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Article

Mathematical modelling and controllability analysis of fractional order coal mill pulverizer model

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Abstract: This paper investigates the controllability of nonlinear dynamical systems and their applications, with a focus on fractional-order systems and coal mill models. A novel theorem is proposed, providing sufficient conditions for controllability, including constraints on the steering operator and nonlinear perturbation bounds. The theorem establishes the existence of a contraction mapping for the nonlinear operator, enabling effective control strategies for fractional systems. The methodology is demonstrated through rigorous proof and supported by an iterative algorithm for controller design. Additionally, the controllability of a coal mill system represented as a nonlinear differential system, is analyzed. The findings present new insights into the interplay of fractional dynamics and nonlinear systems, offering practical solutions for real-world control problems.

Keywords: fractional system; nonlinear systems; coal mill pulverizer model; pulverizer; coal mill; power plant; controllability analysis

AMS Classfication: 93B05; 34H05; 93C10; 93C15

1. Introduction

In recent decades, there has been a growing interest in integral equations and fractional differential and their importance in various scientific fields, including science and engineering [1–3]. The appeal of fractional calculus lies in its ability to effectively describe memory and hereditary properties of diverse materials and processes through fractional derivatives. Many real-world systems are better characterized by non-integer order dynamic models derived from fractional calculus, such as the Basset problem, the Bagley-Torvik equation, and various fluid dynamics models. Fractional dynamical systems have recently attracted significant interest in control system communities, even though fractional-order control problems have been studied since the 1960s. Extending traditional controllers or control schemes to non-integer orders introduces more tuning parameters and enhances flexibility in adjusting a control system's response time. As a result, fractional-order control systems have notably impacted practical applications across all areas of control theory [4–8]. Despite this, the study of fractional-order dynamical systems in the context of control theory has been limited due to a lack of suitable mathematical methods. However, several researchers have made successful attempts in this field [9–12]. Recently, Kaczorek [13] explored fractional control problems in SISO and MIMO systems, highlighting that fractional-order controllers often exhibit superior performance compared to their integer-order counterparts [13,14] Consequently, research on fractional-order systems remains an active and expanding area world wide. A control system is composed of interconnected components designed to produce a desired response. Controllability, a key structural property of dynamical systems, signifies the ability to steer a system from any initial state to any final state using a set of permissible controls. Investigating the controllability of fractional dynamical systems is essential for various applied problems, as fractional order derivatives and integrals often yield more effective results in control theory than their integerorder counterparts. Despite its importance, recent contributions to the controllability of fractional dynamical systems have been relatively scarce [15].

The controllability of linear fractional dynamical systems was studied by Matignon and D'Andréa-Novel [16] , while Vinagre et al. [17] introduced key concepts for fractional-order systems. Bettayeb and Djennoune [18] examined controllability using rank conditions. Chen et al. [19] concentrated on robust controllability for uncertain fractional-order linear time-invariant systems formulated in state-space representation. Guermah et al. [20] discussed the controllability and observability of discrete-time fractional-order systems. Mozyrska and Torres [21, 22] derived results on controllability and introduced modified energy control approaches for fractional linear systems using Riemann-Liouville and Caputo derivatives. In recent studies, Balachandran et al. [23–27] analyzed the controllability of both linear and nonlinear fractional dynamical systems, providing sufficient conditions for controllability in systems with fractional orders $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$. Govindraj and George [27] analyzed the controllability of semilinear systems through a functional analytic approach, assuming that the nonlinear term meets Lipschitzian and monotonicity conditions. This research underscores the controllability of semilinear fractional dynamical systems where the nonlinear term does not include a controller.

This manuscript examines the controllability conditions for a nonlinear Caputo fractional system described as follows:

$$\begin{aligned} {}^{c}D^{\alpha}_{t_0}x(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{F}(t, x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{1}$$

for $0 < \alpha \leq 1$. Here, $x(t)$, defined in the Hilbert space $\mathbb{X}$, represents the state vector for all $t \in [t_0, t_1]$, and $u(t) \in \mathbb{L}^2([t_0, t_1], \mathbb{U})$ denotes the control input of the system Equation (1). The operators $\mathcal{A}$ and $\mathcal{B}$ are linear. These controllability conditions are further applied to analyze the controllability of the coal mill pulverizer model, which is governed by a nonlinear system of the form Equation (1) as we emphasize the need for a fractional model to represent the Coal Mill Pulverizer system accurately. Traditional integer-order models fail to capture the inherent nonlinearities, time-varying dynamics, and memory effects present in the system. The fractional model, by its nature, can account for these complexities, offering a more accurate representation of the system's behavior. For instance, in the coal mill pulverization process, we observe that the dynamics of coal feed rate and pulverizer pressure exhibit non-integer-order dependencies that the fractional model captures effectively. This capability is especially important for processes exhibiting long-term memory and time delays.

