_V(\hat{x})$, so $f_U(\hat{x}) - f_V(\hat{x}) = O(\varepsilon^m)$. It can be expected to get from Wasserstein distance estimation of the same order $W(U, V) = O(\varepsilon^m)$ which gives a coupling with $E(U - V)^2 = O(\varepsilon^{2m})$ [6].

1.2. Coupling

Let (X, μ) and (Y, ν) be two probability spaces. Coupling μ and ν means constructing two random variables $\overline{X}$ and $\overline{Y}$ on some probability space (Ω, P), such that $\text{law}(\overline{X}) = \mu$, $\text{law}(\overline{Y}) = \nu$. The couple $(\overline{X}, \overline{Y})$ is called a *coupling* of (μ, ν). By abuse of language, the law of $(\overline{X}, \overline{Y})$ is also called a *coupling* of (μ, ν) [10].

2. Two-step Order $\frac{3}{2}$ Strong Scheme for Stochastic Differential Equations

The computation of numerical solutions by multi-step methods is more efficient than using one-step method because only one evaluation of the right hand side is required. In addition, the stability of the multi-step methods is sometimes better for larger time steps, although there are unstable multi-step methods. For SDEs (1.1), we recall the following formula presented in [12]:

$$\begin{aligned}\hat{x}^{(j+1)} &= \hat{x}^{(j-1)} + 2ah \\ &\quad - \frac{\sqrt{h}}{2}\sum_{k=1}^{m}\{a((j-1)h,\ \Upsilon_+^k(j-1)) - a((j-1)h,\ \Upsilon_-^k(j-1))\} \\ &\quad \times \Delta W^k(j-1) + D(j) + D(j-1) \qquad (2.1)\end{aligned}$$

with

$$\begin{aligned}D(j) &= \sum_{k=1}^{m}\left[b_k\Delta W^k(j) + \frac{1}{h}\{b_k(j+1)h,\ \hat{x}^{(j)} - b_k\}I_{(0,k)}(jh,\ (j+1)h)\right] \\ &\quad + \frac{1}{2\sqrt{h}}\sum_{l=0}^{m}\sum_{k=1}^{m}\{b_l(jh,\ \Upsilon_+^k(j)) - b_l(jh,\ \Upsilon_-^k(j))\}I_{(k,l)}(jh,\ (j+1)h) \\ &\quad + \frac{1}{2h}\sum_{k,l=1}^{m}\{b_l(jh,\ \Upsilon_+^k(j)) - 2b_l + b_l(jh,\ \Upsilon_-^k(j)\}I_{(0,l)}(jh,\ (j+1)h) \\ &\quad + \frac{1}{2h}\sum_{k,l,n=1}^{m}\{b_n(jh, \Phi_+^{(k,l)}(j)) - b_n(jh, \Phi_-^{(k,l)}(j)) \\ &\qquad - b_n(jh, \Upsilon_+^k(j)) + b_n(jh, \Upsilon_-^k(j))\}I_{(k,l,n)}(jh, ((j+1)h)),\end{aligned}$$

where

$$\Upsilon_{\pm}^{k}(j) = \hat{x}^{(j)} + \frac{1}{m}ah \pm b_k\sqrt{h},$$

$$\Phi_{\pm}^{(k,l)}(j) = \Upsilon_{+}^{k}(j) \pm b_l(jh, \Upsilon_{+}^{k}(j))\sqrt{h}.$$

Because of having difficulty in generating these iterated stochastic integrals I_α in (2.1) directly when Brownian motion has dimension > 1, the concept of our method is to replace these iterated stochastic integrals I_α in (2.1) by random variables and then using perturbation and coupling techniques to get numerical solutions of the desired order. The implementation of the method requires first the assumption that the existence and uniqueness theorem is satisfied:

(i) Lipschitz condition: For all $x, y \in R^d$ and $f \in [t_0, T]$, we have

$$|a(t, \hat{x}) - a(t, \hat{y})| \leq M|\hat{x} - \hat{y}| \tag{2.2}$$

and

$$|b(t, \hat{x}) - b(t, \hat{y})| \leq M|\hat{x} - \hat{y}|. \tag{2.3}$$

(ii) Linear growth bound: For all $(x, t) \in R^d \times [t_0, T]$, we have

$$|a(t, \hat{x})|^2 \leq M^2|1 + |\hat{x}|^2| \tag{2.4}$$

and

$$|b(t, \hat{x})|^2 \leq M^2|1 + |\hat{x}|^2|.$$

Let $\hat{x}(0)$ be a random variable which is independent of the σ-algebra $\mathcal{F}_{\infty}^{(m)}$ generated by $W_s(.)$, $s \geq 0$ such that $E|\hat{x}_0|^2 < \infty$. Then there is a unique t-continuous solution $x_t(\omega)$ of the initial value problem

$$d\hat{x}_t = a(t, \hat{x}_t)dt + b(t, \hat{x}_t)dW_t, \quad \hat{x}_0 = \hat{x}(0), \quad 0 \leq t \leq T, \tag{2.5}$$

with the property that $\hat{x}_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^{\hat{x}(0)}$ generated by $\hat{x}(0)$ and $W_s(.)$, $s \le t$.

Then, we replace the iterated stochastic integrals $\bar{I}_\alpha$ in (2.1) by random variables using this definition

$$\bar{I}_\alpha = h^{\frac{(l(\alpha)+n(\alpha))}{2}} \sum_{\beta=(i_1,\ldots,i_l)} L_\beta \prod_{k:i_k<j_k} \overline{V}_{j_k}, \tag{2.6}$$

where we calculate the sum over all $\beta = (i_1, ..., i_l)$ such that for each $k \in \{1, ..., l\}$, we have either $i_k = j_k$ or $i_k = 0 < j_k$. The random variables L_β and $\overline{V}_k \sim N(0, 1)$ are generated independently.

Thus, (2.1) is rewritten by using (2.6) as

$$\begin{aligned}\hat{x}^{(j+1)} = \hat{x}^{(j-1)} + 2ah \\ - \frac{\sqrt{h}}{2} \sum_{k=1}^{m} \{a((j-1)h, \Upsilon_+^k(j-1)) - a((j-1)h, \Upsilon_-^k(j-1))\} \\ \times \Delta W^k(j-1) + D(j) + D(j-1)\end{aligned} \tag{2.7}$$

with

$$D(j) = \sum_{k=1}^{m} \left[b_k \Delta W^k(j) + \frac{1}{h} \{b_k((j+1)h, \hat{x}^{(j)} - b_k\} \left(\frac{1}{2} \overline{V}_k + L_{0k} \right) \right]$$

$$+ \frac{1}{2\sqrt{h}} \sum_{l=0}^{m} \sum_{k=1}^{m} \{b_l(jh, \Upsilon_+^k(j)) - b_l(jh, \Upsilon_-^k(j))\}$$

$$\times \left(\frac{1}{2} \overline{V}_l \overline{V}_k + L_{0k} \overline{V}_l - L_{0l} \overline{V}_k + L_{lk} \right)$$

$$+\frac{1}{2h}\sum_{k,l=1}^{m}\{b_l(jh,\,\Upsilon_+^k(j))-2b_l+b_l(jh,\,\Upsilon_-^k(j)\}\left(\frac{1}{2}\overline{V}_l-L_{0l}\right)$$

$$+\frac{1}{2h}\sum_{k,l,n=1}^{m}\{b_n(jh,\,\Phi_+^{(k,l)}(j))-b_n(jh,\,\Phi_-^{(k,l)}(j))$$

$$-b_n(jh,\,\Upsilon_+^k(j))+b_n(jh,\,\Upsilon_-^k(j))\}(\overline{V}_n\overline{V}_l\overline{V}_k-\delta_{lk}\overline{V}_n-\delta_{nk}\overline{V}_l-\delta_{nl}\overline{V}_k),$$

where

$$\Upsilon_\pm^k(j)=\hat{x}^{(j)}+\frac{1}{m}ah\pm b_k\sqrt{h},$$

$$\Phi_\pm^{(k,l)}(j)=\Upsilon_+^k(j)\pm b_l(jh,\,\Upsilon_+^k(j))\sqrt{h}$$

for $k, l, n > 0$.

In order to generate the random variables in (2.7), we need to calculate the moments using the following three lemmas:

Lemma 1. *Let* $\beta=(jj\cdots j)$ *with length* $l\geq 2$. *Then* (i) *if* $j=0$, *then* $L_\beta=\frac{1}{l!}$, *and* (ii) *if* $j>0$, *then* $L_\beta=0$ *if* l *is odd, while* $L_\beta=\frac{(-1)^r}{2^r r!}$ *if* $l=2r$.