2. Preliminaries

This section presents fundamental concepts from fractional calculus, controllability of linear systems, and non-linear functional analysis, which form the foundation for this work.

Definition 1. *The Riemann-Liouville fractional integral operator of order $\nu > 0$ for a function $g \in L_1(\mathbb{R}_+)$ is expressed as :*

$$J^{\nu}_{a+}g(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-\tau)^{\nu-1} g(\tau)\, d\tau,$$

provided the integral on the right-hand side converges. Here, $\Gamma(\cdot)$ represents the gamma function [28].

Definition 2. *The Caputo fractional derivative of order $\nu > 0$, where $m-1 < \nu < m$ and $m \in \mathbb{N}$, is defined as :*

$${}^{c}D^{\nu}_{a+}g(t) = \frac{1}{\Gamma(m-\nu)} \int_a^t (t-\tau)^{m-\nu-1} \frac{d^m g(\tau)}{d\tau^m}\, d\tau$$

provided the integral exists, where $m = \lfloor \nu \rfloor + 1$. [28]

Definition 3. *The one-parameter and two-parameter Mittag-Leffler functions are defined, respectively, as:*

$$E_{\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + 1)}, \quad E_{\nu,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + \mu)}$$

where $\nu, \mu > 0$ and $z \in \mathbb{C}$.

Definition 4. *Let $\mathcal{T}_{\alpha}(t)$ and $\mathcal{T}_{\alpha,\beta}(t)$, for $t \geq 0$, denote families of operators mapping $\mathbb{X}$ into itself* [28]. *These operators, generated by a linear operator $A : \mathbb{X} \to \mathbb{X}$, satisfy the following conditions :*

1) $\mathcal{T}_{\alpha}(0) = I$ *and* $\mathcal{T}_{\alpha,\beta}(0) = I$, *where I is the identity operator.*
2) $\mathcal{T}_{\alpha}(t)$ *satisfies the linear fractional equation ${}^{c}D^{\alpha}x(t) = A(t)x(t)$ in a Banach space $\mathbb{X}$.*
3) $\lim_{\mu \to 1} \mathcal{T}_{\alpha,\beta}(t) = \mathcal{T}_{\alpha}(t)$.

Now we will discuss the Controllability of Linear Systems as the controllability of a nonlinear system depends on the controllability of a corresponding linear system. Therefore, we have first discussed the controllability of the corresponding linear system

$$\begin{aligned} {}^{c}D^{\alpha}_{t_0}x(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

where x is the state vector and u is the controller of the system Equation (2).

The solution of the system Equation (2) is given by

$$x(t) = \mathcal{T}_\alpha(t - t_0)x_0 + \int_{t_0}^{t} (t - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t - s)\mathcal{B}u(s)ds. \tag{3}$$

The linear system Equation (2) is controllable over the interval $[t_0, t_1]$ if there exists a controller $u(t)$ steers the initial state x_0 to desired final state x_1 at time t_1. This means the solution Equation (3) at time $t = t_1$ steers

$$x_1 = x(t_1) = \mathcal{T}_\alpha(t_1 - t_0)x_0 + \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_1 - s)\mathcal{B}u(s)ds$$

For discussion of controllability of the linear system Equation (2) define an operator $\mathcal{C} : \mathbb{L}^2([t_0, t_1], \mathbb{U}) \to \mathbb{X}$ by

$$\mathcal{C}u(t) = \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_1 - s)\mathcal{B}u(s)ds \tag{4}$$

whose adjoint $\mathcal{C}^* : \mathbb{X} \to \mathbb{L}^2([t_0, t_1], \mathbb{U})$

$$\mathcal{C}^*\omega = (t_1 - t)^{1-\alpha}\mathcal{B}^*\mathcal{T}_{\alpha,\alpha}(t_1 - t)\omega \tag{5}$$

Finally defining the operator $\mathcal{W} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{W}\omega = \int_{t_0}^{t_1} \mathcal{T}_{\alpha,\alpha}(t_1 - s)\mathcal{B}\mathcal{B}^*\mathcal{T}^*_{\alpha,\alpha}(t_1 - s)\omega ds \tag{6}$$

forgoing theorem gives a characterization for the controllability of linear system over $[t_0, t_1]$ Equation (2).

Theorem 1. *The following statements are equivalent* [27]*:*

1) *The system Equation* (2) *is controllable.*
2) $Range(\mathcal{C}) = \mathbb{X}$.
3) *There exist* $\gamma > 0$ *such that* $||\mathcal{C}^*\omega||^2 \geq \gamma^2||\omega||^2$ *for all* $\omega \in \mathbb{X}$.
4) *There exist* $\gamma > 0$ *such that* $< \mathcal{W}\omega, \omega > \geq \gamma^2||\omega||^2$ *for all* $\omega \in \mathbb{X}$.
5) $Kernel(\mathcal{C}^*) = \{0\}$ *and* $Range(\mathcal{C}^*)$ *is closed.*

A controller that steers given initial state x_0 *to desired final state* x_1 *is given by*

$$u(t) = (t_1 - t)^{1-\alpha}\mathcal{B}^*\mathcal{T}^*_{\alpha,\alpha}(t_1 - t)\mathcal{W}^{-1}\big[x_1 - \mathcal{T}_{\alpha,\alpha}(t_1 - t_0)x_0\big]$$

Corollary 1. *If the system Equation* (2) *is controllable on* $[t_0, t_1]$ *then, there exists steering operator* $\mathcal{S} : \mathbb{X} \to L^2([t_0, t_1], \mathbb{U})$ *define by* $\mathcal{S}\omega = \mathcal{C}^*\mathcal{W}^{-1}\omega$ *is the right inverse of* $\mathcal{C}$ [27]. *This means* $\mathcal{C} \circ \mathcal{S} = I$.

3. Controllability of nonlinear system

This section discusses the controllability of the Caputo fractional nonlinear system. The system governed by

$$\begin{aligned} {}^cD_{\alpha,t_0}x(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{F}(t, x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{7}$$

where $0 < \alpha \leq 1$, $x(t)$ represents the state, and $u(t)$ denotes the controller of the system described by Equation (7). models a Caputo fractional nonlinear system where $0 < \alpha \leq 1$ is the fractional order. In this context, $x(t)$ represents the state vector, capturing the system's dynamic states, while $u(t)$ denotes the control input applied to influence the system's behavior. The system matrix $\mathcal{A}$ encapsulates the inherent dynamics, such as damping or stiffness effects, and the control matrix $\mathcal{B}$ describes how the control inputs impact the system's states. The term $\mathcal{F}(t, x(t), u(t))$ accounts for nonlinearities, including effects such as friction or saturation, which depend on the system state and control. The use of the Caputo fractional derivative ${}^{c}D_{\alpha,t_0}$ introduces memory effects, meaning that the current state depends not only on the present dynamics but also on the entire past behavior. Equation (8) provides the mild solution of Equation (8), derived using fractional calculus principles. The solution is expressed as Assuming the nonlinear function $\mathcal{F}$ is sufficiently well-behaved so that the system Equation (7) admits a unique mild solution:

$$x(t) = \mathcal{T}_\alpha(t - t_0)x_0 + \int_{t_0}^{t} (t - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t - s)\left[\mathcal{B}u(s) + \mathcal{F}(s, x(s), u(s))\right] ds \quad (8)$$

for all fixed u.

The system described by Equation (7) is considered controllable over the interval $[t_0, t_1]$ if there exists a controller $u(t)$ that drives the initial state x_0 to the desired final state x_1 at time t_1. This means

$$x_1 = x(t_1) = \mathcal{T}_\alpha(t-t_0)x_0 + \int_{t_0}^{t_1} (t_1-s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_1-s)\left[\mathcal{B}u(s) + \mathcal{F}(s, x(s), u(s))\right] ds$$

The above equation represents the fractional-order system's free-response operator. The first term, $\mathcal{T}_\alpha(t - t_0)x_0$, captures the free evolution of the state starting from the initial condition x_0, while the integral term incorporates the effects of the control input $u(t)$ and the nonlinearities $\mathcal{F}(t, x(t), u(t))$. The kernel $(t - s)^{\alpha-1}$ reflects the memory characteristic inherent in fractional-order systems.