Proof. We refer the reader to [6, Lemma 5]. □

Lemma 2. *If* $\beta_1, ..., \beta_s\in\mathcal{M}$ *and if some* $j\geq 1$ *occurs an odd number of times in the concatenated multi-index* $\beta_1, ..., \beta_s$, *then* $\mathbb{E}(L_{\beta_1}, ..., L_{\beta_s})=0$.

Proof. We refer the reader to [6, Lemma 6]. □

Lemma 3. (i) *If* $0\leq k<l$, *then* $L_{lk}=-L_{kl}$ *and* $\mathbb{E}(L_{kl}^2)=\frac{1}{12}$.

(ii) *If* $k>0$, *then* $\mathbb{E}L_{0kk}=\mathbb{E}L_{k0k}=\mathbb{E}L_{kk0}=-\frac{1}{6}$.

Proof. We refer the reader to [6, Lemma 7]. □

Now, MATLAB can be used to implement the method by generating L_{kl} with $k < l$ using normal distribution with mean zero, variance $\frac{1}{12}$ being uncorrelated. But this is not the only way of generating L_β. This can be also done by setting $L_{lk} = -L_{kl}$ for $0 \le k < 1$, $L_{00} = \frac{1}{2}$, $L_{kk} = \frac{1}{2}$, $L_{000} = \frac{1}{6}$, $L_{0kk} = L_{k0k} = L_{kk0} = -\frac{1}{6}$ for $k > 0$, and all other L_β of length 3 to 0.

2.1. Illustrative example

The following system of SDEs had been simulated by using (2.7) for number of steps $N = 200$:

$$\begin{cases} d\hat{x}_1 = \hat{x}_1 dt + (\sin^2(\hat{x}_1) + 1) dW_t - \cos^2(\hat{x}_2) dV_t, \\ d\hat{x}_2 = \dfrac{t+2}{1+\hat{x}_2^2} dt + \cos^2(\hat{x}_1) dW_t + (\sin^2(\hat{x}_2) + 1) dV_t, \\ \hat{x}_1(0) = 1, \\ \hat{x}_2(0) = 2, \\ 0 \le t \le 1. \end{cases} \tag{2.8}$$

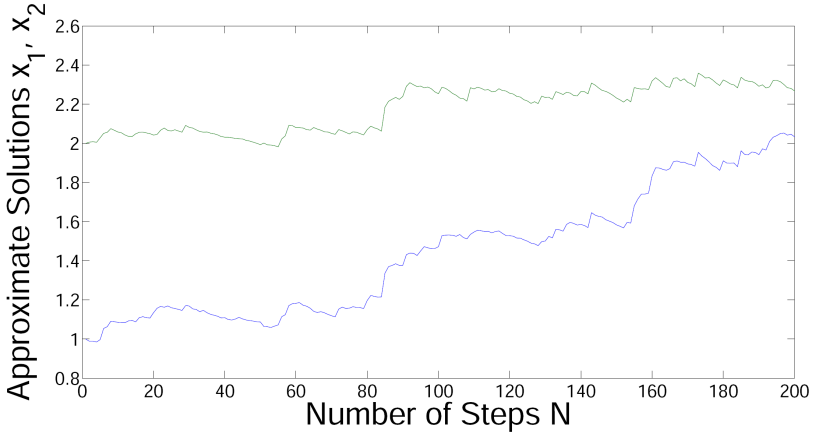

Figure 1. Approximate solutions of (2.8).

Figure 1 shows the piecewise linear curve of the approximate solutions on interval $[0, 1]$ at time steps $h = \frac{1}{N}$ for $N = 200$ by using (2.7). Practically, the error of the approximate solutions for our method is estimated statistically using computer. Since the explicit solutions of stochastic differential equations are usually unknown, it would estimate the error of the method between two levels of approximate solutions using the weak approximation bound. From Kantorovich-Rubinstein theorem, it follows that

$$\mathbb{W}_1(x_h, x_{\frac{h}{2}}) = \sup|\mathbb{E}f(x_h) - \mathbb{E}f(x_{\frac{h}{2}})| = O(\varepsilon^m), \tag{2.9}$$

where x_h and $x_{\frac{h}{2}}$ are simulating approximations at two levels for the same driving path at step sizes h and $\frac{h}{2}$, respectively, which provides (2.7). The infimum is taken over all possible coupling and the supremum over functions $f : \mathbb{R}^q \to \mathbb{R}$, satisfying $|f(u) - f(v)| \le |u - v|$ for all $u, v \in \mathbb{R}^q$. To implement that, first it is needed to plug the approximate solutions which comes from approximating (2.8) in any Lipschitz function such as