To analyze the controllability of a nonlinear system, we define the operator $\mathcal{G} : \mathbb{L}^2([t_0, t_1], \mathbb{U}) \to \mathbb{X}$ by

$$\begin{aligned} \mathcal{G}u &= \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_1 - s)\left[\mathcal{B}u(s) + \mathcal{F}(s, x(s), u(s))\right] ds \\ &= \mathcal{C}u + \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_1 - s)\mathcal{F}(s, x(s), u(s))ds \end{aligned} \quad (9)$$

then operator determines whether the system is controllable by mapping the input space to the state space, with controllability achieved if and only if $\mathcal{G}$ is onto. The nonlinear term $\mathcal{F}(t, x(t), u(t))$ must satisfy certain regularity conditions, such as Lipschitz continuity, to ensure the existence and uniqueness of the mild solution, which is necessary for practical implementation of control strategies.

Assuming controllability of the corresponding linear system, define the operator $\bar{\mathcal{G}} : \mathbb{X} \to \mathbb{X}$ by

$$\begin{aligned}\bar{\mathcal{G}}\omega &= \mathcal{G}\circ\mathcal{S}\omega = \mathcal{C}\circ\mathcal{S}\omega + \int_{t_0}^{t_1}(t_1-s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_1-s)\mathcal{F}(s,x_\omega(s),\mathcal{S}\omega(s))ds \\ &= \omega + \mathcal{H}\omega = (\mathcal{I}+\mathcal{H})\omega\end{aligned} \tag{10}$$

where $\mathcal{I}$ is the identity operator Equation(7).

Theorem 2. *The system Equation* (7) *is controllable if the operator* $\bar{\mathcal{G}}$ *is non-singular, and the controller that steers the initial state to the desired final state* x_1 *at time* t_1 *is given by*

$$u(t) = \mathcal{C}^*\mathcal{W}^{-1}\bar{\mathcal{G}}^{-1}\left[x_1 - \mathcal{T}_\alpha(t_1-t_0)x_0\right]$$

Proof. Since the operator $\bar{\mathcal{G}}$ is non-singular, substituting the controller into the Equation (8), the state of the system at $t = t_1$ becomes:

$$\begin{aligned}x(t_1) &= \mathcal{T}_\alpha(t-t_0)x_0 + \mathcal{G}u \\ &= \mathcal{T}_\alpha(t-t_0)x_0 + (\mathcal{I}+\mathcal{H})(\mathcal{C}\mathcal{C}^*)\mathcal{W}^{-1}(\mathcal{I}+\mathcal{H})^{-1}\left[x_1 - \mathcal{T}_\alpha(t_1-t_0)x_0\right] \\ &= x_1\end{aligned}$$

Hence, the system is controllable over the interval $[t_0, t_1]$. □

Therefore, the controllability of system Equation (7) reduces to the invertibility of the operator $\bar{\mathcal{G}}$. The following theorem derives the conditions under which the operator $\bar{\mathcal{G}}$ is non-singular.

Theorem 3. *If the operator* $\mathcal{H}^{(n)}$ *is a contraction for some* $n \geq 1$*, then* $\bar{\mathcal{G}}$ *is non-singular.*

Proof. If $\mathcal{H}^{(n)}$ is a contraction for some $n \geq 1$, then by the Banach fixed point theorem, the operator equation $\omega = -\mathcal{H}\omega$ has a unique solution. This implies that the equation $(\mathcal{I}+\mathcal{H})\omega = 0$ has a unique trivial solution. Hence, the operator $\bar{\mathcal{G}}$ is non-singular. □

The next theorem discusses the controllability of the nonlinear system Equation (7).

Theorem 4. *If*

1) *The corresponding linear system is controllable.*
2) *There exist constants* f_1 *and* f_2 *such that*

$$||\mathcal{F}(t,x_1,u_1) - \mathcal{F}(t,x_2,u_2)|| \leq f_1||x_1-x_2|| + f_2||u_1-u_2||$$

for all $x, u \in \mathbb{B}_{r_0}(x_0,u_0)$ *for some* $r_0 > 0$ *where*

$$u_0(t) = (t_1-t)^{1-\alpha}\mathcal{B}^*\mathcal{T}^*_{\alpha,\alpha}(t_1-t)\mathcal{W}^{-1}\left[x_1 - \mathcal{T}_{\alpha,\alpha}(t_1-t_0)x_0\right]$$

then the nonlinear system Equation (7) *is controllable over* $[t_0, t_1]$ *and the controller which drives the system Equation* (7) *to desired final state* x_1 *at* $t = t_1$ *is given by*

$$u(t) = \mathcal{C}^*\mathcal{W}^{-1}\bar{\mathcal{G}}^{-1}\left[x_1 - \mathcal{T}_\alpha(t_1-t_0)x_0\right]$$

Proof. To prove the system is controllable, it is sufficient to prove that the operator $\mathcal{H}^{(n)}$

is a contraction for some $n \geq 1$. Therefore, for $x_1, x_2, u_1, u_2 \in \mathbb{B}(x_0, u_0)$, consider

$$||\mathcal{H}^{(n)}x_1 - \mathcal{H}^{(n)}x_2|| \leq \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} ||\mathcal{T}_{\alpha,\alpha}|| \, ||\mathcal{H}^{(n-1)}x_1 - \mathcal{H}^{(n-1)}x_2|| \, ds$$

$$\leq M^2(f_1 + f_2 S) \int_{t_0}^{t_1} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1}(t_1 - s_1)||\mathcal{H}^{(n-2)}x_1 - \mathcal{H}^{(n-2)}x_2|| \, ds_1 \, ds$$

$$\leq M^3(f_1 + f_2 S)^2 \int_{t_0}^{t_1} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1}(s - s_1)(s_1 - s_2)^{\alpha-1}||\mathcal{H}^{(n-3)}x_1 - \mathcal{H}^{(n-3)}x_2|| \, ds_2 \, ds_1 \, ds$$

and continuing this process gives

$$||\mathcal{H}^{(n)}x_1 - \mathcal{H}^{(n)}x_2|| \leq M^n (f_1 + f_2 S)^{(n)} \int_{t_0}^{t_1} (t_1 - t_0)^{n(\alpha-1)} \frac{(t_1 - s)^{n-1}}{(n-1)!} \, ds \, ||x_1 - x_2||$$

$$\leq \frac{M^n (f_1 + f_2 S)^{(n)} (t_1 - t_0)^{n\alpha}}{n!} ||x_1 - x_2||$$

where S is a bound for the steering operator $\mathcal{S}$ and is finite as the corresponding linear system is controllable.

Since

$$\frac{M^n (f_1 + f_2 S)^{(n)} (t_1 - t_0)^{n\alpha}}{n!} \to 0 \quad \text{as} \quad n \to \infty$$

there exists an n such that the operator $\mathcal{H}$ is a contraction. Hence, the system Equation (7) is controllable, and the controller $u(t)$ which drives the system Equation (7) to the desired final state x_1 is given by

$$u(t) = \mathcal{C}^* \mathcal{W}^{-1} \bar{\mathcal{G}}^{-1} \left[x_1 - \mathcal{T}_\alpha(t_1 - t_0)x_0 \right]$$

This completes the proof. □

The algorithm to find the controller and the state is given by

$$\begin{aligned} u^{(n)}(t) &= \mathcal{C}^* \mathcal{W}^{-1} (\mathcal{I} + \mathcal{H})^{-1} \left[x_1 - \mathcal{T}_\alpha(t_1 - t_0)x_0 \right] \\ x^{(n+1)}(t) &= \mathcal{T}_\alpha(t - t_0)x_0 + \int_{t_0}^{t} (t - s)^{\alpha-1} \mathcal{T}_{\alpha,\alpha}(t - s) \Big[\mathcal{B}u^{(n)}(s) + \mathcal{F}(s, x^{(n)}(s), u^{(n)}(s)) \Big] \, ds \\ \mathcal{H}^{(n)} x_1^{(n)} &= \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \mathcal{T}_{\alpha,\alpha}(t_1 - s) \mathcal{F}(s, x^{(n-1)}(s), \mathcal{S}x^{(n-1)}(s)) \, ds \end{aligned} \tag{11}$$

where

$$u_0(t) = (t_1 - t)^{1-\alpha} \mathcal{B}^* \mathcal{T}^*_{\alpha,\alpha}(t_1 - t) \mathcal{W}^{-1} \left[x_1 - \mathcal{T}_{\alpha,\alpha}(t_1 - t_0)x_0 \right]$$