$$f(x_1, x_2) = |x_1| + |x_2|.$$

Then, to construct a confidence interval for the error, it is required to estimate the expectation:

$$\mathbb{E}(f(x_1, x_2)_h) = \frac{\sum_{r=1}^{R}(f(x_1, x_2)_h)(r)}{R},$$

standard deviation

$$\text{S.D} = \sqrt{\frac{1}{R-1}\sum_{r=1}^{R}(f(x_1, x_2)_h(r) - \mathbb{E}(f(x_1, x_2)_h))^2}$$

and the standard error

$$\text{S.E}(f(x_1,\, x_2)) = \sqrt{\frac{(\text{S.D}(f(x_1,\, x_2)_h))^2 + (\text{S.D}(f(x_1,\, x_2)_{2h}))^2}{R}},$$

where R is the number of simulations. Then, the standard error is multiplied by the value of the confidence level such as in our example 95% which is 1.96 using the formula:

$$\text{C.I.} = [(\mathbb{E}(f(x_1,\, x_2))_h - \mathbb{E}(f(x_1,\, x_2))_{2h}) - 1.96 \times \text{S.E};$$

$$(\mathbb{E}(f(x_1,\, x_2))_h - \mathbb{E}(f(x_1,\, x_2))_{2h}) + 1.96 \times \text{S.E}].$$

Table I shows the estimation of the LHS of (2.9) in the time interval $[0, 1]$ for $N = 2^9,\ 2^{10},\ 2^{11}$, step sizes $h = 1/2^9,\ 1/2^{10},\ 1/2^{11}$ and the simulation number $R = 3000000$. Figure 2 shows the confidence intervals along the blue line of the least square for the expectations against step sizes. The level of the confidence interval is 95%. The green line is the upper confidence bounds, while the red line is the lower confidence bounds. The red line has just two points because the lower bound of one of the confidence intervals is negative. Therefore, we swap the negative value in the confidence interval with zero, so when we take the log for this confidence interval, then we get $-\infty$ as the lower bound for this confidence interval, but it is not shown in the graph. Therefore, we have used this method to handle the negative value in the confidence interval because if we do not do this, then the least square regression line does not pass through confidence intervals as it should be. This scheme aims to give approximation of order $\frac{3}{2}$, where the slope of the line is 1.80.

Table I. Truncation error

N	Truncation error	Confidence intervals
2^9	0.0104	0.0069 : 0.0139
2^{10}	0.0039	0.0005 : 0.0073
2^{11}	0.0014	-0.002 : 0.0048

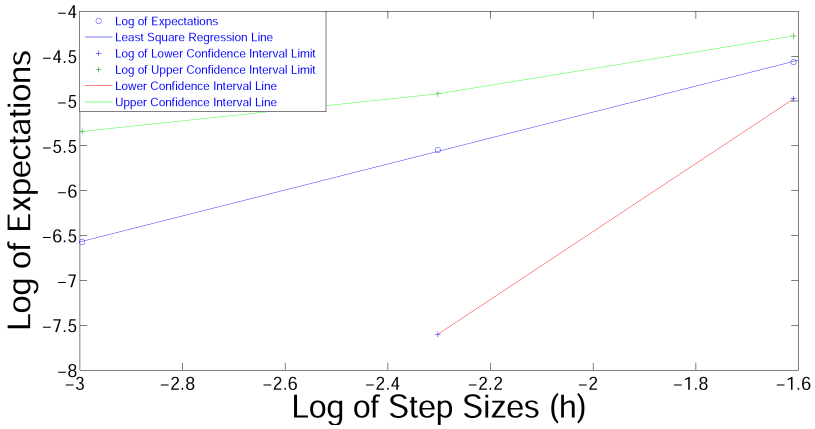

Figure 2. Weak bound of final time approximation.

3. Conclusion

The newly developed method is needed in many fields since so far no efficient method is available to compute higher-order numerical solutions in two-steps for SDEs. It can be used to compute a strong approximation with any dimension of SDEs because it does not involve a computational cost. Also, it is possible to extend this new method to achieve a strong approximation with a higher error rate, such as 2, by adding additional terms from the Itô-Taylor expansion, which would be achieved in the next paper.

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