4. Modelling and controllability analysis of the coal mill system

This section provides an overview of the coal mill process and its modeling and Controllability Analysis. A simplified schematic of a roll mill is shown in **Figure 1**. In the system, raw coal is transported via conveyor belts and fed into the mill, where rollers crush it on a grinding table. Fine coal particles are carried upward by primary air introduced from the mill's base, which directs them toward the classifier section. The classifier selectively allows the smallest particles to exit the mill while larger particles are returned to the grinding table for further processing. Rotary classifiers, when employed, enable adjustment of coal output by modifying the rotational speed, allowing larger particles to pass if needed.The general flow of coal particles within the mill is illustrated in **Figure 1**. The mathematical model developed for this system primarily captures the nominal grinding process [29]. However, it is also robust enough to represent the dynamic behavior during start-up and shutdown operations. The key aspect of the model is the circulation of coal particles within the mill, as depicted in the layout.

As illustrated in **Figure 1**, the circulation of coal particles forms a fundamental aspect of the model.

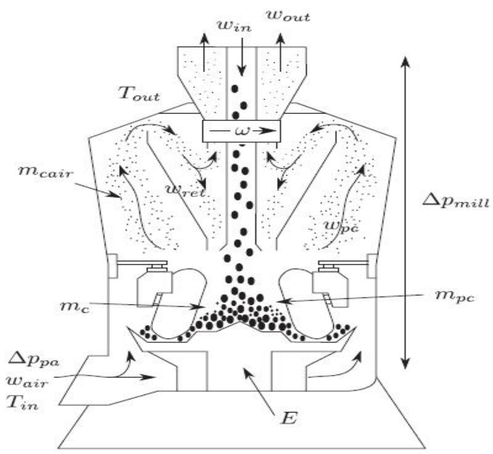

Figure 1. Flow of coal particles in a roll wheel mill [30].

To develop the mathematical model, the following parameters and variables are defined [29], as illustrated in **Figure 2:**

- $m_c(t)$: Mass of raw coal awaiting pulverization.
- $m_{pc}(t)$: Mass of pulverized coal present on the grinding table.
- $m_{cair}(t)$: Mass of coal particles transported pneumatically within the mill.
- $w_{in}/w_c(t)$: Mass flow rate of raw coal entering the mill.
- $w_{ret}(t)$: Mass flow rate of particles rejected by the classifier and returned for further grinding.
- w_{pc}: Mass flow rate of coal picked up from the table by primary air.
- w_{out}: Mass flow rate of pulverized coal exiting the mill.
- w_{air}: Mass flow rate of primary air.
- ω: Rotational speed of the classifier.

Using the principle of continuity, the rate of change of mass of coal (m_c) to be pulverized is equal to the mass flow of raw coal (w_c/w_{in}) and the return flow of the particles rejected by the classifier (w_{ret}) and the grinding rate which is proportional to the mass of raw coal at the grinding table (m_c).

$$\frac{d}{dt}m_c(t) = w_c(t) + w_{ret}(t) - \theta_1 m_c(t) \tag{12}$$

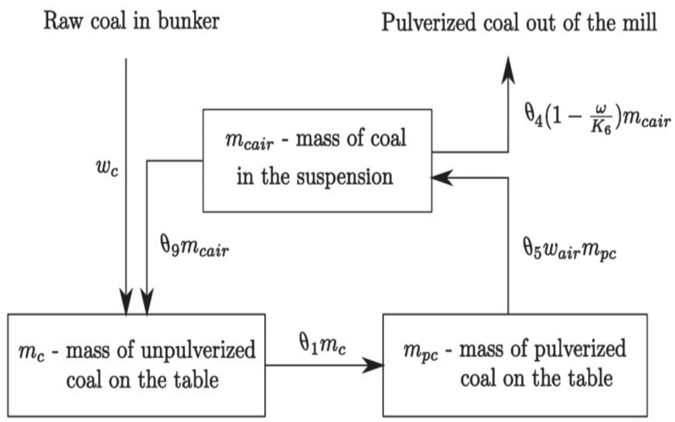

Figure 2. Schematic representation of coal particle flow dynamics in a mill [30].

The rate of change of pulverized coal (m_{pc}) at the table is equal to the amount of grinded raw coal(m_c) minus the amount of coal picked up by the primary air from the table (w_{pc})

$$\frac{d}{dt}m_{pc}(t) = \theta_1 m_c(t) - w_{pc}(t) \tag{13}$$

The mass flow of the coal particles collected from the grinding table by primary air (w_{pc}) minus the fuel out of the mill (w_{out}) and the return flow of the rejected particles to the table (w_{ret}) are equal to the rate of change in the mass of the particles in the pneumatic transport up (m_{cair}) in the mill

$$\frac{d}{dt}m_{cair}(t) = w_{pc}(t) - w_{out}(t) - w_{ret}(t) \tag{14}$$

The primary air mass flow $(wair)$ and the mass of pulverised coal on the table (mpc) are proportional to the mass flow of pulverized particles picked up by the primary air (w_{pc}) to be transferred towards the classifier.

$$\frac{d}{dt}w_{pc}(t) = \theta_5 w_{air}(t) m_{pc}(t) \tag{15}$$

The mass flow of pulverized coal out of the mill $(wout)$ is proportional to the mass of coal lifted off the table $(mcair)$ and is influenced by the classifier speed $omega$.

$$w_{out}(t) = \theta_4 m_{cair}(t)(1 - \frac{\omega(t)}{\theta_6}) \tag{16}$$

where $0 < \omega(t) < \theta_6.\theta_6$

The mass of coal in the pneumatic transport m_{cair} is equal to the mass flow of coal returning to the grinding table proportional to

$$w_{ret}(t) = \theta_9 m_{cair}(t) \tag{17}$$

Niemczyk et al. [30] proposed a mathematical model for the coal mill pulverizer problem, described by a nonlinear system of differential Equations (12)–(14) can be written as:

$$\begin{aligned}\frac{d}{dt}m_c(t) &= w_c(t) + \theta_9 m_{cair}(t) - \theta_1 m_c(t)\\ \frac{d}{dt}m_{pc}(t) &= \theta_1 m_c(t) - \theta_5 w_{air}(t) m_{pc}(t)\\ \frac{d}{dt}m_{cair}(t) &= \theta_5 w_{cair}(t) m_{pc}(t) - \theta_4 m_{cair}(t)\left(1 - \frac{\omega(t)}{\theta_6}\right) - \theta_9 m_{cair}(t)\end{aligned} \tag{18}$$

where the variables represent the mass of coal to be pulverized $m_c(t)$, the mass of pulverized coal on the table $m_{pc}(t)$, and the mass of particles in pneumatic transport $m_{cair}(t)$, while $w_c(t)$ and $w_{air}(t)$ are the respective mass flows, and $\omega(t)$ denotes the classifier speed.

Let us define the system states as $x_1(t) = m_c(t)$, $x_2(t) = m_{pc}(t)$, and $x_3(t) = m_{cair}(t)$, while the control inputs are $u_1(t) = w_c(t)$, $u_2(t) = w_{air}(t)$, and $u_3(t) = \omega(t)$. With this, the model in Equation (18) becomes:

$$\begin{aligned}\dot{x}_1 &= -\theta_1 x_1 + \theta_9 x_3 + u_1\\ \dot{x}_2 &= \theta_1 x_1 - \theta_5 u_2 x_2\\ \dot{x}_3 &= -(\theta_4 + \theta_9)x_3 + \theta_5 u_2 x_2\end{aligned} \tag{19}$$

where the constants θ_1, θ_4, θ_5, θ_6, and θ_9 are known system parameters.

The coal mill pulverizer system exhibits highly nonlinear behaviors, time-varying dynamics, and long-term memory effects, making its control and modeling particularly challenging. Traditional integer-order models often fail to accurately capture these complexities due to their inability to represent the system's hereditary and memory-dependent properties. Fractional-order models, on the other hand, inherently incorporate memory effects and fractional dynamics, offering a more precise representation of such systems. For example, the grinding process in a pulverizer involves not only instantaneous changes in the coal flow but also long-term effects of grinding and accumulation, which are naturally modeled by fractional derivatives. This study, therefore, leverages fractional-order modeling to better characterize the coal mill's dynamics, providing a foundation for more effective control strategies. In the context of coal mill systems, fractional calculus provides significant advantages over traditional integer-order models by accurately capturing the inherent complexities of the process. Coal pulverization involves non-linearities, time-varying dynamics, and considerable memory effects due to delays in grinding and material transport. While integer-order models can approximate some dynamics, they often fail to address the long-term dependencies and intricate transient behaviors observed in real-world systems. Therefore, we now propose a fractional-order version of the system in Equation (19), given by:

$$\begin{aligned}{}^cD^\alpha x_1 &= -\theta_1 x_1 + \theta_9 x_3 + u_1\\ {}^cD^\alpha x_2 &= \theta_1 x_1 - \theta_5 u_2 x_2\\ {}^cD^\alpha x_3 &= -(\theta_4 + \theta_9)x_3 + \theta_5 u_2 x_2 + \theta_4 x_3 \frac{u_3}{\theta_6}\end{aligned} \tag{20}$$

Next, we focus on the controllability analysis of the fractional system in Equation

(20), assuming a constant control input $u_2 = 0.02$. This leads to the simplified system:

$$\begin{aligned} {}^{c}D^{\alpha}x_1 &= -\theta_1 x_1 + \theta_9 x_3 + u_1 \\ {}^{c}D^{\alpha}x_2 &= \theta_1 x_1 - 0.02\theta_5 x_2 \\ {}^{c}D^{\alpha}x_3 &= 0.02\theta_5 x_2 - (\theta_4 + \theta_9)x_3 + \theta_4 x_3 \frac{u_3}{\theta_6} \end{aligned} \tag{21}$$

This can be expressed in a more compact form as:

$${}^{c}D^{\alpha}x(t) = \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{F}(t, x(t), u(t)) \tag{22}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ represents the state vector and $u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ is the control input. The system matrices $\mathcal{A}$, $\mathcal{B}$, and the nonlinear term $\mathcal{F}(t, x(t), u(t))$ are given by:

$$\mathcal{A} = \begin{bmatrix} -\theta_1 & 0 & \theta_9 \\ \theta_1 & -0.02\theta_5 & 0 \\ 0 & -0.02\theta_5 & -(\theta_4 + \theta_9) \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathcal{F}(t, x(t), u(t)) = \begin{bmatrix} 0 \\ 0 \\ \frac{\theta_4}{\theta_6} x_3 u_3 \end{bmatrix}$$

Since the rank of the controllability matrix $P = [B : AB : A^2B]$ is 3 therefore the controllability Gramian of the linearized system is non-singular, the system is controllable in the linear case. Moreover, the nonlinear term $\mathcal{F}(t, x(t), u(t))$ satisfies a Lipschitz condition, with constants $f_1 = \frac{\theta_4}{\theta_6}|u_3^0|$ and $f_2 = \frac{\theta_4}{\theta_6}|x_3^0|$, ensuring that:

$$||\mathcal{F}(t, x(t), u(t)) - \mathcal{F}(t, y(t), v(t))|| \leq f_1||x - y|| + f_2||u - v||$$

By the controllability theorem, this implies that the system is controllable over a finite time interval. The transition from traditional integer-order models to fractional-order systems provides enhanced controllability insights for the coal mill pulverizer system. By accounting for memory and hereditary effects, fractional dynamics enable the model to capture gradual changes and interdependencies that would otherwise be overlooked in integer-order approaches. For instance, the system's response to varying coal loads and grinding pressures is significantly influenced by past operational states, which are naturally incorporated through fractional derivatives.

5. Conclusion and future direction

This study presents a comprehensive analysis of the controllability of nonlinear dynamical systems, with a focus on fractional-order systems and coal mill models. A theorem establishing sufficient conditions for controllability was rigorously derived, demonstrating the role of contraction mappings in ensuring system control. The proposed iterative algorithm offers a practical framework for designing controllers capable of driving nonlinear systems to desired states. Furthermore, the application to the coal mill system highlights the adaptability of the methodology to real-world industrial processes. The results emphasize the significance of fractional dynamics in advancing the understanding and control of complex systems, laying a foundation for further research and development in this field. Looking ahead, future research will extend this methodology to more complex systems and incorporate additional control strategies. The application of fractional dynamics to other industrial processes and emerging technologies, such as robotics and autonomous systems, will also be explored. Moreover, future work will focus on refining the iterative algorithm to enhance computational efficiency and robustness, particularly for large-scale systems.

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